Name: ______________________________

MAC 2312 — Homework 3

Directions: Complete the following problems (front and back) for a homework grade. Problems *must* be neatly written up and presented in a professional manner in order to receive credit, and answers given without showing work will not be eligible to receive partial credit.
Date Due: Monday, October 31.

1. Go to the #past_exams channel in our course's SLACK room (see course homepage for the URL) and either ask about a question from a past exam (Exam 1 or Exam 2) or give feedback to someone else's question thereon.

2. Solve each of the following separable differential equations (DEs) and/or separable DE initial value problems (IVPs).

   (a) $xy^2y' = x + 1$

   (b) $\dfrac{dx}{dt} = \dfrac{e^x \sin^2(t)}{x \sec t}$

   (c) $\dfrac{dy}{dx} = 2xy - 2y + 2x - 2$, $y(1) = 0$

   (d) $x^2 \dfrac{dy}{dx} = \sqrt{1 - y^2}$

   (e) $e^y \left(\dfrac{dy}{dx}\right) = 1 + e^{2y} - xe^{2y} - x$, $y(0) = 1$

3. (a) Write the differential equation modeling the following scenario: *The rate of growth of a population P over time is directly proportional to the population.*

   (b) Show that the solution to the equation in (a) is $P(t) = Ce^{kt}$ where k is the constant of proportionality.

4. (a) Let M be a constant and let k denote a constant of proportionality. Show that the solution to the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

has the form

$$P(t) = M\,\frac{Ce^{kt}}{1+Ce^{kt}}.$$

(b) Write the solution of the initial value problem

$$\frac{dP}{dt} = 0.08P\left(1-\frac{P}{1000}\right) \qquad P(0) = 100.$$

(c) Show that if P satisfies the logistic equation in (a), then the second derivative $\frac{d^2P}{dt^2}$ satisfies the following:

$$\frac{d^2P}{dt^2} = k^2P\left(1-\frac{P}{M}\right)\left(1-\frac{2P}{M}\right).$$

5. Solve each of the following linear differential equations and/or linear DE IVPs.

(a) $\frac{dy}{dx} = y\sin x - 2\sin x$

(b) $xy' = e^x - y$, $y(1) = 0$

(c) $y' = \frac{y}{x} + x$, $y(1) = 1$

(d) $(1+t^2)y' + 4ty = (1+t^2)^{-2}$

6. A rocket with initial mass M_0 (kilograms) blasts off at time $t = 0$, and after taking off, the mass of the rocket decreases with time because the fuel is being spent at a constant burning rate r (kg per sec). If thrust is a constant force F and velocity is v, Newton's second law gives

$$\frac{d}{dt}\left((M_0 - rt)v\right) = F - (M_0 - rt)g \tag{1}$$

where $g = 9.8$ is the gravitational constant. (a) Solve the equation (1) for $v = v(t)$, given that $v = 0$ when $t = 0$. (b) If the mass of the rocket at burnout is $M_0 - rt = M_1$, compute the velocity of the rocket at burnout.

Hint: Once you have the solution in part (a), replace $M_0 - rt$ by M_1 *and* replace rt by the appropriate expression in terms of M_0 and M_1 obtained by solving the equation $M_0 - rt = M_1$ for rt.