Feb 2, 2017

Exam 1

MAC 2313—Calculus III, Spring 2017

(Neatly!) Print Name: ~~~ KEY ~~~

Read all of what follows carefully before starting!

1. This test has **4 problems** (14 parts total) and is worth **100 points**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students.
3. No calculators may be used on this exam!
4. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
5. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
6. You **do not** need to simplify results, unless otherwise stated.
7. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.

Question	1 (16 pts)	2 (25 pts)	3 (35 pts)	4 (24 pts)	Total (100 pts)
Points					

Do not write in these boxes! If you do, you get 0 points for those questions!

1. (*4 pts ea.*) Let $\mathbf{v} = \langle 2, -1, 1 \rangle$ and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$. Compute each of the following or state that it does not exist; if it does not exist, clearly explain why.

(a) $|\mathbf{v} - \mathbf{w}|$

$= |\langle -1, -2, 2 \rangle|$

$= 3$

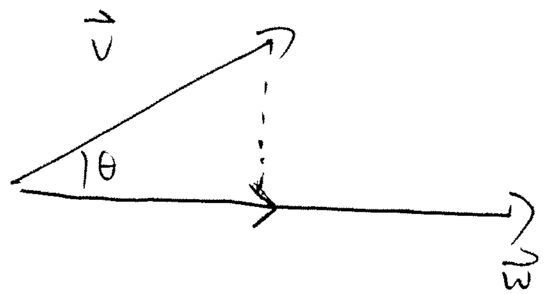
(b) $|\mathbf{v} - \mathbf{w}| \times \mathbf{w}$

DNE : Can't cross scalar w/ vector

Part (d) is on the next page

$$\vec{v} = \langle 2, -1, 1 \rangle$$
$$\vec{w} = \langle 3, 1, -1 \rangle$$

(c) $\text{proj}_{\mathbf{w}}\, \mathbf{v}$



$$\text{proj}_{\vec{w}}\, \vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \right) \vec{w}$$

$$= \frac{4}{11} \langle 3, 1, -1 \rangle$$

$$= \left\langle \frac{12}{11}, \frac{4}{11}, \frac{-4}{11} \right\rangle$$

(d) The angle between $\mathbf{w}$ and $\mathbf{v} - \text{proj}_{\mathbf{w}}\, \mathbf{v}$.

$$\vec{v} - \text{proj}_{\vec{w}}\, \vec{v} = \langle 2, -1, 1 \rangle - \left\langle \frac{12}{11}, \frac{4}{11}, \frac{-4}{11} \right\rangle$$

$$= \left\langle \frac{10}{11}, \frac{-15}{11}, \frac{15}{11} \right\rangle \stackrel{\text{def}}{=} \vec{u} \leadsto \theta = \cos^{-1}\left(\frac{\vec{w} \cdot \vec{u}}{|\vec{w}||\vec{u}|} \right)$$

$$\rightarrow \theta = \cos^{-1}\left(\frac{0}{\cdots} \right) = \frac{\pi}{2}.$$

2. (a) (*15 pts*) Find the equation of the plane containing the vectors $\mathbf{a} = \langle -1, -1, 1\rangle$ and $\mathbf{b} = \langle 1, 2, -2\rangle$ and passing through the point $(0, 0, 2)$.

SOLUTION:

So $\vec{n} = \vec{a} \times \vec{b} \perp$ plane. Now.

$$\vec{n} = \vec{a} \times \vec{b} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix} = \cdots = \langle 0, -1, -1\rangle$$

$\Rightarrow$ normal vect $= \langle 0, -1, -1\rangle$ w/ pt $(0,0,2)$

$\Rightarrow$ plane: $0(x-0) + -1(y-0) + -1(z-2) = 0$

$\Rightarrow \quad -y - z + 2 = 0.$

Part (b) is on the next page

$\perp$ vec + ~~pla~~ point — 5pts
cross prod ⟶ 5pts
plane equation ⟶ 5pts

(b) (*10 pts*) Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ with the plane from part (a).

SOLUTION:

Intersectial lives on cylinder $x^2+y^2=1$

$\Rightarrow$ $x = \cos t$ $\quad y = \sin t$.

