Parametric Curves You Should Know

Straight Lines

Let a and c be constants which are not both zero. Then the parametric equations determining the straight line passing through (b, d) with slope c/a (i.e., the line $y - d = c/a(x - b)$) are:

$$\begin{cases} x(t) = at + b \\ y(t) = ct + d \end{cases}, -\infty < t < \infty.$$

Note that when $c = 0$, the line is horizontal of the form $y = d$, and when $a = 0$, the line is vertical of the form $x = b$.

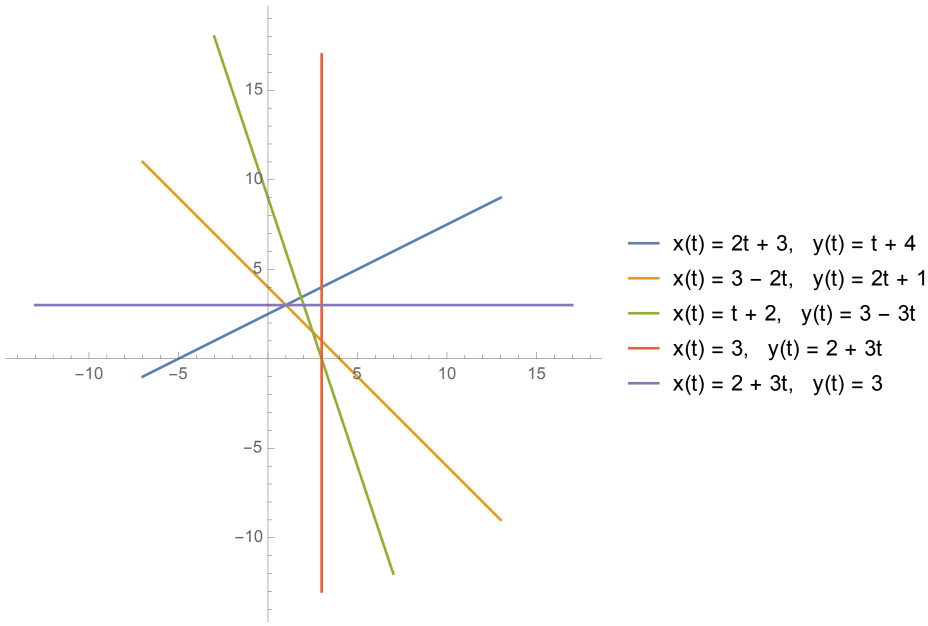


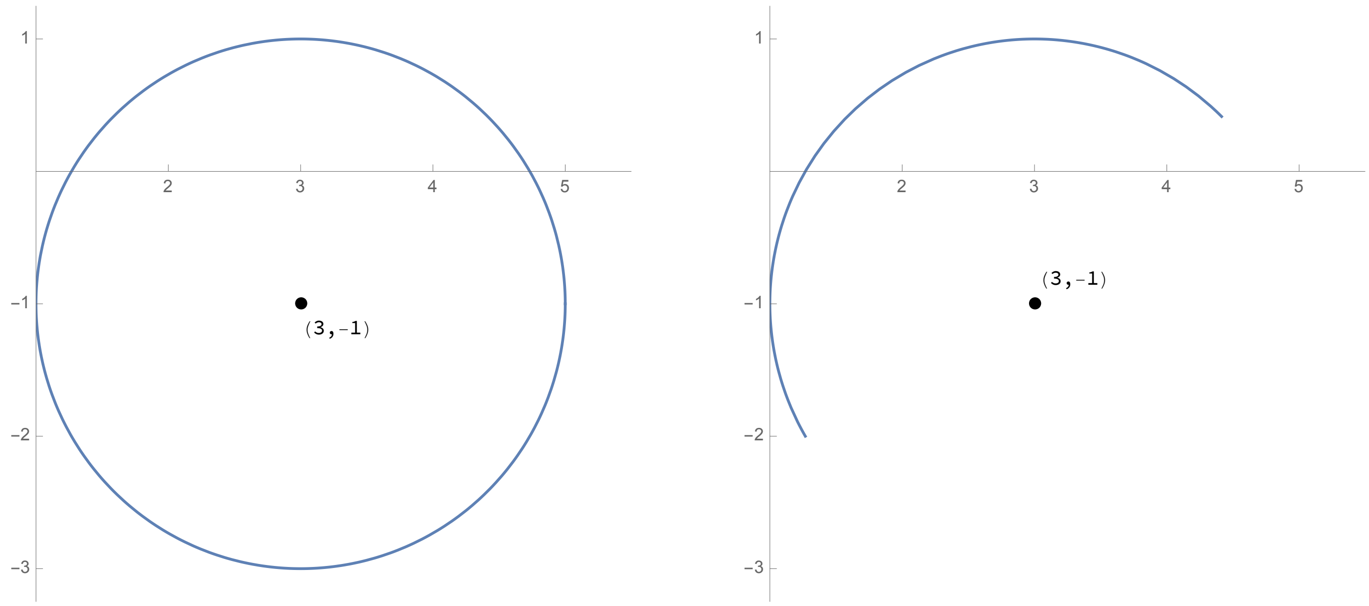
Figure 1
A collection of lines whose parametric equations are given as above.

