MAC 2312 — Homework 5

———SOLUTIONS———

1. Nothin' to see here, folks....

2. (a) $\leftrightarrow$ (iii); (b) $\leftrightarrow$ (i); (c) $\leftrightarrow$ (iv); (d) $\leftrightarrow$ (ii)

3. $\left\{1, 2, 4, \sqrt{21}, \sqrt{41}, \sqrt{78}, \sqrt{140}, \sqrt{259}\right\}$

4. (a) Converges to 12

   (b) Diverges to $-\infty$

   (c) Converges to 1

   (d) Diverges to $-\infty$

   (e) Converges to $\tan^{-1}(1) = \dfrac{\pi}{4}$

   (f) Converges to $\arccos(1/2) = \pi/3$

   (g) Converges to 1

   (h) Converges to 0

5. Note the inductive formulation $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$, $n \geq 1$.

   (a) Assume $a_n < 2$. Then $a_{n+1} \stackrel{\text{def}}{=} \sqrt{2a_n} < \sqrt{2 \cdot 2}$, by assumption. Hence, $a_{n+1} < \sqrt{4} = 2$.

   (b) Assume again that $a_n < 2$. Then $a_{n+1} \stackrel{\text{def}}{=} \sqrt{2a_n} > \sqrt{a_n \cdot a_n}$, (again) by assumption. Thus, $a_{n+1} > \sqrt{a_n^2} = |a_n| = a_n$, since the terms of $\{a_n\}$ are clearly non-negative.

   (c) By (a), the sequence is bounded; by (b), the sequence is monotone. **All bounded, monotone sequences have a limit!**

   (d) Suppose $a_n \to L$. Then $a_{n+1} \to L$ as well, and hence $L = \sqrt{2L}$. Solving for L: $L = \sqrt{2L}$ implies $L^2 = 2L$ implies $L^2 - 2L = 0$ implies $L(L-2) = 0$. Hence, either $L = 0$ or $L = 2$, and because the sequence is increasing from $a_1 = \sqrt{2}$, $L \neq 0$. Thus, $a_n \to 2$.

   (e) You do this! The recursive definition is $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3a_n}$, $n \geq 1$.

   (f) You do this! The recursive definition is $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$, $n \geq 1$.

6. Let $a_n = \dfrac{n}{n+1}$. Find a number M such that:

(a) $|a_n - 1| \leq 0.001$ for $n \geq M$.

Solution: Set up the absolute value inequality $-0.001 \leq a_n - 1 \leq 0.001$ and note that $a_n = \dfrac{n}{n+1}$. Plugging in for a_n in the absolute value inequality yields

$$-0.001 \leq \frac{n}{n+1} - 1 \leq 0.001,$$

and solving with respect to the $\geq$ inequality (since we want $n \geq M$ for some M) yields

$$-0.001 \leq \frac{n}{n+1} - 1 \implies 0.999 \leq \frac{n}{n+1} \implies 0.999(n+1) \leq n.$$

Solving for n yields $n \geq 999$, so $M = 999$ will do the trick.

(b) $|a_n - 1| \leq 10^{-5}$ for $n \geq M$.

Solution: You do this! The answer is $n \geq 99,999$.

7. (a) $a_n = \left(\dfrac{5}{2}\right)^{n-1} \longleftrightarrow \text{series} = \displaystyle\sum_{n=1}^{\infty} \left(\dfrac{5}{2}\right)^{n-1}$.

(b) $a_n = \dfrac{1 + \frac{1+(-1)^{n+1}}{2}}{n^2+1} \longleftrightarrow \text{series} = \displaystyle\sum_{n=1}^{\infty} \dfrac{1 + \frac{1+(-1)^{n+1}}{2}}{n^2+1}$

8. (a) $s_2 = -\dfrac{1}{2}$; $s_4 = -\dfrac{7}{12}$; $s_6 = -\dfrac{37}{60}$

(b) $s_2 = \dfrac{3}{2}$; $s_4 = \dfrac{41}{24}$; $s_6 = \dfrac{1237}{720}$

(c) $s_2 = -\dfrac{14}{3}$; $s_4 = -\dfrac{44}{5}$; $s_6 = -\dfrac{90}{7}$

(d) $s_2 = \dfrac{42}{121}$; $s_4 = \dfrac{5460}{14641}$; $s_6 = \dfrac{664062}{1771561}$

9. (a) $\dfrac{1}{9}$

(b) $\dfrac{13}{99}$

(c) $\dfrac{217}{999}$

(d) $\dfrac{1234}{9999}$

10. (a) Series notation: $\sum_{k=3}^{\infty} \frac{1}{k(k-1)}$;

partial fraction: $\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$;

partial sum: $s_n = \frac{1}{2} - \frac{1}{n}$ for $n \geq 3$;

total sum: $\lim_{n\to\infty} s_n = 1/2$.

(b) Series notation: $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$;

partial fraction: $\frac{1}{(2k-1)(2k+1)} = \frac{1/2}{2k-1} - \frac{1/2}{2k+1}$;

partial sum: $s_n = \frac{1}{2} - \frac{1}{2(2n+1)}$

total sum: $\lim_{n\to\infty} s_n = 1/2$.

(c) Series notation: $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$;

partial fraction: $\frac{1}{k(k+1)(k+2)} = \frac{1/2}{k} - \frac{1}{k+1} + \frac{1/2}{k+2}$;

partial sum: $s_n = \frac{n^2+3n}{4(n^2+3n+2)}$

total sum: $\lim_{n\to\infty} s_n = 1/4$.

