MAC 2312 — Homework 3

———SOLUTIONS———

1. Nossing, Lebowski, nossing.

2. Solve each of the following separable differential equations (DEs) and/or separable DE initial value problems (IVPs).

   (a) $xy^2y' = x + 1$

   Solution: Rewrite y' as dy/dx and separate:

   $$xy^2\frac{dy}{dx} = x + 1 \quad \Longrightarrow \quad y^2 dy = \frac{x+1}{x}dx.$$

   Now, rewrite $\frac{x+1}{x}$ as $1 + \frac{1}{x}$ and integrate:

   $$\int y^2\,dy = \int 1 + \frac{1}{x}\,dx \quad \Longrightarrow \quad \frac{y^3}{3} = x + \ln|x| + C.$$

   Now, solve for y:

   $$\frac{y^3}{3} = x + \ln|x| + C \quad \Longrightarrow \quad \boxed{y = (3x + 3\ln|x| + 3C)^{1/3}}.$$

   Note: You can rewrite $3C$ as C! In class, that's what we did! Here, I'm leaving it this way to be explicit!

   (b) $\dfrac{dx}{dt} = \dfrac{e^x\sin^2(t)}{x\sec t}$

   Solution: Separate, do some algebra, and bring in the integrals:

   $$\frac{dx}{dt} = \frac{e^x\sin^2(t)}{x\sec t} \implies \frac{x}{e^x}dx = \frac{\sin^2(t)}{\sec t}dt \implies \int xe^{-x}\,dx = \int \sin^2(t)\cos t\,dt.$$

   For the integral on the left, you have to use **Integration by Parts (IBP)** with $u = x$ and $v' = e^{-x}$; on the right, you can use **u-substitution** with $u = \sin(t) \implies du = \cos(t)\,dt$. Doing so shows that

   $$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C \quad \text{and} \quad \int \sin^2(t)\cos t\,dt = \frac{1}{3}\sin^3(t).$$

   To finish, set these two values equal and solve for t by (a) multiplying both sides by 3, (b) taking a cube root (i.e. a 1/3 power) of both sides, and (c) taking arcsin of both sides:

   $$-xe^{-x} - e^{-x} + C = \frac{1}{3}\sin^3(t) \implies \boxed{t = \arcsin\left(3\left(-xe^{-x} - e^{-x} + C\right)\right)^{1/3}}.$$

(c) $\dfrac{dy}{dx} = 2xy - 2y + 2x - 2,\ y(1) = 0$

Solution: This problem is tremendously hard if you don't realize you need to factor the right-hand side! When you have four terms, think "factor by grouping":

$$\begin{aligned} 2xy - 2y + 2x - 2 &= (2xy - 2y) + (2x - 2) \\ &= 2y\boxed{(x-1)} + 2\boxed{(x-1)} \\ &= (x-1)(2y+2)\,. \end{aligned}$$

Now, we separate and introduce integrals:

$$\frac{dy}{dx} = (x-1)(2y+2) \implies \frac{dy}{2y+2} = (x-1)dx \implies \frac{1}{2}\int \frac{dy}{y+1} = \int (x-1)\,dx.$$

Using u**-substitution** on the left and **basic integration** on the right, we have

$$\frac{1}{2}\ln(y+1) = \frac{1}{2}x^2 - x + C,$$

and so solving for y yields the general solution:

$$y = -1 + e^{x^2-2x+2C}. \tag{1}$$

Since this is an IVP, we use the initial condition $y(1) = 0$ to solve for C:

$$y = -1 + e^{x^2-2x+2C} \implies 0 = -1 + e^{1^2-2(1)+2C} \implies 0 = -1 + e^{-1+2C}.$$

Now, solving for C yields

$$1 = e^{-1+2C} \implies \ln(1) = -1 + 2C \implies C = \frac{1}{2},$$

and so plugging back into (1) gives the particular solution

$$\boxed{y = -1 + e^{x^2-2x+1}}.$$

(d) $x^2\frac{dy}{dx} = \sqrt{1-y^2}$

Solution: Because dx is on the bottom on the side with the x term, we flip everything:

$$x^2\frac{dy}{dx} = \sqrt{1-y^2} \implies \frac{1}{x^2}\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}.$$

