Mar 30, 2017

Exam 3

MAC 2313—Calculus III, Spring 2017

(Neatly!) Print Name: ~ KEY ~

Read all of what follows carefully before starting!

1. This test has **6 problems** (12 parts total) and is worth **100 points**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
7. Regions D in $\mathbb{R}^2$ may be one and/or both of:

   Type I if $D = \{(x, y) : a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}$ for continuous functions f_1, f_2

   Type II if $D = \{(x, y) : g_1(y) \leq x \leq g_2(y), c \leq y \leq d\}$ for continuous functions g_1, g_2.
8. Regions E in $\mathbb{R}^3$ may be any and/or all of:

   Type IB if $E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$ for continuous functions u_1, u_2

   Type IIB if $E = \{(x, y, z) : (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$ for continuous functions u_1, u_2

   Type III if $E = \{(x, y, z) : (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$ for continuous functions u_1, u_2.

   Recall that "$\in$" means "is an element of".

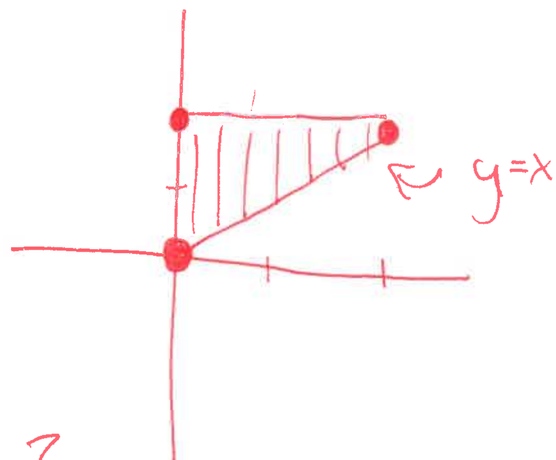
Question	1 (10 pts)	2 (25 pts)	3 (25 pts)	4 (20 pts)	5 (10 pts)	6 (10 pts)	Total (100 pts)
Points							

Do not write in these boxes! If you do, you get 0 points for those questions!

1. (*5 pts ea.*) Sketch each of the following regions and write each as the type designated. **There should be no integrals in your answers!**

(a) T_1 = the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,2)$, $(2,2)$; as a Type I region.

$\{(x,y): 0 \leq x \leq 2,\ x \leq y \leq 2\}$



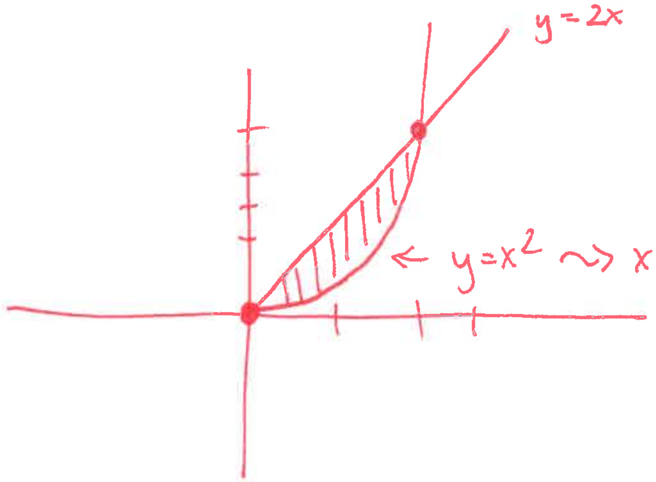
(b) T_2 = the same triangle as in (a); as a Type II region.

$\{(x,y): 0 \leq x \leq y,\ 0 \leq y \leq 2\}$

2. (a) (*20 pts*) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. **Simplify fully.**

SOLUTION:

