Feb 28, 2017

Exam 2

MAC 2313—CALCULUS III, SPRING 2017

(NEATLY!) PRINT NAME: ~ KEY ~

Read all of what follows carefully before starting!

1. This test has **4 problems** (12 parts total) and is worth **100 points**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam: you may also use the backs of pages or get more scratch paper from me.

Question	1 (20 pts)	2 (15 pts)	3 (30 pts)	4 (35 pts)	Total (100 pts)
Points					

Do not write in these boxes! If you do, you get 0 points for those questions!

1. Let $f(x, y) = e^{x+xy}$. $= e^{x(1+y)}$

(a) (*10 pts*) Find the equation of the tangent plane to f at the point $(1, 0, e)$.

SOLUTION: $f_x = (1+y)e^{x(1+y)} \Rightarrow f_x(1,0) = e$

$f_y = xe^{x(1+y)} \Rightarrow f_y(1,0) = e$

$\Rightarrow$ plane: $z - e = e(x-1) + e(y-0)$

$\Rightarrow z = e(x-1) + ey + e.$

Part (b) is on the next page

- f_x & f_y $= 4$
- @ pt $= 1$
- Eq. $= 4$

$$f_x = (1+y)e^{x(1+y)} \qquad f_y = xe^{x(1+y)}$$

(b) (*10 pts*) Prove that $f(x,y)$ is a solution to the *partial differential equation*

$$\overbrace{xf_y - f_x}^{\text{LHS}} = \overbrace{f_{yy} - \frac{f_{xx}}{1+y}}^{\text{RHS}}.$$

Hint: This means that if you compute f_x, f_y, f_{xx} and f_{yy}, they should satisfy that equation.

Solution: <u>Note:</u> $f_{xx} = (1+y)^2 e^{x(1+y)}$

$$f_{yy} = x^2 e^{x(1+y)}$$

$$\Rightarrow \text{RHS} = x^2 e^{x(1+y)} - \frac{(1+y)^2 e^{x(1+y)}}{(1+y)} = x^2 e^{x(1+y)} - (1+y)e^{x(1+y)}$$

$$= x\left[xe^{x(1+y)}\right] - (1+y)e^{x(1+y)}$$

$$= x f_y - f_x$$

$$= \text{LHS}. \quad \blacksquare$$

2. Let $f(w, x, y, z) = w + e^x - \sin y + z\cos z - xyz + 4$, where

$$w = r^2 s - t \qquad x = t\sin s + r \qquad y = r(t - s) \qquad z = e^{e^t} + r - s.$$

(a) (*5 pts*) Find f_{zyxw}, citing any results you may use to simplify your computations.

SOLUTION: By Clairaut [which we can use because all partials of f are defined & continuous everywhere....]

$f_{zyxw} = f_{wzyx} = (f_w)_{zyx}$.

Now, $f_w = 1 \Rightarrow (f_w)_{zyx} = 0$.