Now, separate and introduce integrals:

$$\frac{1}{x^2}\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}} \implies \frac{dx}{x^2} = \frac{dy}{\sqrt{1-y^2}} \implies \int \frac{dx}{x^2} = \int \frac{dy}{\sqrt{1-y^2}}.$$

For the left side, you're integrating x^{-2}, which is simple; on the right, you need **trig substitution** with $y = \sin\theta$ (or, you may have memorized the integral of the right-hand side). Upon finishing the integral (**you should do this integration yourself!**), you should have

$$\frac{-1}{x} + C = \arcsin y \implies \boxed{y = \sin\left(\frac{-1}{x} + C\right)}.$$

(e) $e^y\left(\frac{dy}{dx}\right) = 1 + e^{2y} - xe^{2y} - x$, $y(0) = 1$

Solution: This is another factor by grouping thing:

$$1 + e^{2y} - xe^{2y} - x = 1\boxed{\left(1+e^{2y}\right)} - x\boxed{\left(e^{2y}+1\right)} = (1-x)\left(1+e^{2y}\right)$$

implies that

$$e^y\left(\frac{dy}{dx}\right) = (1-x)\left(1+e^{2y}\right).$$

Now, separate and write integrals:

$$e^y\left(\frac{dy}{dx}\right) = (1-x)\left(1+e^{2y}\right) \implies \frac{e^y}{1+e^{2y}}\,dy = (1-x)\,dx \implies \int \frac{e^y}{1+e^{2y}}\,dy = \int (1-x)\,dx.$$

The right integral is obvious; for the left integral, let $u = e^y \implies du = e^y\,dy$ and notice that the denominator is $1 + e^{2y} = 1 + (e^y)^2 = 1 + u^2$. So,

$$\int \frac{e^y}{1+e^{2y}}\,dy = \int \frac{du}{1+u^2} = \arctan u = \arctan(e^y),$$

and thus,

$$\arctan(e^y) = x - \frac{1}{2}x^2 + C \implies y = \ln\left(\tan\left(x - \frac{1}{2}x^2 + C\right)\right). \tag{2}$$

Now, use the initial condition $y(0) = 1$ to deduce that $1 = \ln(\tan C) \implies C = \tan^{-1} e$; plugging into (2) yields the final solution:

$$\boxed{y = \ln\left(\tan\left(x - \frac{1}{2}x^2 + \tan^{-1} e\right)\right)}.$$

3. (a) Write the differential equation modeling the following scenario: *The rate of growth of a population P over time is directly proportional to the population.*

   Solution: $\boxed{\dfrac{dP}{dt} = kP}$.

   (b) Show that the solution to the equation in (a) is $P(t) = Ce^{kt}$ where k is the constant of proportionality.

   Solution: This is worked out in detail in §9.4, subsection "The Law of Natural Growth."

4. (a) Let M be a constant and let k denote a constant of proportionality. Show that the solution to the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

   has the form

$$P(t) = M\,\frac{Ce^{kt}}{1 + Ce^{kt}}.$$

   Solution: This is worked out in detail in §9.4, subsection "The Logistic Model."

   (b) Write the solution of the initial value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100.$$

   Solution: Notice that this problem looks identical to the one in (a) with the values $k = 0.08$ and $M = 1000$; thus, the answer in (a) yields a general solution of the form

$$P(t) = 1000\left(\frac{Ce^{0.08t}}{1 + Ce^{0.08t}}\right). \tag{3}$$

   Now, use the condition $P(0) = 100$ to solve for C:

$$100 = 1000\left(\frac{Ce^{0.08(0)}}{1 + Ce^{0.08(0)}}\right) = \frac{1000C}{1 + C} \implies C = \frac{1}{9}.$$

   Substituting back in to (3) yields the result:

$$\boxed{P(t) = 1000\left(\frac{\frac{1}{9}e^{0.08t}}{1 + \frac{1}{9}e^{0.08t}}\right)}.$$

(c) Show that if P satisfies the logistic equation in (a), then the second derivative $\frac{d^2P}{dt^2}$ satisfies the following:

$$\frac{d^2P}{dt^2} = k^2 P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right).$$

Solution: Find the derivative (with respect to t) of dP/dt using the product rule, noting that anything other than P and t are constants. The result is immediate.