$y=2x \rightsquigarrow x=\frac{y}{2}$



$2x=x^2$

$\Rightarrow x=0$ or $x=2$

$D=\{(x,y): 0\le x\le 2,\ x^2\le y\le 2x\}$

$=\{(x,y): \frac{y}{2}\le x\le \sqrt{y},\ 0\le y\le 4$

$$\text{Vol} = \iint_D x^2+y^2\,dA = \int_0^2\int_{x^2}^{2x} x^2+y^2\,dy\,dx$$

$$= \int_0^2 \left[\left[x^2y+\frac{1}{3}y^3\right]_{y=x^2}^{y=2x} \right] dx$$

$2x^3+\frac{8}{3}x^3 - x^4 - \frac{1}{3}x^6$

$$= \int_0^2 -\frac{1}{3}x^6 - x^4 + \overbrace{2x^3 + \frac{8}{3}x^3}^{\frac{14}{3}x^3}\,dx$$

$$= -\frac{1}{21}x^7 - \frac{1}{5}x^5 + \frac{14}{12}x^4\Big]_0^2$$

$$= \frac{-2^7}{21} - \frac{2^5}{5} + \frac{7(16)}{6} = \cdots$$

Part (b) is on the next page

(b) (*5 pts*) Now, set up the integral you would use to get the volume of the solid in part (a) using *the other* order of integration. **Do not integrate!**

Hint: In (a), you found the solution by expressing D as *either* a Type I *or* a Type II region; here, your answer should be the integral which expresses D as *the other* type.

SOLUTION:

$$\int_0^4 \int_{y/2}^{\sqrt{y}} x^2+y^2 \, dx \, dy$$

Right func w/ $x \leftrightarrow y$ 4

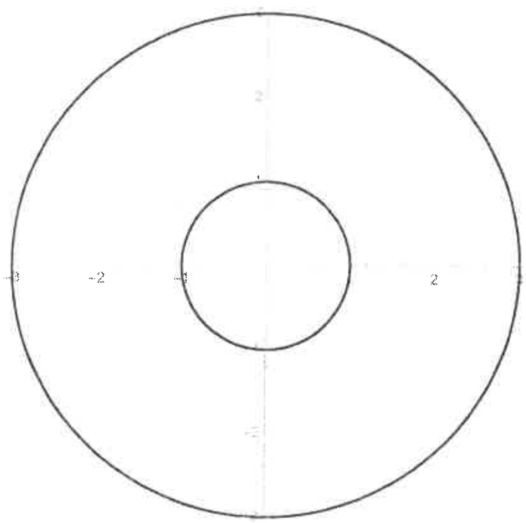
$x \leftrightarrow y$ w/ no new func 2

Right func + var switch, wrong order 3

3. (a) (5 *pts*) Let D be a region in the xy-plane, let $z = f(x, y)$ be the equation of a surface S, and let $A(S)$ denote the surface area of the portion of S over the region D. Write the formula for $A(S)$.

$$A(S) = \iint_D \sqrt{1 + [f_x]^2 + [f_y]^2}\, dA$$

(b) (5 *pts*) Express the below region in polar coordinates.



SOLUTION:

$\{(r, \theta) : 1 \leq r \leq 3, \ 0 \leq \theta \leq 2\pi\}$

Part (c) is on the next page

(c) (*15 pts*) Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$ and above the plane $z = 1$. **Hint**: This is part three of a problem; perhaps the other two parts help?

SOLUTION:

$f_x = 2x \quad f_y = 2y$

@ $z=1$: $\begin{matrix} x^2+y^2=1 \\ z=1 \end{matrix} \xrightarrow[\text{xy-plane}]{\text{proj to}} x^2+y^2=1 \rightsquigarrow r^2=1$

$\Rightarrow r=1$

@ $z=9$: $\begin{matrix} x^2+y^2=9 \\ z=9 \end{matrix} \xrightarrow[\text{xy-plane}]{\text{proj to}} x^2+y^2=9 \rightsquigarrow r^2=9$

$\Rightarrow r=3$

$z = x^2+y^2 = r^2$ in cylindrical/polar.