Circles

Let $r > 0$ and let x_0 and y_0 be real numbers. Then the parametric equations determining the circle of radius r centered at (x_0, y_0) are:

$$\begin{cases} x(t) = r\cos t + x_0 \\ y(t) = r\sin t + y_0 \end{cases}, \quad 0 < t < 2\pi. \tag{1}$$

To get a circular arc instead of the full circle, restrict the t-values in (1) to $t_1 < t < t_2$.



(a) $0 \le t \le 2\pi$ **(b)** $\pi/4 \le t \le 7\pi/6$

Figure 2
The circle $x(t) = 2\cos t + 3$, $y(t) = 2\sin t - 1$ and a circular arc thereof. Note that the center is at $(3, 1)$.

Circles tangent to the x- and/or y-axes occur as special one-petal cases of the roses covered below.

Ellipses

Ellipses are algebraically similar to circles, and so their parametric equations are qualitatively similar. In particular, let $r_1, r_2 > 0$ and let x_0 and y_0 be real numbers. Then the parametric equations determining the ellipse of horizontal radius r_1 and vertical radius r_2 centered at (x_0, y_0) are:

$$\begin{cases} x(t) = r_1 \cos t + x_0 \\ y(t) = r_2 \sin t + y_0 \end{cases}, \quad 0 < t < 2\pi. \tag{2}$$

Obviously, when $r_1 = r_2$ in (2), your ellipse is a circle with parametric equation (1).

Spirals

One way to describe a spiral qualitatively is to say that its the shape formed when the angle $(\cos t, \sin t)$ is proportional to the value t, i.e. its parametric equations have the form:

$$\begin{cases} x(t) = c_1\, t \cos t + x_0 \\ y(t) = c_2\, t \sin t + y_0 \end{cases}, \quad 0 < t < t_{\text{terminal}} \quad (c_1, c_2 \text{ constants}; x_1, x_2 \text{ real numbers}). \tag{3}$$

Here, the spiral will cross the horizontal line $y = y_0$ (e.g. the x-axis, when $y_0 = 0$) precisely t_{terminal} times. In the event that $x_0 = y_0 = 0$ and $c_1 = c_2 = 1$, the spiral is a "circular spiral" spiraling about the origin: Figure 10 below shows how the spiral "grows" as t gets larger.

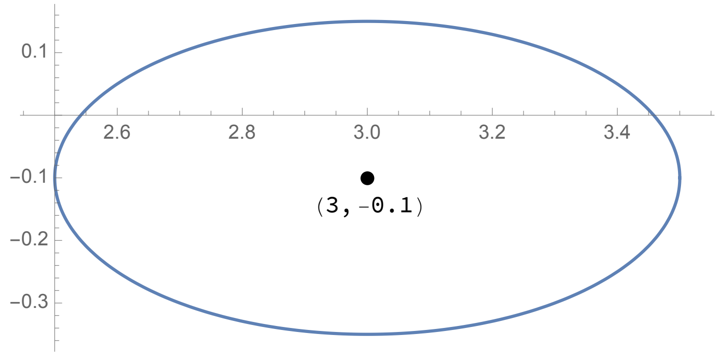


Figure 3

The ellipse $x(t) = 0.5\cos t + 3$, $y(t) = 0.25\sin t - 0.1$, $0 \le t \le 2\pi$. Note that the center is at $(3, 1)$.

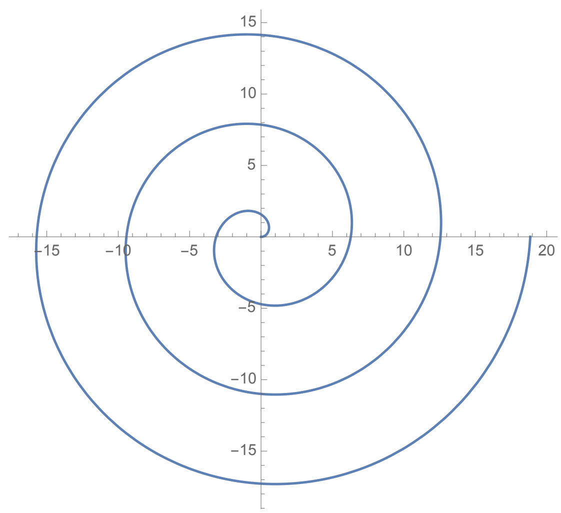
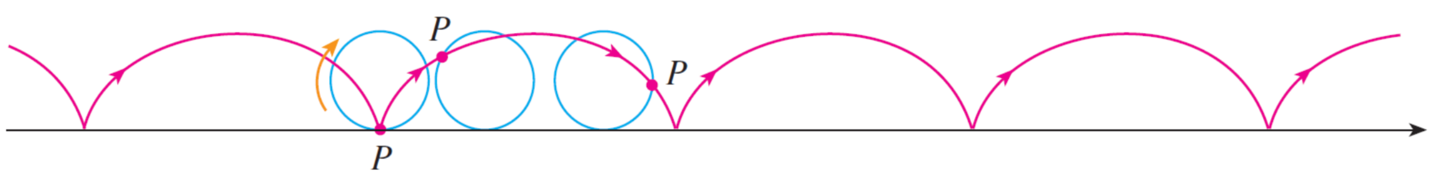


Figure 4

The spiral whose parametric equations are given by $x(t) = t\cos t$, $y(t) = t\sin t$, $0 \le t \le 6\pi$.