5. Solve each of the following linear differential equations and/or linear DE IVPs.

Recall: The goal for each of these problems is to write the DE in the "standard form"

$$\frac{dy}{dx} + P(x)y = Q(x),$$

to use $P(x)$ to define the integrating factor

$$I(x) = e^{\int P(x)\,dx},$$

and to multiply both sides of the original DE by $I(x)$. **Don't forget *the trick*:**

$$I(x)\left(\frac{dy}{dx} + P(x)y\right) = \frac{d}{dx}\left(y\,I(x)\right).$$

(a) $\frac{dy}{dx} = y\sin x - 2\sin x$

Solution: Rewriting the DE as

$$\frac{dy}{dx} - y\sin x = -2\sin x,$$

it follows that $P(x) = -\sin x$ and hence that

$$I(x) = e^{\int(-\sin x)\,dx} = e^{\cos x}.$$

Thus:

$$\begin{aligned}
\frac{dy}{dx} - y\sin x = -2\sin x \quad &\Longrightarrow \quad I(x)\left(\frac{dy}{dx} - y\sin x\right) = -2I(x)\sin x \\
&\Longrightarrow \quad \underbrace{e^{\cos x}\left(\frac{dy}{dx} - y\sin x\right)}_{\text{Don't forget the trick!}} = -2e^{\cos x}\sin x \\
&\Longrightarrow \quad \frac{d}{dx}\left(ye^{\cos x}\right) = -2e^{\cos x}\sin x
\end{aligned}$$

Now, integrate both sides with respect to x:

$$\frac{d}{dx}\left(ye^{\cos x}\right) = -2e^{\cos x}\sin x \implies \int\left(\frac{d}{dx}\left(ye^{\cos x}\right)\right)\,dx = \int -2e^{\cos x}\sin x\,dx.$$

On the left, the fundamental theorem of Calculus (FTC) says the integral cancels the derivative; on the right, let $u = \cos x \implies du = -\sin x\,dx$ to get:

$$ye^{\cos x} = 2e^{\cos x} + C.$$

Now, solve for y:

$$\boxed{y = 2 + \frac{C}{e^{\cos x}}}.$$

(b) $xy' = e^x - y$, $y(1) = 0$

Solution: The first steps are mechanical:

$$\begin{aligned} xy' = e^x - y \quad &\implies \quad \frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x} && \text{(divide by } x \text{ and rearrange)} \\ &\implies \quad P(x) = \frac{1}{x} \quad \text{and} \quad I(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x \\ &\implies \quad x\left(\frac{dy}{dx} + \frac{1}{x}y\right) = x\left(\frac{e^x}{x}\right) && \text{(multiply both sides by } I(x) = x\text{)} \\ &\implies \quad \frac{d}{dx}\left(yx\right) = e^x. && \text{(use \textit{the trick})} \end{aligned}$$

Now, integrate both sides with respect to x and solve for y:

$$\frac{d}{dx}\left(yx\right) = e^x \implies \int\left(\frac{d}{dx}\left(yx\right)\right)\,dx = \int e^x\,dx \implies yx = e^x + C,$$

and so

$$y = \frac{e^x}{x} + \frac{C}{x}. \tag{4}$$

Now, use the initial value $y(1) = 0$ to find C:

$$0 = e + C \implies C = -e.$$

Finally, plug back into (4):

$$\boxed{y = \frac{e^x}{x} - \frac{e}{x}}.$$

(c) $y' = \frac{y}{x} + x$, $y(1) = 1$

Solution: The solution of the DE is similar to that in (b):

$$y' = \frac{y}{x} + x \implies \frac{dy}{dx} - \frac{1}{x}y = x \implies I(x) = e^{\int(-1/x)\,dx} = e^{-\ln x}.$$

Now, write $-\ln x = -1 \cdot \ln x$ so that

$$I(x) = \underbrace{e^{-1\cdot\ln x} = \left(e^{\ln x}\right)^{-1}}_{a^{bc} = (a^b)^c} = x^{-1} = \frac{1}{x}.$$