Now, on the plane $-y-z+2=0$,

$$-\sin t - z + 2 = 0$$

$$\Rightarrow z = 2 - \sin t$$

$\Rightarrow$ curve $= \langle \cos t, \sin t, 2-\sin t \rangle$.

3. (*5 pts ea.*) Let $\mathbf{r}(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle$ and note that:

$$\begin{aligned} \mathbf{r}'(t) &= \langle e^t, e^t \cos t + e^t \sin t, e^t \cos t - e^t \sin t \rangle, \\ \mathbf{r}''(t) &= \langle e^t, 2e^t \cos t, -2e^t \sin t \rangle, \text{ and} \\ |\mathbf{r}'(t) \times \mathbf{r}''(t)| &= \sqrt{6} e^{2t}. \end{aligned}$$

Find each of the following.

(a) The domain of $\mathbf{r}$

$\mathbb{R}$

(b) The velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t)$

$\vec{v} = \vec{r}' = \langle e^t, e^t\cos t + e^t \sin t, e^t \cos t - e^t \sin t \rangle$

$v = |\vec{v}| = \cdots = \sqrt{3}\, e^t$

$\vec{a} = \vec{r}'' = \langle e^t, 2e^t \cos t, -2e^t \sin t \rangle$

Part (c) is on the next page

(c) The unit tangent vector $\mathbf{T}(t)$ of $\mathbf{r}$

$$T = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}}{v} = \frac{1}{\sqrt{3}} \langle 1, \cos t + \sin t, \cos t - \sin t \rangle$$

(d) The unit normal vector $\mathbf{N}(t)$ of $\mathbf{r}$

$$T' = \frac{1}{\sqrt{3}} \langle 0, -\sin t + \cos t, -\sin t - \cos t \rangle$$

$$N = \frac{T'}{|T'|}$$

$$\Rightarrow \underbrace{|T'| = \sqrt{\frac{\sin^2 + \cos^2 + \sin^2 + \cos^2}{3}}}_{\text{b/c middle FOIL terms cancel}} = \sqrt{\frac{2}{3}}$$

$$= \frac{1}{\sqrt{2}} \langle 0, -\sin t + \cos t, -\sin t - \cos t \rangle$$

(e) The curvature $\kappa(t)$ of $\mathbf{r}$

$$K = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{|\vec{v} \times \vec{a}|}{v^3} = \frac{\sqrt{6}e^{2t}}{(\sqrt{3})^3 e^{3t}}$$

$$= \sqrt{\frac{6}{27}}\, e^{-t}$$

$$= \frac{\sqrt{2}}{3e^t}.$$

Part (f) is on the next page

(f) The tangential component of the acceleration $\mathbf{a}(t)$

$$a_T = \frac{\vec{v} \cdot \vec{a}}{v} = \cdots = \sqrt{3}\, e^t$$

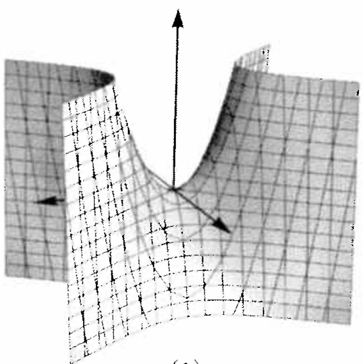
$$\boxed{a_N = \frac{|\vec{v} \times \vec{a}|}{v} = \frac{\sqrt{6}\, e^{2t}}{\sqrt{3}\, e^t} = \sqrt{2}\, e^t}$$

(g) The normal component of $\mathbf{a}(t)$

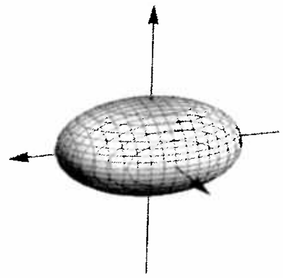
alle

m

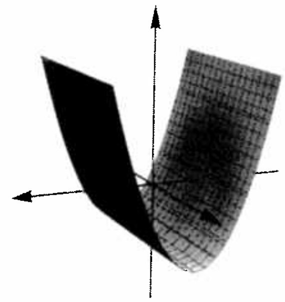
4. (*4 pts ea.*) Match the following cylinder/quadric surface graphs with the equations which define them by writing (i) — (vi) beneath the appropriate equation(s). **You do not need to show work!**