Cycloids

Cycloids are curves traced out by a point P on the circumference of a circle as the circle rolls along a straight line. The picture below shows what this looks like with a (pink) point P on the circumference of a (blue) circle, rolling along the (black) straight line (/axis).



There's a complicated procedure by which you can figure out the parametric equations producing various cycloids, but the gist is this: The parametric equations defining a cycloid formed when a circle of radius r is rolled n times is

$$\begin{cases} x(t) = r(t - \sin t) \\ y(t) = r(1 - \cos t) \end{cases}, \quad 0 < t < nr\pi.$$

So, for example, a circle of radius 2 which is rolled 12 times will have the parametric equations

$$\begin{cases} x(t) = 2(t - \sin t) \\ y(t) = 2(1 - \cos t) \end{cases}, \quad 0 < t < 24\pi.$$

The following figure shows a number of cycloids.

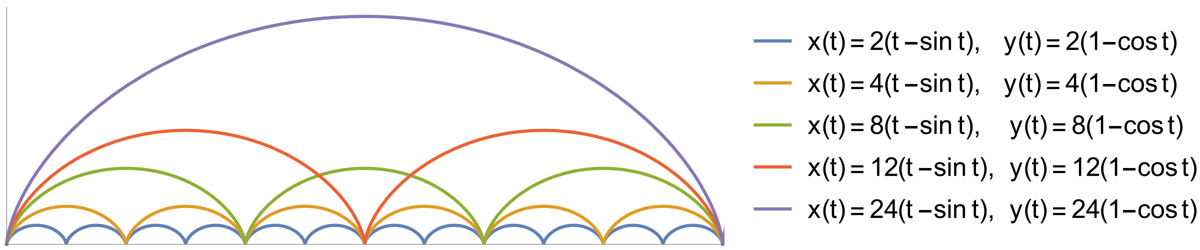
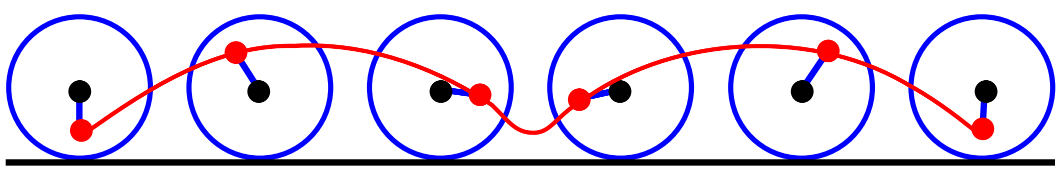


Figure 5

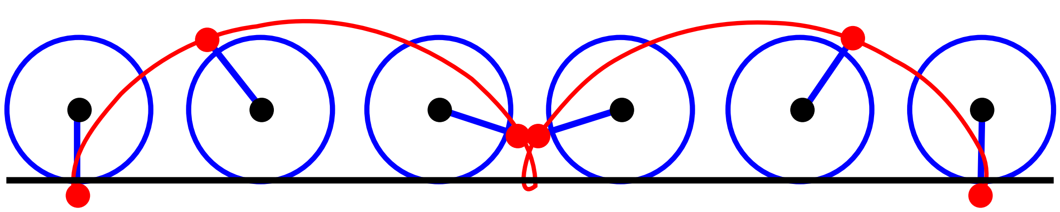
A collection of cycloids whose parametric equations are given as above, all plotted for $0 < t < 24\pi$. Notice that the number of "rollings" is given by $24/r$ for the various radii provided (so that $r = 2$ is rolled 12 times but $r = 24$ is rolled once).

Related Curves

Interestingly, the cycloid described above is one of a number of *cycloid-type* parametric curves which are defined similarly and which therefore have similar parametric representations. For example, the **curtate cycloid** is a curve traced out by a (red) point P on the *interior* of a given (blue) circle, rolling along the (black) straight line (/axis):



There's also the **prolate cycloid**, formed by a (red) point P on the *exterior* of a given (blue) circle, rolling along the (black) straight line (/axis):



As it happens, the curtate cycloid is defined by parametric equations of the form

$$\begin{cases} x(t) = at - b\sin t \\ y(t) = a - b\cos t \end{cases}, \quad 0 < t < t_{\text{terminal}} \tag{4}$$

for $a > b$, while the prolate cycloid is defined by the same parametric equations (4) with $a < b$.

