Name: ______________________________

MAC 2312 — Homework 5

Directions: Complete the following problems for a homework grade. Solutions *must* be presented in a neat and professional manner in order to receive credit, answers given without showing work will not be eligible to receive partial credit, and *work for the problems **must** be done on scratch paper and not on this handout!* **Date Due:** Monday, November 28.

1. Go to the #talk_about_your_break channel in our course's Slack room (see course homepage for the URL) and talk about your break!

2. Match each sequence with its general term:

$a_1, a_2, a_3, a_r, \ldots$	General Term
(a) $1, -1, 1, -1, \ldots$	(i) $\cos(\pi n)$
(b) $-1, 1, -1, 1, \ldots$	(ii) $\dfrac{n!}{2^n}$
(c) $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$	(iii) $(-1)^{n+1}$
(d) $\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{3}{4}, \dfrac{3}{2}, \ldots$	(iv) $\dfrac{n}{n+1}$

3. Write the first eight terms of the following sequence:

$$b_1 = 1, \qquad b_2 = 2, \qquad b_3 = 4, \qquad b_n = \sqrt{b_{n-1}^2 + b_{n-2}^2 + b_{n-3}^2} \quad \text{for } n \geq 4.$$

4. Determine the limit of each of the following sequences or state that the sequence diverges.

(a) $a_n = 12$

(b) $\left\{\ln\left(\dfrac{12n+2}{-9+4n^2}\right)\right\}_{n=1}^{\infty}$

(c) $b_n = 10^{-1/n}$

(d) $\{\ln(\sin n) - \ln n\}$

(e) $s_n = \tan^{-1}\left(e^{e^{-n}}\right)$

(f) $\left\{\arccos\left(\dfrac{n^3}{2n^3+1}\right)\right\}$

(g) $k_n = \dfrac{n}{n + n^{1/n}}$

(h) $\left\{\dfrac{(-1)^n n^3 + 2^{-n}}{3n^3 + 4^{-n}}\right\}$

5. In this problem, you're going to formalize the proof of the result we did in class showing that

$$\sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}} \longrightarrow 2.$$

Note the inductive formulation $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$, $n \geq 1$.

(a) Use the inductive formulation to show that $a_{n+1} = \sqrt{2a_n} < 2$ if $a_n < 2$. (**Note:** Because $a_1 = \sqrt{2} < 2$, this shows that the entire sequence is *bounded above* by 2).

(b) Using part (a) along with the inductive formulation, show that $\sqrt{2a_n} > \sqrt{a_n \cdot a_n}$ if $a_n < 2$. Use this to conclude that $a_{n+1} > a_n$ (i.e., that the entire sequence is *monotone*).

(c) Use the results from parts (a) and (b) to conclude that the sequence $\{a_n\}$ has a limit.

(d) Find the value L for which $a_n \to L$ as $n \to \infty$.

(e) Repeat parts (a)–(d) with the expression $\sqrt{3\sqrt{3\sqrt{3\sqrt{\cdots}}}}$.

(f) Repeat parts (a)–(d) with the expression $\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$

6. Let $a_n = \dfrac{n}{n+1}$. Find a number M such that:

(a) $|a_n - 1| \leq 0.001$ for $n \geq M$.

(b) $|a_n - 1| \leq 10^{-5}$ for $n \geq M$.

7. For each of the following series, (a) find a formula for the general term a_n (**not the partial sum!**) and (b) write in summation notation.

(a) $1 + \dfrac{5}{2} + \dfrac{25}{4} + \dfrac{125}{8} + \cdots$

(b) $\dfrac{2}{1^2+1} + \dfrac{1}{2^2+1} + \dfrac{2}{3^2+1} + \dfrac{1}{4^2+1} + \cdots$

Hint: Numerators are either $1 = 1 + \dfrac{0}{2}$ or $2 = 1 + \dfrac{2}{2}$....