Thus:

$$\frac{1}{x}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = \frac{1}{x}(x) \implies \underbrace{\frac{d}{dx}\left(\frac{1}{x}y\right) = 1 \implies \frac{1}{x}y = x + C}_{\text{integrate both sides with respect to } x} \implies y = x^2 + Cx.$$

Now, $y(1) = 1$ implies $C = 0$, and so

$$\boxed{y = x^2}.$$

(d) $(1+t^2)y' + 4ty = (1+t^2)^{-2}$

Solution: Divide by $(1+t^2)$ to get

$$\frac{dy}{dt} + \frac{4t}{1+t^2}y = \frac{1}{(1+t^2)^3}$$

so that $P(x) = 4t(1+t^2)^{-1}$ and hence

$$I(x) = \underbrace{e^{\int 4t(1+t^2)^{-1}\,dt} = e^{2\ln(1+t^2)}}_{\text{let } u = 1+t^2 \implies du = 2t\,dt} = \left(e^{\ln(1+t^2)}\right)^2 = \left(1+t^2\right)^2.$$

Now,

$$\frac{dy}{dt} + \frac{4t}{1+t^2}y = \frac{1}{(1+t^2)^3} \implies \left(1+t^2\right)^2\left(\frac{dy}{dt} + \frac{4t}{1+t^2}y\right) = \left(1+t^2\right)^2\left(\frac{1}{(1+t^2)^3}\right),$$

and hence,

$$\frac{d}{dt}\left(y\left(1+t^2\right)^2\right) = \frac{1}{1+t^2} \implies \int\left[\frac{d}{dt}\left(y\left(1+t^2\right)^2\right)\right]dt = \int\frac{1}{1+t^2}\,dt.$$

The integral of the right-hand side is $\arctan t$, and so

$$y\left(1+t^2\right)^2 = \tan^{-1}t + C \implies \boxed{y = \frac{\tan^{-1}t + C}{(1+t^2)^2}}.$$

6. **Solution:** For (a), the goal is to solve the DE

$$\frac{d}{dt}\left((M_0 - rt)v\right) = F - (M_0 - rt)g$$

for $v = v(t)$. This is already separated, so integrating both sides with respect to t is sufficient:

$$\begin{aligned}
\frac{d}{dt}\left((M_0 - rt)v\right) = F - (M_0 - rt)g \quad &\Longrightarrow \quad \int \left[\frac{d}{dt}\left((M_0 - rt)v\right)\right] dt = \int \left(F - (M_0 - rt)g\right) dt \\
&\Longrightarrow \quad v(M_0 - rt) = Ft - M_0 gt - \frac{grt^2}{2} + C \\
&\Longrightarrow \quad v = \frac{1}{M_0 - rt}\left(Ft - M_0 gt - \frac{grt^2}{2} + C\right).
\end{aligned}$$

Now, to finish part (a), note that $t = 0$ implies $v = 0$, i.e. $C = 0$. Hence,

$$\boxed{v = \frac{1}{M_0 - rt}\left(Ft - M_0 gt - \frac{grt^2}{2}\right) = \frac{Ft}{M_0 - rt} - \frac{g}{M_0 - rt}\left(M_0 t - \frac{rt^2}{2}\right)}. \tag{5}$$

To do (b), note that at burnout, $M_1 = M_0 - rt$, and per the hint,

$$rt = M_0 - M_1 \implies t = \frac{M_0 - M_1}{r}.$$

The goal will be to plug into (5) and to solve for v (without t's):

$$\begin{aligned}
\boxed{v} \quad &= \quad \frac{Ft}{M_0 - rt} - \frac{g}{M_0 - rt}\left(M_0 t - \frac{rt^2}{2}\right) \\
&= \quad \frac{Frt}{r(M_0 - rt)} - \frac{g}{M_0 - rt}\left(M_0 t - \frac{r^2t^2}{2r}\right) \qquad \text{(replace } t \text{ with } rt \text{ by adding extra } r\text{'s)} \\
&\boxed{= \frac{F(M_0 - M_1)}{rM_1} - \frac{g}{M_1}\left[M_0\left(\frac{M_0 - M_1}{r}\right) - \frac{(M_0 - M_1)^2}{2r}\right]}.
\end{aligned}$$

This can be simplified some, but there really is no need.