Limaçons and Cardioids

Curves defined by parametric equations of the form

$$\begin{cases} x(t) = (1 + c\sin t)\cos t \\ y(t) = (1 + c\sin t)\sin t \end{cases}, \quad 0 < t < 2\pi \quad (c \text{ constant}) \tag{5}$$

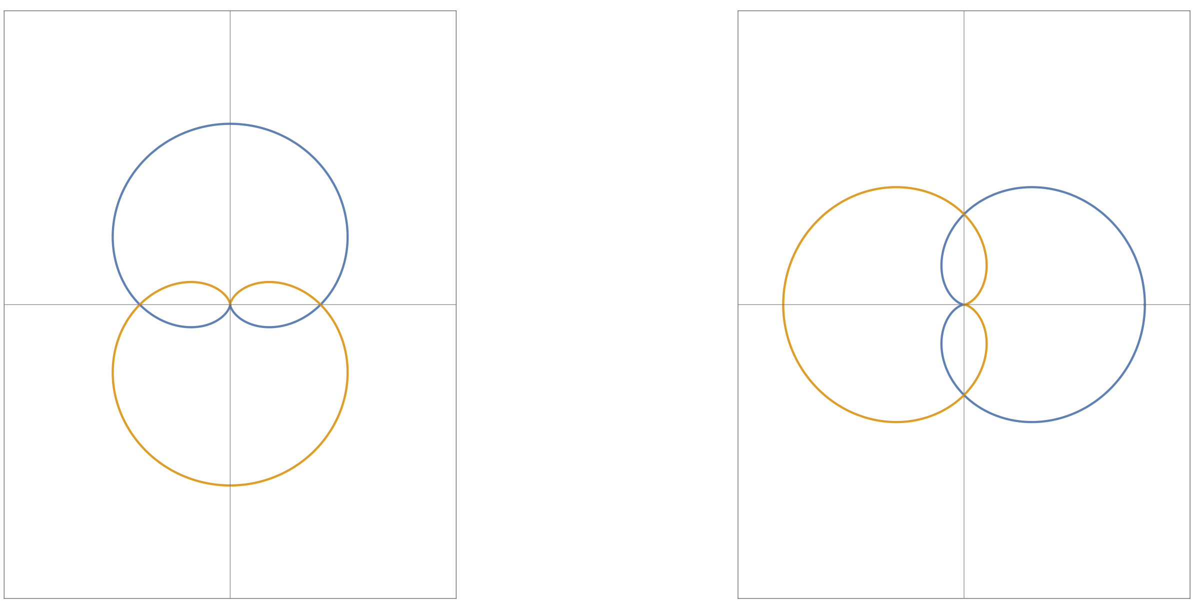
are called *limaçons*. Figure 11 below shows limaçons for various values c. When $c = \pm 1$, the equations in (5) become

$$\begin{cases} x(t) = (1 \pm \sin t)\cos t \\ y(t) = (1 \pm \sin t)\sin t \end{cases}, \quad 0 < t < 2\pi, \tag{6}$$

and the associated curve is called a *cardioid*. Alternatively, "horizontal analogues" of (6) can be obtained by replacing $(1 + \sin t)$ in (6) by $(1 + \cos t)$:

$$\begin{cases} x(t) = (1 \pm \cos t)\cos t \\ y(t) = (1 \pm \cos t)\sin t \end{cases}, \quad 0 < t < 2\pi. \tag{7}$$

The curves obtained from (6) and (7) are shown in figure 6 below.



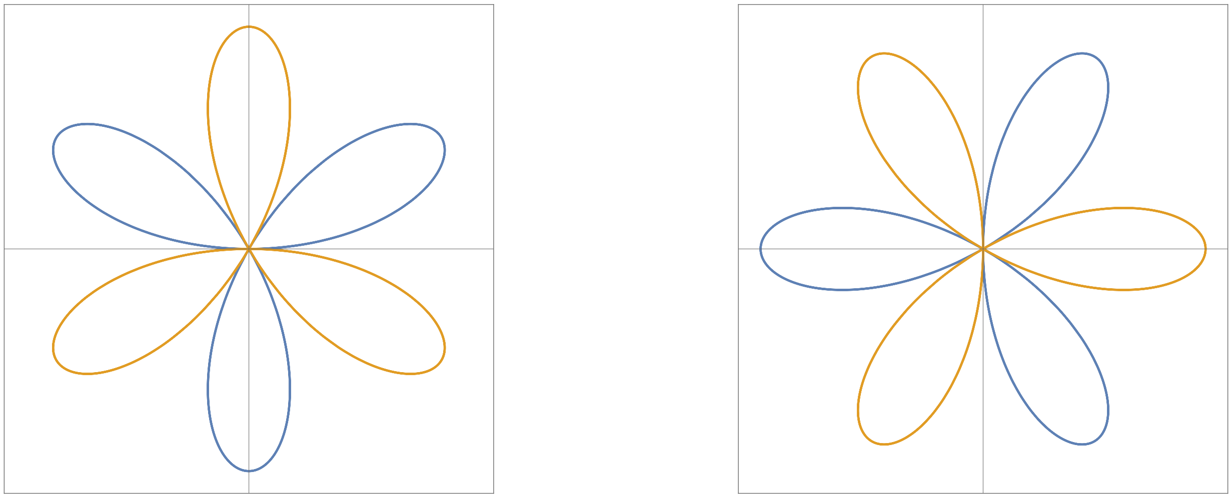
(a) $x(t) = (1 \pm \sin t)\cos t$, $y(t) = (1 \pm \sin t)\sin t$ (plus in blue, minus in orange)