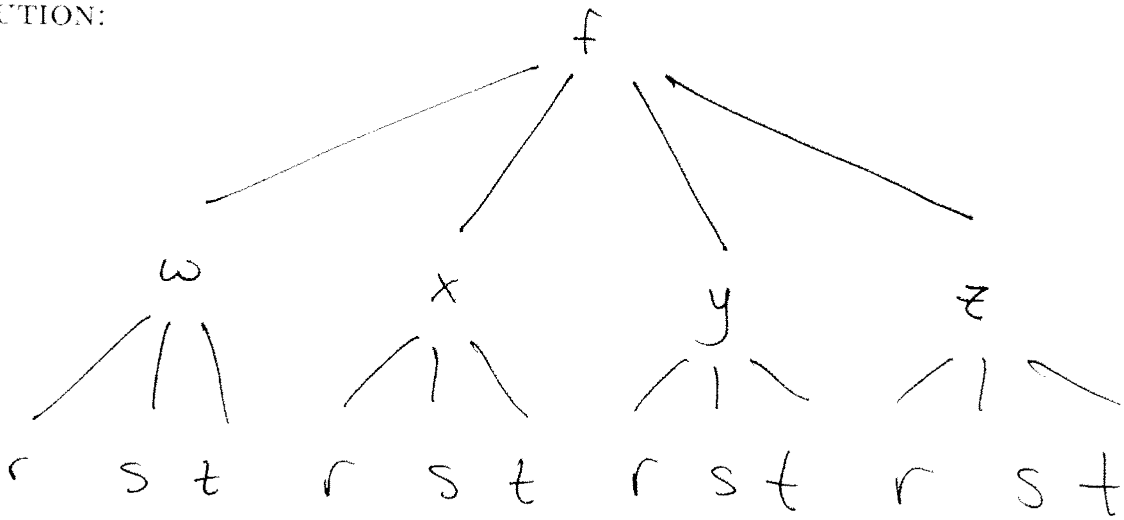
Part (b) is on the next page

Say Clairaut → 1 pt
Justify Clairaut → 1 pt
Right deriv. → 3 pts

or Doing all derivatives right → 5 (1 pt ea)

(b) (*10 pts*) Find $\dfrac{\partial f}{\partial t}$.

SOLUTION:



$$\Rightarrow \frac{\partial f}{\partial t} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= 1(-1) + (e^x - yz)(\sin s) + (-\cos y - xz)(r) + e^t e^{e^t}$$

$\uparrow$ now, plug in $w, x, y, z \ldots$

$\uparrow$ $(-z\sin z + \cos z - xy)$

Tree/Eq $\rightarrow$ 2 pts

Right partials $\Rightarrow$ 2 pts ea ($\times 4$)

3. Let $g(x, y) = xe^y$, let $\theta = \pi/6$, and let $\mathbf{u}$ be the unit vector given by θ.

(a) (*5 pts*) Is g differentiable? Why or why not?

g is differentiable everywhere because g_x, g_y both exist and are continuous everywhere.

Mostly all or nothing: +1 for graph comment

(b) (*10 pts*) Using any technique you know (e.g. witchcraft, voodoo magic, limits, shortcuts we learned in class,...), find the directional derivative of g in the direction of $\mathbf{u}$.

$$\vec{u} = \langle \cos\tfrac{\pi}{6}, \sin\tfrac{\pi}{6} \rangle = \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle$$

$$\nabla g = \langle g_x, g_y \rangle = \langle e^y, xe^y \rangle$$

$$\Rightarrow D_{\vec{u}} g(x,y) = \nabla g \cdot \vec{u}$$

$$= \langle e^y, xe^y \rangle \cdot \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle$$

$$= \frac{\sqrt{3}}{2} e^y + \frac{1}{2} x e^y.$$

Find $\vec{u} \rightarrow$ 2 pts

Grad $g \rightarrow 3$ | limit form $\rightarrow 3$

Formula $\rightarrow 2$ | Work $\rightarrow 5$

dot/ans $\rightarrow 3$

Part (c) is on the next page

(c) (*10 pts*) At the point $P(1,1)$, in what direction does g have the maximum rate of change? What is this maximum rate of change?

SOLUTION:

