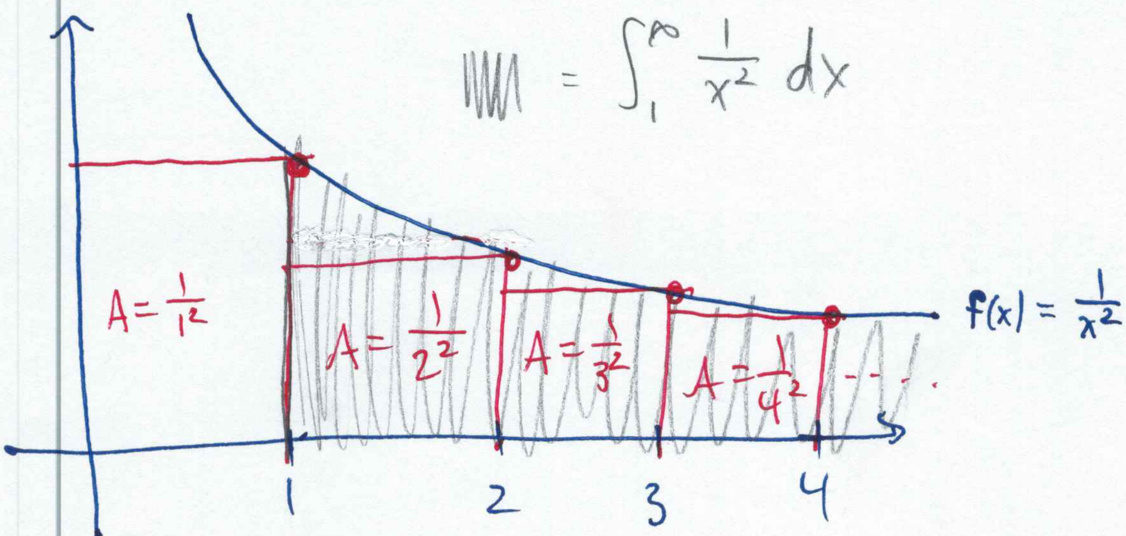
§ 11.3 – Integral Test

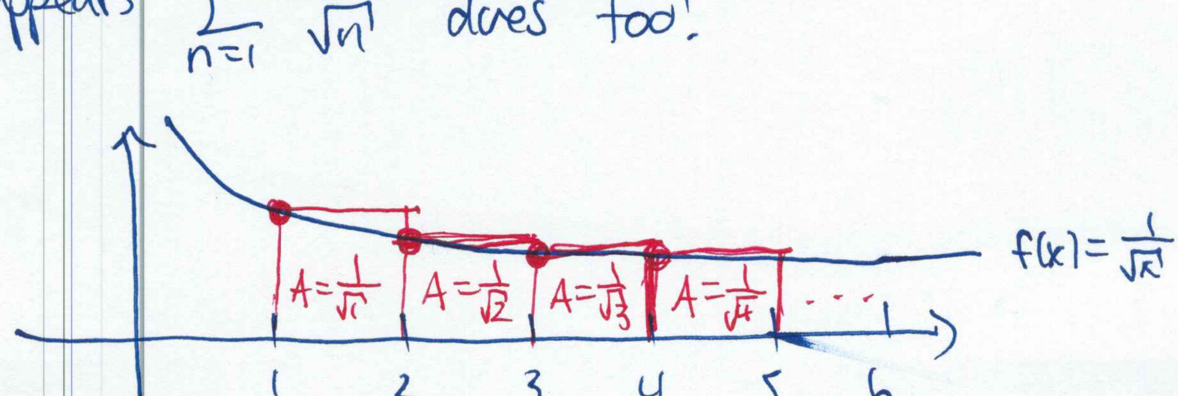
<u>Idea:</u> You want to think of sums as being "like integrals".

<u>Ex:</u> By using a computer, can determine that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. We <u>KNOW</u> $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges, and geometrically, we expect a relationship.



It appears $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\,dx + \frac{1}{1^2} = 1 + 1 = 2.$

On the other hand: $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ <u>diverges</u> (by p-test) and it appears $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ does too!



We can make this precise.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then:

① If $\int_1^\infty f(x)\,dx$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

② If $\int_1^\infty f(x)\,dx$ diverges, $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Note:</u> ① $n=1$ not important! To test convergence of $\sum_{n=32}^{\infty} \frac{1}{(n-2)^2}$, use $\int_{32}^{\infty} \frac{1}{(x-2)^2}\,dx$.

② f may not be <u>always</u> decreasing, but the test works as long as f is <u>ultimately</u> decreasing.

③ Finding antiderivatives is hard/impossible, so this isn't a great test.

④ The sum of the series is almost guaranteed <u>not</u> to equal the integral! <u>Ex:</u> $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ but $\int_1^\infty \frac{1}{x^2}\,dx = 1$.

Ex: Discuss convergence / divergence:

① $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

② $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \rightarrow \int_1^{\infty} \frac{1}{x^p}\,dx$ converges iff $p>1$

$$\Downarrow$$

The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$

③ $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n} \rightarrow$ i) $f(x) = \frac{\ln(x)}{x}$ always positive, continuous for $x>1$

ii) Test decreasing:

$$f'(x) = \frac{1-\ln x}{x^2} < 0 \quad \text{for } x > e$$

$\Rightarrow$ eventually decreasing

iii) $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx = \dots = \infty.$

$\Rightarrow \sum \frac{\ln n}{n}$ diverges by integral test.