(b) $x(t) = (1 \pm \cos t)\cos t$, $y(t) = (1 \pm \sin t)\cos t$ (plus in blue, minus in orange)

Figure 6
Vertical and horizontal cardioids

Roses

Consider the parametric curves shown in the following figures:



(a) **(b)**

Figure 7
Various roses.

Unsurprisingly, these curves are called *roses*. There are four general forms for the parametric curves defining roses: The equations defining the roses in figure (7a) have the form

$$\begin{cases} x(t) = \pm\sin(ct)\cos t \\ y(t) = \pm\sin(ct)\sin t \end{cases}, \quad 0 < t < 2\pi \quad (c \text{ constant}) \tag{8}$$

(where $+\sin(ct)$ is in blue and $-\sin(ct)$ is in orange), while the equations defining those in (7b) have the form

$$\begin{cases} x(t) = \pm\sin(ct)\sin t \\ y(t) = \pm\sin(ct)\cos t \end{cases}, \quad 0 < t < 2\pi \quad (c \text{ constant}). \tag{9}$$

(with $+\sin(ct)$ and $-\sin(ct)$ again in blue and orange, respectively).

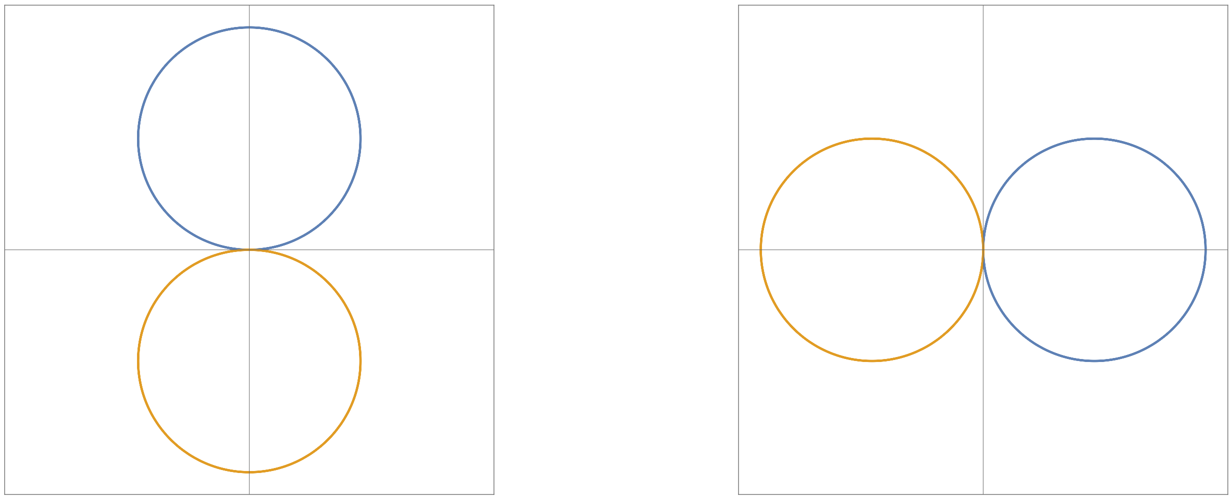
When c is odd, the roses in (8) and (9) have c petals; when c is even, they have $2c$ petals. Figure 12 below shows curves corresponding to (8) for varying values of c.

In the special case that $c = 1$, the result is four "roses with one petal," i.e. circles. In contrast to the circles shown in figure 2 above, the one-petal roses are tangent to the x- and y-axes. Figure 8 illustrates this case.

Important Note

As we saw in class, replacing $\cos t$ and $\sin t$ with $\cos(kt)$ and $\sin(kt)$, respectively, in (1) will yield a curve which **graphically** is the same as that rendered by (1) but which loops around k times instead of once.

Figure 9 below demonstrates precisely that via the parametric equations $x(t) = \cos(kt)$, $y(t) = \sin(kt)$, $0 \leq t \leq 2\pi$, for the values $k = 1$, $k = 5$, and $k = 50$.