8. Calculate the partial sums s_2, s_4, and s_6 for each of the following.

(a) $\sum_{k=1}^{\infty} (-1)^k k^{-1}$

(b) $\sum_{j=1}^{\infty} \frac{1}{j!}$

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$

(d) $\sum_{r=1}^{\infty} \left(\frac{3}{11} \right)^{-r}$

9. Write each of the following repeating decimals as rational numbers in lowest form.

(a) $0.111111\ldots$

(b) $0.131313\ldots$

(c) $0.217217217\ldots$

(d) $0.1234123412341234\ldots$

10. Use partial fraction decomposition to find each of the following sums.

(a) $\sum_{n=3}^{\infty} \frac{1}{n(n-1)}$

(b) $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$.

(c) $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots$.

(d) $\sum_{j=1}^{\infty} \frac{1}{n^2 + 3n + 2}$

11. Determine the sum of each of the following series or state that the series does not exist.

(a) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

(b) $\sum_{n=1}^{\infty} (-1)^n n^2$

(c) $\cos 1 + \cos 1/2 + \cos 1/3 + \cos 1/4 + \cdots$

(d) $\frac{4^3}{5^3} + \frac{4^4}{5^4} + \frac{4^5}{5^5} + \cdots$

(e) $\frac{2}{3} + \frac{3^2}{2^2} + \frac{2^3}{3^3} + \frac{3^4}{2^4} + \frac{2^5}{3^5} + \cdots$

(f) $\sum_{n=0}^{\infty} e^{3-2n}$

(g) $\sum_{n=3}^{\infty} \frac{3(-2)^2 - 5^n}{8^n}$

(h) $\frac{25}{9} + \frac{5}{3} + 1 + \frac{3}{5} + \frac{9}{25} + \frac{27}{125} + \cdots$

(i) $\frac{7}{8} - \frac{49}{64} + \frac{343}{512} - \frac{2401}{4096} + \cdots$

(j) $\sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}$

12. A **subsequence** of a sequence $\{a_n\}$ is a sequence $\{b_n\}$ such that, for all n, $b_n = a_k$ for some increasing collection of k's (i.e., if $b_1 = a_{12}$, then $b_2 = a_k$ for some $k \geq 12$). Throughout, let

$$a_n = \frac{(-1)^n n!}{2^n}.$$

   (a) Write the first three terms of the sequence a_n.

   (b) Write the first three terms of the subsequence b_n where, for each n, $b_n = a_{2n}$.

   (c) Write the first three terms of the subsequence c_n where, for each n, $c_n = (a_n + a_{n+1})/b_n$.

13. Write an example of a sequence which satisfies each of the following conditions or state that no such sequence exists. If both cases, justify your claim using results from class and/or "formal" proofs.

   (a) Is increasing; is bounded below; is not bounded above.

   (b) Is increasing; is bounded above; is not bounded below; converges.

   (c) Is decreasing; is bounded neither above nor below; converges.

   (d) Is neither increasing nor decreasing; is bounded; converges to 4.

   (e) Is neither increasing nor decreasing; is not bounded; converges.

   (f) Is monotone; is bounded; converges to 6 **and** -12 simultaneously.

   (g) Does not converge but has a subsequence which *does* converge.

   (h) Converges, but has a subsequence which *does not* converge.

   (i) Converges, and has a subsequence which converges to a *different* limit.

   (j) Has two *different* subsequences which converge to -7 and to 7, respectively.

   (k) Has nine *different* subsequences which converge to $1, 2, 3, \ldots, 9$ respectively.

   (l) Diverges; has a subsequence which is increasing; has a different subsequence which diverges.

14. Write an example of two sequences $\{a_n\}$ and $\{b_n\}$ which satisfy each of the following conditions or state that no such sequence exists. If both cases, justify your claim using results from class and/or "formal" proofs.

   (a) Both a_n and b_n converge but $\{a_n + b_n\}$ fails to converge.

   (b) Neither a_n nor b_n converge but $a_n - b_n$ converges.

   (c) a_n converges, b_n diverges, and a_n/b_n converges.

   (d) $a_n \to L$, $b_n \to L$, and $a_n/b_n \to L$.

   (e) a_n diverges, $b_n \to \sqrt{2}$, and $a_n b_n \to 19$.

   (f) $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ diverges but $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

   (g) $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges but $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ diverges.