Direction $= \nabla g(1,1)$

$$= \langle e^y, xe^y \rangle \Big|_{(1,1)}$$

$$= \langle e, e \rangle.$$

Rate: $|\nabla g(1,1)| = |\langle e, e \rangle|$

$$= \sqrt{e^2 + e^2}$$

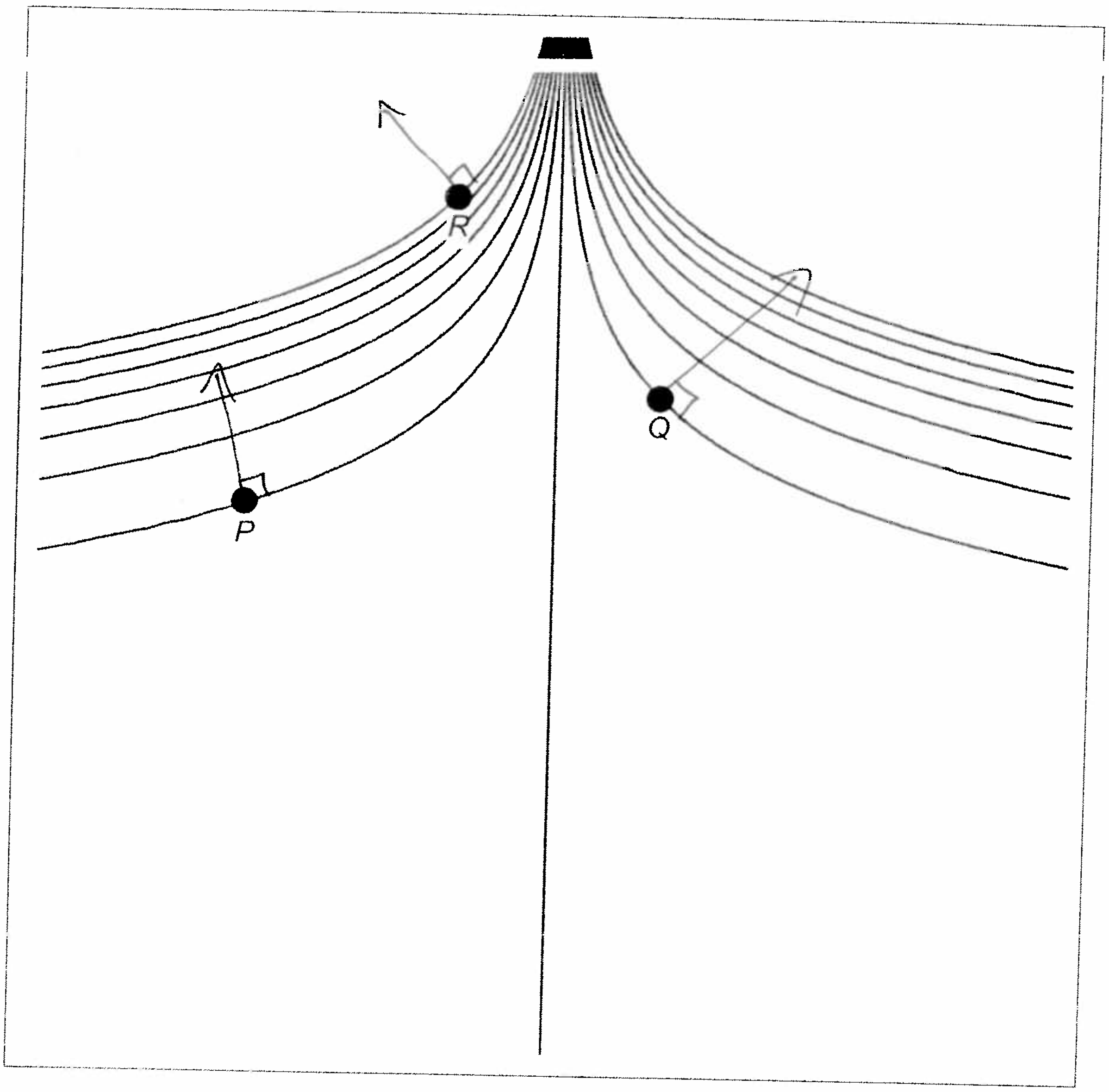
$$= \sqrt{2}\, e.$$

- Direction $\rightarrow$ 5
  - ↳ fact about grad $\rightarrow$ 3
  - plug in pt $\rightarrow$ 2
- Rate $\rightarrow$ 5
  - ↳ fact about grad $\rightarrow$ 3
  - mag $\rightarrow$ 2

Part (d) is on the next page

(d) (*5 pts*) Below is a contour plot of $g(x, y) = k$ for various values of k (a constant).