(a) $x(t) = \pm \sin t \cos t$, $y(t) = \pm \sin t \sin t$
(plus in blue, minus in orange)

(b) $x(t) = \pm \sin t \sin t$, $y(t) = \pm \sin t \cos t$
(plus in blue, minus in orange)

Figure 8
Various one-petal roses (i.e., circles tangent to the x- and y-axes)

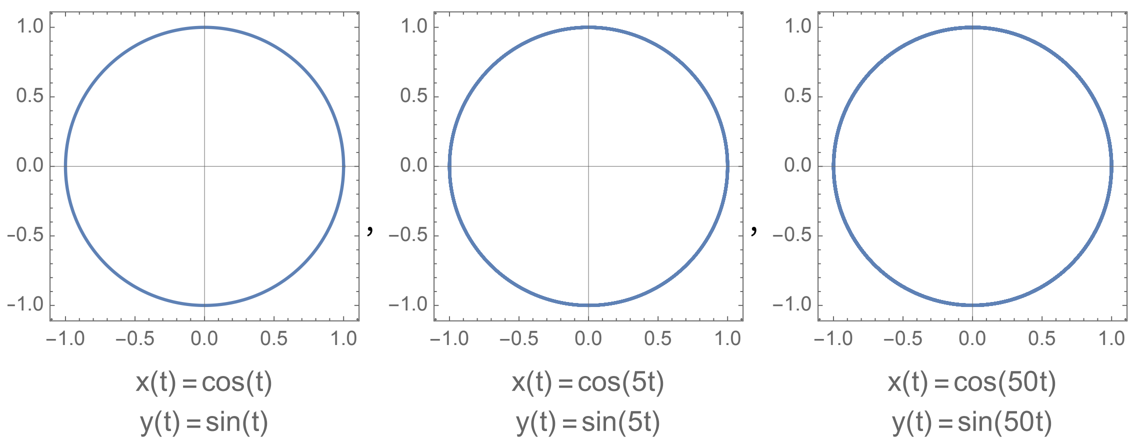


Figure 9
Curves which are identical as curves but which are *different* as parametric curves (i.e., curves + their associated parametric equations).

Note, too, that for these parametrizations, varying the terminal value t_{terminal} may also yield the same curve: For example, $x(t) = \cos(kt)$ and $y(t) = \sin(kt)$ yield graphs which are unit circles for all k *and* for all parameter values $0 \leq t \leq 2\pi + c$ for all values $c \geq 0$.

This tells us that parametric curves aren't unique; even more concretely, it tells us that there are often multiple parametrizations which yield the same curve. This fact will be very important when we start doing calculus (e.g., definite integration), because presumably, the mechanisms we employ there should yield correct results while allowing for different parametrizations. These are ideas we'll revisit later.