$A(S) = \iint_D \sqrt{1+(2x)^2+(2y)^2}\, dA$ where $D = \{(r,\theta): 1 \le r \le 3,\ 0 \le \theta \le 2\pi\}$

$= \iint_D \sqrt{1+4(x^2+y^2)}\, dA$

$= \int_0^{2\pi} \int_1^3 \sqrt{1+4r^2}\, r\, dr\, d\theta$

$u = 1+4r^2 \quad du = 8r\,dr \quad \frac{du}{8} = r\,dr$

$\int \sqrt{\cdots}\, dr = \frac{1}{8} \int \sqrt{u}\, du$

$\frac{2}{3} u^{\frac{3}{2}}$

$= \frac{2}{3} \cdot \frac{1}{8} \int_0^{2\pi} \left(1+4r^2\right)^{3/2} \Big]_{r=1}^{r=3} d\theta$

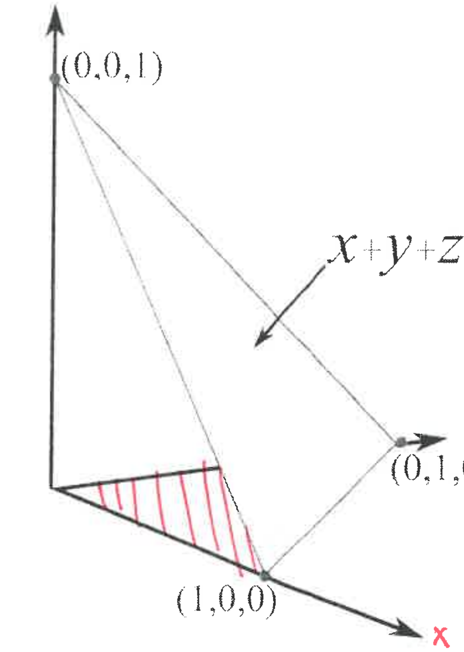
$= \frac{1}{12} \int_0^{2\pi} (37^{3/2} - 5^{3/2})\, d\theta = \frac{\pi}{6}\left(37^{3/2} - 5^{3/2}\right).$

Region: 5

→ Polar: 5

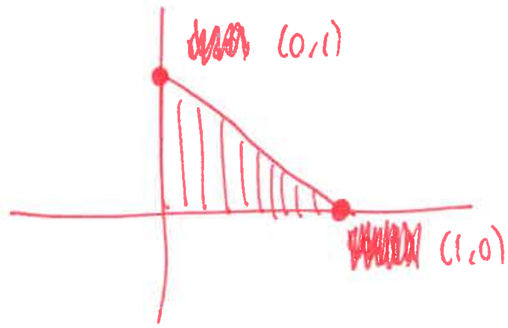
Int/Ans: 4 / 1

4. (*20 pts*) Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$ (see below).



SOLUTION:

$z: 0 \to 1-x-y$



xy-plane

$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$$V(T) = \iiint_E 1\, dV \quad \text{where} \quad E = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}$$

$$= \iint_D \left(\int_0^{1-x-y} 1\, dz \right) dA$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

$$-\tfrac{1}{2}(1-2x+x^2)$$

$$\cancel{1}-x-\cancel{x}+x^2-\cancel{\tfrac{1}{2}}+\cancel{x}-\tfrac{1}{2}x^2$$

$$\tfrac{1}{2}-x+\tfrac{1}{2}x^2$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_0^1 \left[y - yx - \tfrac{1}{2}y^2 \right]_{y=0}^{y=1-x} dx = \int_0^1 1-x-(1-x)x - \tfrac{1}{2}(1-x)^2\, dx$$

$$= \int_0^1 \tfrac{1}{2} - x + \tfrac{1}{2}x^2\, dx = \tfrac{1}{2}x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3$$

$$= \boxed{\tfrac{1}{6}}$$

5. (*5 pts ea.*) Rewrite each of the rectangular iterated integrals in the coordinate system specified. **Do not integrate!**

(a) $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$; in cylindrical coordinates.

SOLUTION:

$z: r^2 \to 2-r^2$

$D = \{(x,y): -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, -1 \le x \le 1\}$

$= \{(r,\theta): 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r\, dz\, r\, dr\, d\theta$$

Part (b) is on the next page

(b) $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$; in spherical coordinates.

