§ 11.2 - Series

<u>Motivation</u>: Spse we have an infinite decimal

$$\pi = 3.14159\ldots$$

What does this mean?

$$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10^4} + \frac{9}{10^5} + \ldots$$

So: adding any finite amt gives finite approx & as finite $\to \infty$, we get closer & closer to π.

<u>Def</u>: An <u>(infinite) series</u> is the sum $a_1 + a_2 + \cdots + a_n + \cdots$ of elements of a sequence $\{a_n\}_{n=1}^{\infty}$. We write

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n .$$

<u>Q</u>: Do we always get a # ?

- $\{n\}_{n=1}^{\infty} \longleftrightarrow 1+2+3+\cdots+n+\cdots$. This sum will be <u>infinite</u>:

$$\text{partial sums} \begin{cases} s_1 = a_1 = 1 \\ s_2 = a_1 + a_2 = 1+2 = 3 \\ s_3 = a_1+a_2+a_3 = 6 \\ \vdots \\ s_n = \frac{n(n+1)}{2} \longrightarrow \infty \text{ as } n \to \infty . \end{cases}$$

- However: $\left\{\frac{1}{2^n}\right\} \longleftrightarrow \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$, and

$s_1 = \frac{1}{2}$ $\quad s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ $\quad s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ $\quad \ldots \quad s_n = 1 - \frac{1}{2^n} \to 1$

as $n \to \infty$.

Def: Given $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n .$$

If $\{S_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists (as a real number), the series is said to be <u>convergent</u> and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

↑ sum of the series

If $\{S_n\}$ is divergent, then we say the series is divergent.

Note: $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i .$

Ex: Spse $S_n = \sum_{i=1}^{n} a_i = \frac{2n}{3n+5}$. Then

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \frac{2}{3}.$$

Ex: Geometric Series

For $a \neq 0$, consider $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$

When does this converge?

- $r=1 \Rightarrow S_n = a + a + \cdots + a = na \to \pm\infty$ as $n \to \infty$.
- $r \neq 1 \Rightarrow S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$\Rightarrow rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

$S_n - rS_n = a - ar^n$

$\Rightarrow S_n = \frac{a - ar^n}{1-r}.$

Geometric Series (Cont'd)

So, $S_n = \dfrac{a(1-r^n)}{1-r}$.

<u>Now</u>:

- If $-1<r<1$, $r^n \to 0$ as $n\to\infty$ (see handout)

$$\Rightarrow \sum a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-\overset{\to 0}{r^n})}{1-r} = \frac{a}{1-r}.$$

i.e. for $|r|<1$, geometric series converges to $a/(1-r)$.

- If $r \le -1$ or $r>1$, $\{r^n\}$ diverges $\Rightarrow$ series diverges.

<u>Result</u>: The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is:

- convergent if $|r|<1$ & its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

- divergent if $|r|\ge 1$.

<u>Ex</u>: Consider $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

$a=5$ $\quad 5\cdot\frac{-2}{3}$ $\quad \frac{-10}{3}\cdot\frac{-2}{3} = 5\left(\frac{-2}{3}\right)^2$ $\quad 5\left(\frac{-2}{3}\right)^3$

$= \sum_{n=1}^{\infty} 5\left(\frac{-2}{3}\right)^n$. Now, $a=5$, $|r| = \left|\frac{-2}{3}\right| < 1$, and so

$$\sum_{n=1}^{\infty} 5\left(\frac{-2}{3}\right)^n = \frac{a}{1-r} = \frac{5}{1-\left(\frac{-2}{3}\right)} = \frac{5}{5/3} = 3.$$

Ex: Is $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent?

Note:

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)}$$

$$= \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot \frac{4^{n-1}}{3^{n-1}}$$

$$= \sum_{n=1}^{\infty} 4 \left(\frac{4}{3}\right)^{n-1}.$$

Since this is geometric & $|r| = \left|\frac{4}{3}\right| > 1$, $\sum \cdots$ diverges

Ex: Write $2.3\overline{17} = 2.3171717\ldots$ as a ratio of ints.

$$\Rightarrow 2.3171717\ldots = 2.3 + \underbrace{\frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots}$$

$$\frac{17}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) = \sum_{n=} \frac{17}{10^3}\left(\frac{1}{10^2}\right)^n$$

$$\sum \cdots = \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} \Leftarrow \left[\begin{array}{l} a = \frac{17}{10^3} \\ r = \frac{1}{10^2} \end{array}\right.$$

$$= 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} = \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}.$$

Ex: Find sum of $\sum_{n=0}^{\infty} x^n$, $|x| < 1$.

$$\sum \cdots = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

$a = 1$ $\quad r = x$ w/ $|r| = |x| < 1$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent. Find its sum.

$\hookrightarrow$ Not geometric ! $\therefore$ Consider partial sums:

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

What to do? Partial Fractions!

$$\frac{1}{i(i+1)} = \frac{A}{i} + \frac{B}{i+1} \Rightarrow A = 1 \quad B = -1$$

$$\hookrightarrow \frac{1}{i} - \frac{1}{i+1}$$

$$\Rightarrow S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

TELESCOPING SUM $\Rightarrow$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n-1}} - \cancel{\tfrac{1}{n}}\right) + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \rightarrow 1 \text{ as } n \to \infty.$$

Ex: Show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. HARMONIC SERIES

$s_1 = 1$, $s_2 = 1 + \frac{1}{2}$, $s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right)$$

$$= 1 + \frac{3}{2} \ \ldots$$

$\Rightarrow S_{2^n} > 1 + \frac{n}{2} \rightarrow \infty$ as $n \to \infty$

Theorem: If $\sum a_n$ converges, $\lim a_n \to 0$.

Ex: Discuss convergence / divergence of $\sum_{n=1}^{\infty} \frac{3n^4}{(n^2+2)(2n^2-1)}$.

NOTE! JUST B/C $\lim a_n \to 0$ DOES NOT SAY THAT $\sum a_n$ CONVERGES!

Ex: $\{\frac{1}{n}\} \to 0$ but $\sum \frac{1}{n}$ diverges.

Thm: If $\sum a_n$, $\sum b_n$ converge, then so is $\sum c a_n$ (c const), $\sum (a_n \pm b_n)$, and

1. $\sum c a_n = c \sum a_n$
2. $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$.

$\hookrightarrow$ Sums behave like integrals

Ex: Find $\sum \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$. $= 3(1) + 1 = 4$.

$= 3 \left(\sum \frac{1}{n} - \frac{1}{n+1} \right)$

$\to 3$

$\sum \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$