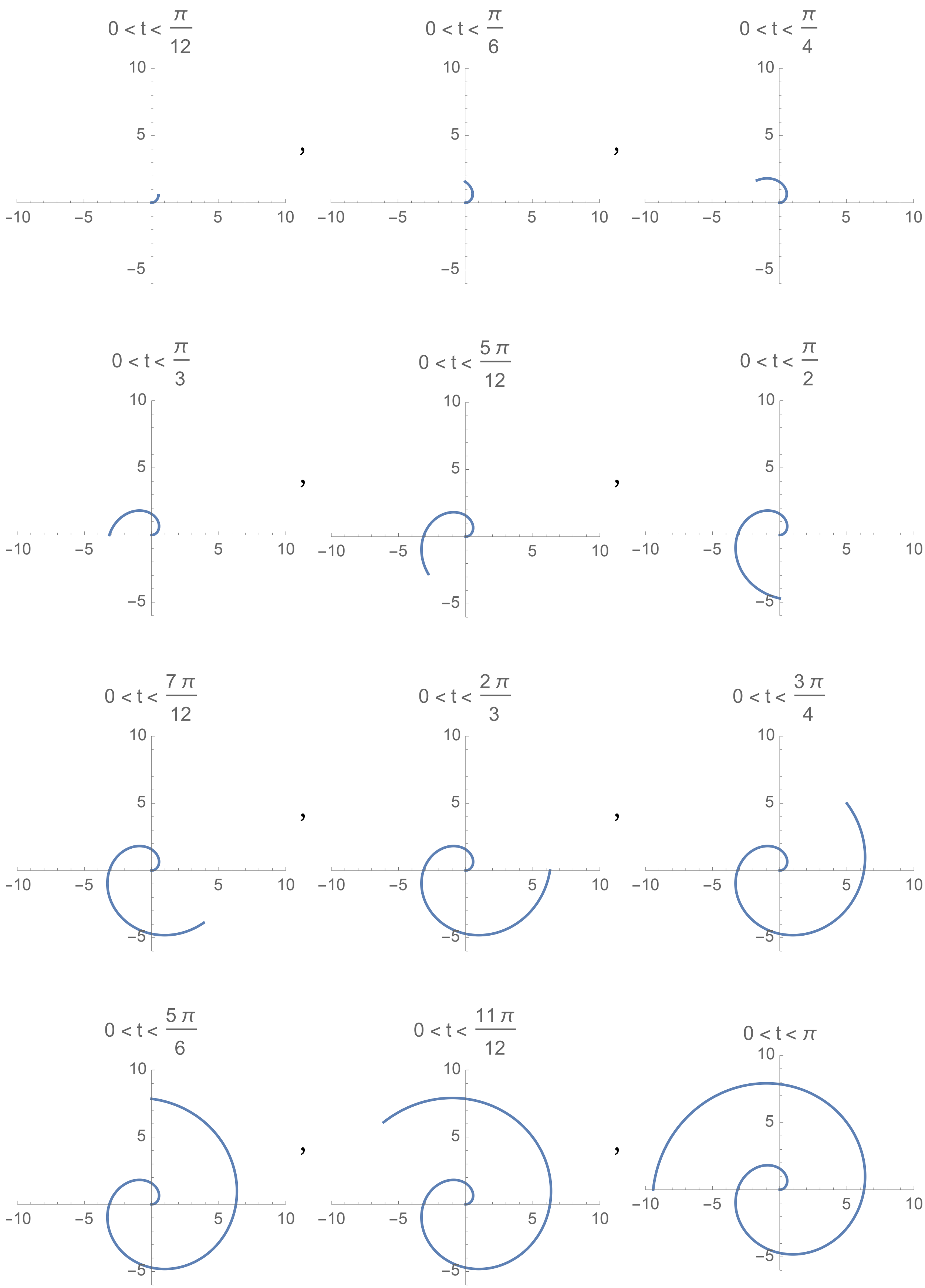


Figure 10

The growth of the spiral whose parametric equations are given by $x(t) = t\cos t$, $y(t) = t\sin t$, for increasing values of t.

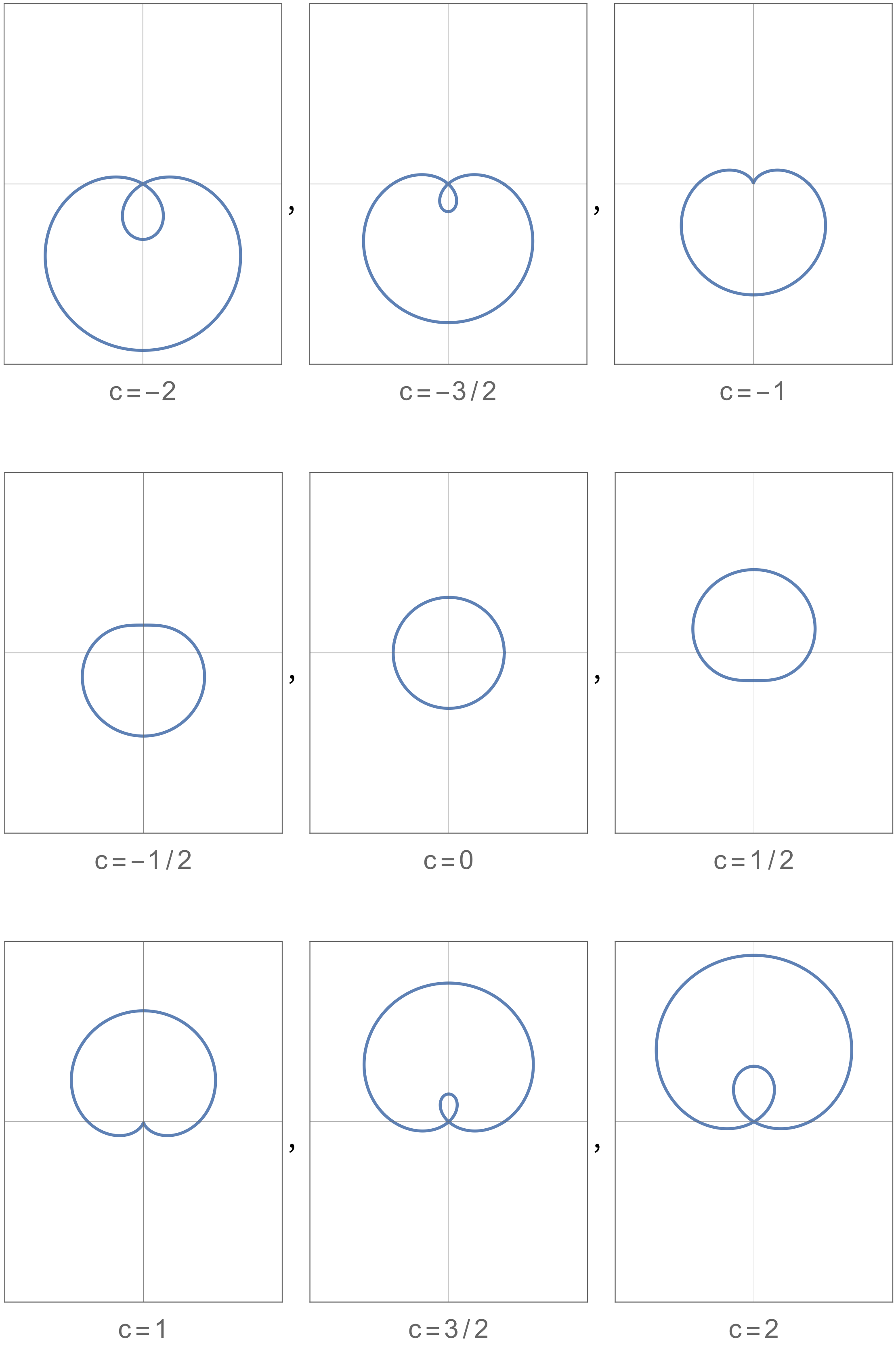


Figure 11
Limaçons of the form $x(t) = (1 + c \sin t) \cos t$, $y(t) = (1 + c \sin t) \sin t$ for various values c.