SOLUTION:

$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

"r" = "ρ"

$z: 0 \to \sqrt{9-x^2-y^2}$

$0 \to \sqrt{9-(r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta)}$

$0 \to \sqrt{9-r^2\sin^2\phi} \Rightarrow z^2 + r^2\sin^2\phi = 9$

$\Rightarrow r^2(\cos^2\phi + \sin^2\phi) = 9$

$r^2 = 9$

$r = 3$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{3} \rho(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$y: -\sqrt{9-x^2} \to \sqrt{9-x^2}$

$: -\sqrt{9-r^2 \sin^2\phi\cos^2\theta} \to \ldots$

$y^2 = 9 - r^2\sin^2\phi\cos^2\theta$

$r^2\sin^2\phi(\sin^2\theta + \cos^2\theta) = 9$

$r^2\sin^2\phi = 9 \underset{r=3}{\leadsto} 9\sin^2\phi = 9 \leadsto \sin^2\phi = 1$

$\leadsto \sin\phi = \pm 1$

$\leadsto \phi: -\frac{\pi}{2} \to \frac{\pi}{2}$

$x: -3 \to 3$

$r^2\sin^2\phi\cos^2\theta : 0 \to 9$

$\cos^2\theta = 1$

$r=3,\ \sin^2\phi = 1$

$\Downarrow$

$\cos\theta = \pm 1 \qquad \theta: 0 \to \pi$

6. (a) (*5 pts*) Compute the Jacobian of the transformation from rectangular coordinates to cylindrical coordinates in $\mathbb{R}^3$. **Simplify fully.**

SOLUTION:

$$x(r,\theta,z) = r\cos\theta$$
$$y(r,\theta,z) = r\sin\theta$$
$$z(r,\theta,z) = z$$

$$J = \det\begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} & \frac{dx}{dz} \\ \frac{dy}{dr} & \frac{dy}{d\theta} & \frac{dy}{dz} \\ \frac{dz}{dr} & \frac{dz}{d\theta} & \frac{dz}{dz} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \cos\theta \cdot \det\begin{pmatrix} r\cos\theta & 0 \\ 0 & 1 \end{pmatrix} - -r\sin\theta \det\begin{pmatrix} \sin\theta & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ 0 \ldots$$

$$= \cos\theta\,[r\cos\theta] + r\sin\theta\,[\sin\theta]$$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

Part (b) is on the next page

(b) Use the result from part (a) to derive the formula for the triple integral in cylindrical coordinates of a function $f(x, y, z)$ over a region E in $\mathbb{R}^3$.

SOLUTION:

$$\iiint_E f(x,y,z)\,dV = \iiint_{E'} f(r\cos\theta, r\sin\theta, z)\,|J|\,dz\,dr\,d\theta$$

$$= \iiint_{E'} \cdots \quad r\,dz\,dr\,d\theta.$$

Bonus 1:

(a) (*4 pts*) Verify that

$$f(x,y) = \begin{cases} 4xy & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

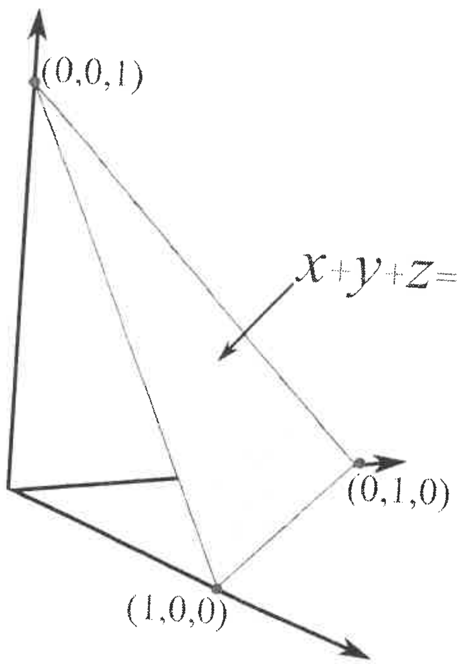
is a joint (probability) density function.

(b) (*3 pts ea.*) Find the expected values of X and Y.

SOLUTION:

Bonus 2: *(2 pts ea.)* In question 4, you used triple integration to find the volume of the tetrahedron T bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$ shown below. Rewrite this integral as an equivalent iterated integral in the other five orders. **Do not integrate!**

Hint: Upon completing this problem, you should have evaluated/written six integrals—one each for $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$, $dy\,dz\,dx$, $dz\,dx\,dy$, and $dz\,dy\,dx$—and each should give the same volume for T.



SOLUTION:

Scratch Paper