At each of the three points P, Q, and R, draw a vector in the same direction as ∇g. *I don't care about the magnitude of the vectors you draw.*



all or nothing

4. Let $f(x, y) = x^2 + 3y^2$.

$f_x = 2x \quad f_y = 6y$

$f_{xx} = 2 \quad f_{xy} = 0 = f_{yx} \quad f_{yy} = 6$

(a) (*5 pts*) Is $(3, 4)$ a critical point of f? How do you know?

No: $f_x(3,4) = 6 \neq 0$

$f_y(3,4) = 24 \neq 0$.

partials → 2; plug in pt → 1; concl → 2

(b) (*10 pts*) The point $P(0, 0)$ is a critical point for f. Use the second derivative test to determine if P is a local maximum, a local minimum, or a saddle point for f.

- $D(x,y) = \det \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} = 12 > 0$

  $\Rightarrow (0,0)$ either max or min.

- $f_{xx}(0,0) = 2 > 0 \Rightarrow \boxed{(0,0) \text{ is min.}}$

2nd der test → 2

$D(x,y)$ → 4

Check f_{xx} → 2

min → 2

Part (c) is on the next page

$$f(x,y) = x^2 + 3y^2 \qquad f_x = 2x \quad f_y = 6y$$

(c) (*10 pts*) Use Lagrange multipliers to find the extreme values of f on the ellipse $g(x,y) = 1$, where

$$g(x,y) = \frac{x^2}{4} + y^2. \qquad g_x = \frac{x}{2} \quad g_y = 2y$$

SOLUTION: Lagrange $\Rightarrow \nabla f = \lambda \nabla g$

$$\Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 1 \end{cases} \Rightarrow \begin{cases} 2x = \lambda\left(\frac{x}{2}\right) & (1) \\ 6y = \lambda(2y) & (2) \\ \frac{x^2}{4} + y^2 = 1 & (3) \end{cases}$$

So: (1) $\Rightarrow 4x = \lambda x \Rightarrow$ either $\lambda = 4$ or $x = 0$.

↳ If $\lambda = 4$: (2) $\Rightarrow 6y = 8y \Rightarrow y = 0$, and (3) $\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$] $(2,0)$ & $(-2,0)$

↳ If $x = 0$: (3) $\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$] $(0,1)$ & $(0,-1)$

Now:

$f(2,0) = 4$
$f(-2,0) = 4$] Maxes

$f(0,1) = 3$
$f(0,-1) = 3$] Mins.

Part (d) is on the next page

Eq's $\rightarrow$ 2

Test pts $\rightarrow$ 4 (1 ea)

Max/Min $\rightarrow$ 4 (1 ea)

(d) (*10 pts*) Find absolute maxima and minima of f on the region Σ, where

$$\Sigma = \left\{ (x, y) \text{ in } \mathbb{R}^2 \text{ such that } \frac{x^2}{4} + y^2 \leq 1 \right\}.$$

Note: If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!

SOLUTION:

Note: f is continuous everywhere and Σ is closed & bounded: By the extreme value theorem, f does have abs max and abs min on Σ; moreover, these must occur either on the boundary of Σ or at critical pts of f in Σ.

- Part (c) tested boundary:

  $f(2,0) = 4$; $f(-2,0) = 4$; $f(0,1) = 3$; $f(0,-1) = 3$.

- The critical pts of f are (x,y) such that

  $$\left.\begin{array}{l} f_x = 2x = 0 \\ f_y = 6y = 0 \end{array}\right] \Rightarrow (0,0) \text{ only crit pt of } f$$

  [AND it lives in Σ, so we test it!]

- $f(0,0) = 0$.

- So: $(0,0)$ abs min (value $= 0$) & $(\pm 2, 0)$ abs max (value $= 4$).

EVT → 4 (1 closed/bounded, 1 cont., 2 for which pts to check)
Refer to (c) → 1 ↳ may be given if those checked explicitly (& only those)
crit pt → 2
2 (1 pt ea)

Scratch Paper