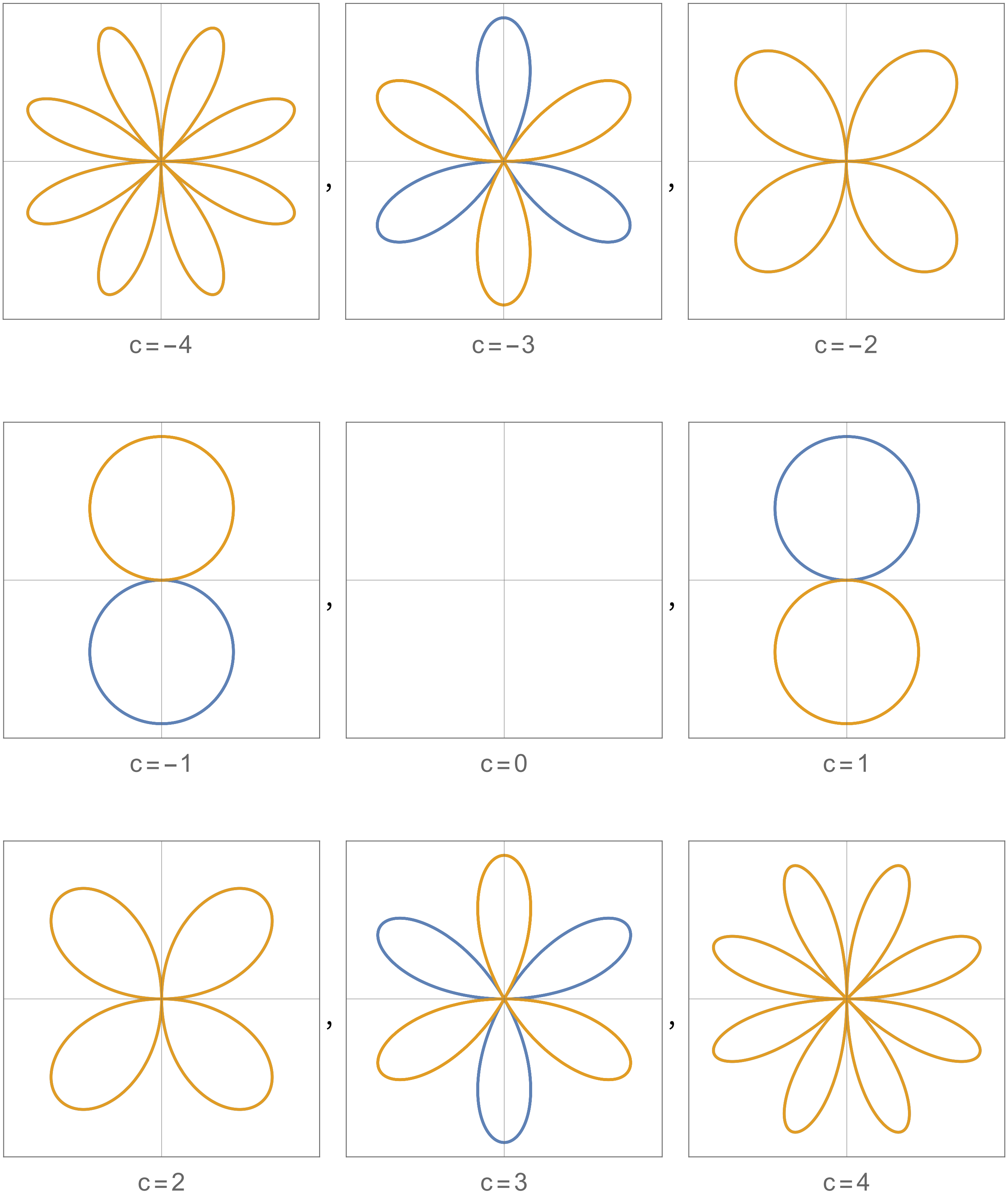


Figure 12

Roses of the form $x(t) = \pm\sin(ct)\cos t$, $y(t) = \pm\sin(ct)\sin t$, $0 < t < 2\pi$, for various values of c. Note that when c is even, there are $2c$ petals and that the blue and orange curves coincide; otherwise, there are c petals and the blue and orange curves are unique. Also, when $c = 0$, $x(t) = 0 = y(t)$.