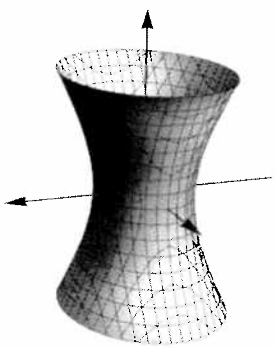
(i)



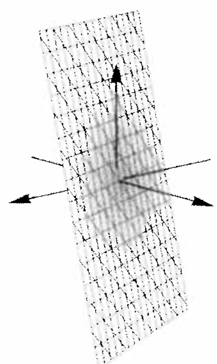
(ii)



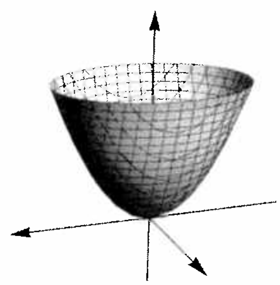
(iii)



(iv)



(v)



(vi)

(a) $\frac{3}{2}x - 9y = 2z - 1$

(v)

(c) $\frac{x^2}{4} + \frac{y^2}{2} = \frac{z}{4}$

(vi)

(e) $z = x^2 - \frac{1}{2}$

(iii)

(b) $\frac{x^2}{0.8} - \frac{y^2}{0.45} = \frac{z}{2}$

(i)

(d) $\frac{x^2}{0.45} + \frac{y^2}{0.8} - \frac{z^2}{2} = 1$

(iv)

(f) $\frac{x^2}{4} + \frac{y^2}{2} + z^2 = 1$

(ii)

Bonus: Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ be arbitrary vectors in $\mathbb{R}^3$. Prove each of the following.

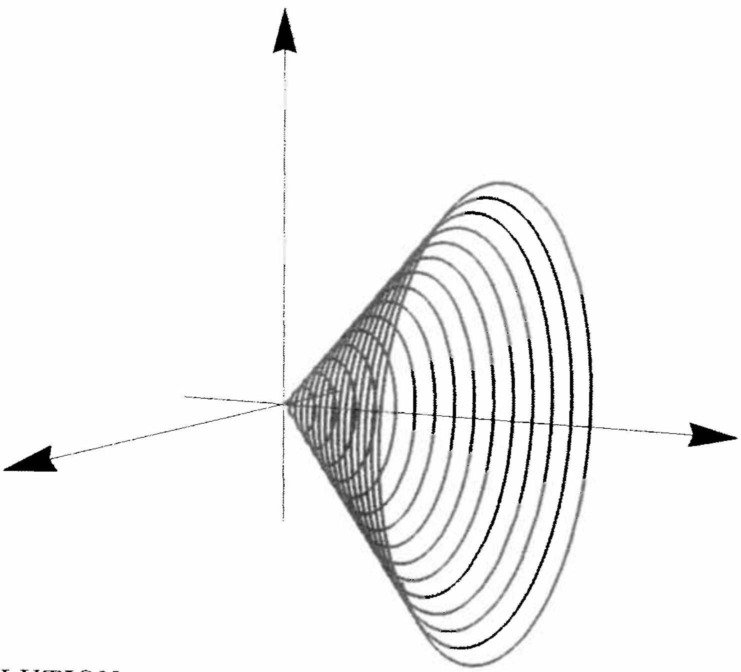
(a) *(3 pts)* $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

(b) *(3 pts)* $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$

(c) *(3 pts)* The vector $\mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b}$ is orthogonal to $\mathbf{a}$

(d) *(6 pts)* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Bonus: Below is the graph of the space curve $\mathcal{C}$ corresponding to $\mathbf{r}(t) = \langle t\cos t, t, t\sin t\rangle$ for $0 \le t \le 100$.



(a) (*2 pts*) Intuitively, do you think the curvature of $\mathcal{C}$ gets larger, gets smaller, or remains constant as t increases from 0 to 100? Explain.

(b) (*5 pts*) Prove that the curvature of $\mathcal{C}$ is equal to

$$\kappa(t) = \frac{\sqrt{t^4 + 5t^2 + 8}}{(t^2+2)^{3/2}}.$$

(c) (*3 pts*) Based on (b), what happens to the curvature of $\mathcal{C}$ as $t \to \infty$?

SOLUTION:

Scratch Paper