§11.4 – Comparison Tests

Recall: Sums are "like integrals," and for indefinite integrals, the comparison test showed

- smaller than $\left(\begin{array}{c}\text{something we}\\ \text{know is}\\ \text{small}\end{array}\right) \Rightarrow$ small
  (i.e. convergent) (convergent)
- bigger than $\left(\begin{array}{c}\text{something we}\\ \text{know is}\\ \text{big}\end{array}\right) \Rightarrow$ big
  (i.e. divergent) (divergent).

we expect the same for series:

Ex: $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ (geometric series w/ $a = \frac{1}{2}$, $r = \frac{1}{2}$)

and $\frac{1}{2^n+1} < \frac{1}{2^n} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n+1} = s < 1$.

we expect this but don't know it yet!

The Comparison Test

Suppose $\sum a_n$, $\sum b_n$ are series w/ positive terms.

① If $\sum b_n$ converges & $a_n \leq b_n$ $\forall n$, then $\sum a_n$ converges.

"for all"

② If $\sum b_n$ diverges & $a_n \geq b_n$ $\forall n$, then $\sum a_n$ diverges.

To use this, we almost always compare w/ a p-series or a geometric series!

Ex: Discuss convergence:

① $\sum \frac{5}{2n^2+4n+3} < \sum \frac{5}{2n^2} = \frac{5}{2}\sum \frac{1}{n^2}$

② $\sum \frac{\ln k}{k} > \sum \frac{1}{k}$ for $k \geq 3$.

Note: Being smaller than a divergent series or larger than a convergent series tells you **nothing**!

↳ Ex: $\sum \frac{1}{2^n-1} > \sum \frac{1}{2^n}$ & this converges but that tells you nothing!

↳ Note, however, that if $a_n = \frac{1}{2^n-1}$ & $b_n = \frac{1}{2^n}$,

then

$$\frac{a_n}{b_n} = \frac{1/(2^n-1)}{1/2^n} = \frac{2^n}{2^n-1} = \frac{1}{1-2^{-n}} \to 1 \text{ as } n\to\infty.$$

As it happens, THIS + $\sum b_n$ converging is enough to conclude that $\sum a_n$ converges.

The Limit Comparison Test.

Suppose $\sum a_n$ & $\sum b_n$ are series w/ positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \quad 0<c<\infty,$$

then either both series converge or both diverge.

Ex: Test $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ for convergence.

TO DO: Let a_n be what you're given & b_n be what it looks like when you squint at it!

$$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}} \qquad b_n = \frac{2n^2}{\sqrt{n^5}} = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}.$$

Now: $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{2n^2+3n}{\sqrt{5+n^5}}\right)\cdot\left(\frac{n^{1/2}}{2}\right)$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}} = \frac{2}{2} = 1.$$

Since $\sum b_n = \sum \frac{2}{\sqrt{n}} = 2\sum 1/n^{1/2}$ is divergent (p-series) and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ w/ $0<c<\infty$, $\sum a_n$ also diverges!