

Tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Throughout, let f be a function satisfying $f(n) = a_n$.

The first test you should ever do, always

The Test for Divergence, aka the “is convergence even possible?!” test:

- *What you know:*
 - If $\lim_{n \rightarrow \infty} a_n \neq 0$ or $\lim_{n \rightarrow \infty} a_n$ does not exist, your series **cannot** converge (i.e., it must diverge).
 - If $\lim_{n \rightarrow \infty} a_n = 0$, your series **may** converge.

Specialty “tests” that work only for very specific cases

1. Geometric series:

- Can only use when your series looks like $\sum_{n=0}^{\infty} ar^n$ and/or $\sum_{n=1}^{\infty} ar^{n-1}$ for constants $a \neq 0$ and r .
- *What you know:*
 - Your series converges **if and only if** $|r| < 1$.
 - If your series converges, it converges to $\frac{a}{1-r}$

2. p -series:

- Can only use when your series looks like $\sum \frac{1}{n^p}$ for p a constant.
- *What you know:*
 - Your series converges **if and only if** $p > 1$.

Tests that work for series with *positive terms only*

1. The integral test:

- Can only use when f is positive, decreasing, and continuous, and when $\int_1^{\infty} f(x) dx$ can be (hopefully-easily) computed.
- *What you know:*
 - $\sum_{n=1}^{\infty} a_n$ converges **if and only if** $\int_1^{\infty} f(x) dx$ converges.

2. The comparison test:

- Can only use when $\{a_n\}$ “looks like” another sequence $\{b_n\}$ where $\{b_n\}$ also has positive terms and where convergence/divergence of $\sum b_n$ is known.
- **Usually**, you want to pick $\{b_n\}$ so that $\sum b_n$ is either a geometric series or a p -series.
- *What you know:*
 - If $a_n \leq b_n$ for all n and if $\sum b_n$ converges, then $\sum a_n$ **converges**.
 - If $a_n \geq b_n$ for all n and if $\sum b_n$ diverges, then $\sum a_n$ **diverges**.
 - **Note:** Knowing that $a_n \geq b_n$ for $\sum b_n$ convergent or that $a_n \leq b_n$ for $\sum b_n$ divergent tells you **nothing!**

3. The limit comparison test:

- As with the comparison test, you can use this when $\{a_n\}$ “looks like” another sequence $\{b_n\}$ where $\{b_n\}$ also has positive terms and where convergence/divergence of $\sum b_n$ is known.

However: This test should be used either (a) when the comparison test **doesn't** work, or (b) if you don't like the comparison test (because inequalities are hard). *If the comparison test works, the limit comparison test will work, but not vice versa!*

- As with the comparison test, you want to pick $\{b_n\}$ so that $\sum b_n$ is either a geometric series or a p -series.
- *What you know:*
 - If $\lim_{n \rightarrow \infty} a_n/b_n = c$ where $0 < c < \infty$, then either $\sum a_n$ and $\sum b_n$ *both* converge or they *both* diverge.
 - If $\lim_{n \rightarrow \infty} a_n/b_n = c$ where $c = 0$ or $c = \infty$, then you know **nothing!** In particular: You may have “squinted wrong” when picking $\{b_n\}$ or your series $\sum a_n$ may genuinely diverge. *You just don't know!*

Tests that work *only* for series with negative terms

The alternating series test:

- To use, $\sum a_n$ must be an *alternating series*, i.e. $\sum a_n$ must be writable as $\sum (-1)^n b_n$ or $\sum (-1)^{n+1} b_n$ for $\{b_n\}$ a sequence with positive terms.
- *What you know:*
 - If b_n is decreasing (i.e. if $b_{n+1} \leq b_n$ for all n) if $\lim_{n \rightarrow \infty} b_n = 0$, then $\sum a_n$ **converges**.

Other tests

1. The ratio test:

- Can use with *any* series.
- *What you know:*
 - If $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| < 1$, then $\sum a_n$ **converges absolutely** (and hence **converges**).
 - If $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| > 1$ (or is ∞), then $\sum a_n$ **diverges**.
 - If $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$, then $\sum a_n$ the ratio test tells you **nothing!**

2. The root test:

- Can use with *any* series.
- *What you know:*
 - If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ **converges absolutely** (and hence **converges**).
 - If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ (or is ∞), then $\sum a_n$ **diverges**.
 - If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, then $\sum a_n$ the root test tells you **nothing!**