(d) Series notation: $\sum_{k=1}^{\infty} \frac{1}{k^2+3k+2} = \sum_{k=1}^{\infty} \frac{1}{(k+2)(k+1)}$;

partial fraction: $\frac{1}{(k+2)(k+1)} = \frac{1}{k+1} - \frac{1}{k+2}$;

partial sum: $s_n = \frac{1}{2} - \frac{1}{n+2}$

total sum: $\lim_{n\to\infty} s_n = 1/2$.

11. Determine the sum of each of the following series or state that the series does not ~~exist~~ **converge**.

(a) Does not converge: $a_n \to 1$ as $n \to \infty$

(b) Does not converge: $a_n \to \pm\infty$ as $n \to \infty$

(c) Does not converge: $a_n \to \pm 1$ as $n \to \infty$

(d) Geometric series: $a = (4/5)^3$, $r = 4/5$:

Converges to $\dfrac{(4/5)^3}{1 - 4/5} = \dfrac{64}{25}$.

(e) Does not converge: $a_n \not\to 0$ as $n \to \infty$

(f) Geometric series: $a = e^3$, $r = e^{-2}$:

Converges to $\dfrac{e^3}{1 - e^{-2}} = \dfrac{64}{25}$.

(g) Converges to $-\dfrac{839}{1344}$;

this is definitely **not** an exam-level question, but if you want to see it worked out, don't hesitate to ask!

(h) Geometric series: $a = 25/9$, $r = 3/5$:

Converges to $\dfrac{25/9}{1 - (3/5)} = \dfrac{125}{18}$

(i) Geometric series: $a = 7/8$, $r = -7/8$:

Converges to $\dfrac{7/8}{1 - (-7/8)} = \dfrac{7}{15}$

(j) Does not converge: p-test with $p = 1/2$.

12. (a) Write the first three terms of the sequence a_n.
Solution: $a_1 = -1/2$; $a_2 = 2/4$; $a_3 = -6/8$.

(b) Write the first three terms of the subsequence b_n where, for each n, $b_n = a_{2n}$.
Solution: $b_1 = a_2 = 2/4$; $b_2 = a_4 = 24/16$; $b_3 = a_6 = 720/64$.

(c) Write the first three terms of the subsequence c_n where, for each n, $c_n = (a_n + a_{n+1})/b_n$.
Solution: $c_1 = (a_1 + a_2)/b_1 = 0$; $c_2 = (a_2 + a_3)/b_2 = -1/6$; $c_3 = (a_3 + a_4)/b_3 = 1/15$

13. (a) Is increasing; is bounded below; is not bounded above.
Solution: $a_n = n^2$

(b) Is increasing; is bounded above; is not bounded below; converges.
Solution: No such sequence exists: All convergent sequences are bounded (this is hard).

(c) Is decreasing; is bounded neither above nor below; converges.
Solution: No such sequence exists: All convergent sequences are bounded (this is hard).

(d) Is neither increasing nor decreasing; is bounded; converges to 4.
Solution: $a_n = 4 + \frac{(-1)^n}{n}$

(e) Is neither increasing nor decreasing; is not bounded; converges.
Solution: No such sequence exists: All convergent sequences are bounded (this is hard).

(f) Is monotone; is bounded; converges to 6 **and** -12 simultaneously.
Solution: No such sequence exists: The limit of a convergent sequence is unique.

(g) Does not converge but has a subsequence which *does* converge.
Solution: $a_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, \ldots\}$. The subsequences $a_{2n-1} = \{-1, -1, -1, \ldots\}$ and $a_{2n} = \{1, 1, 1, \ldots\}$ both converge.

(h) Converges, but has a subsequence which *does not* converge.
Solution: No such sequence exists: Every subsequence of a convergent sequence converges.

(i) Converges, and has a subsequence which converges to a *different* limit.
Solution: No such sequence exists: Every subsequence of a convergent sequence converges to the same limit as the larger sequence.

(j) Has two *different* subsequences which converge to -7 and to 7, respectively.
Solution: $a_n = (-7)^n = \{-7, 7, -7, 7, \ldots\}$.

(k) Has nine *different* subsequences which converge to $1, 2, 3, \ldots, 9$ respectively.
Solution: $a_n = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots\}$. For $k = 1, 2, 3, \ldots, 9$, the subsequence $a_{kn} = \{k, k, k, \ldots\}$ converges to k.

(l) Diverges; has a subsequence which is increasing; has a different subsequence which diverges.
Solution: $a_n = (-1)^n n$. The subsequence $a_{2n} = 2n$ is increasing and the subsequence $a_{2n-1} = -(2n-1)$ is a different sequence which diverges.

14. (a) Both a_n and b_n converge but $\{a_n + b_n\}$ fails to converge.
Solution: No such sequences exists: The sum of convergent sequences converges.

(b) Neither a_n nor b_n converge but $a_n - b_n$ converges.
Solution: $a_n = n, b_n = n-1 \implies a_n - b_n = n - (n-1) = 1$ converges to 1.

(c) a_n converges, b_n diverges, and a_n/b_n converges.
Solution: $a_n = 1, b_n = n \implies a_n/b_n = 1/n$ converges to 0.

(d) $a_n \to L$, $b_n \to L$, and $a_n/b_n \to L$.
Solution: $a_n = 1, b_n = 1 - 1/n \to 1 \implies a_n/b_n = 1/(1-1/n)$ also converges to 1.

(e) a_n diverges, $b_n \to \sqrt{2}$, and $a_n b_n \to 19$.
Solution: No such sequences exist.

(f) $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ diverges but $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

Solution: $a_n = -n$, $b_n = n$.

(g) $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges but $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ diverges.

Solution: $a_n = 1/n^3$, $b_n = 1/n^2$. $\sum a_n$ and $\sum b_n$ both converge as p-series. However,

$$\frac{a_n}{b_n} = \frac{1/n^3}{1/n^2} = \frac{1}{n}$$

and $\sum 1/n$ diverges.