Sample Questions for Approximate Integration

There are a number of good, test-worthy questions that can be asked regarding approximate integration. Here's a sample.

Example 1. Let $f(x) = e^{-x^2}$. Given that $|f''(x)| \leq 2$ and that $|f^{(4)}(x)| \leq 12$ on $[-3, 3]$, answer the following questions regarding numerical approximations to $\int_{-3}^{3} f(x)\,dx$:

1. Bound the values $|E_T|$, $|E_M|$, and $|E_S|$ for $n = 8$.
2. Find the number of subintervals needed for the trapezoidal rule to be accurate within 10^{-6}.

Solution:

1. Here, we use the formulas
$$|E_T| \leq \frac{K_1(b-a)^3}{12n^2},$$
$$|E_M| \leq \frac{K_1(b-a)^3}{24n^2},$$
and
$$|E_S| \leq \frac{K_2(b-a)^5}{180n^4}$$
with $a = -3$, $b = 3$, $n = 8$, $K_1 = 2$, and $K_2 = 12$: For example,
$$|E_T| \leq \frac{K_1(b-a)^3}{12n^2} = \frac{2(6)^3}{12(8)^2} = \frac{9}{16} = 0.5625.$$
Similarly, $|E_M| \leq 9/32 = 0.28125$ and $|E_S| \leq 81/640 = 0.1265625$.

2. Here, we use the formula
$$|E_T| \leq \frac{K_1(b-a)^3}{12n^2}$$
a little bit differently: In particular, we know we want the error to be **less than or equal to** 10^{-6}, and we know from the question that we need to solve for n. To do that, we set up
$$\frac{K_1(b-a)^3}{12n^2} \leq 10^{-6}$$
and solve for n (given that with $a = -3$, $b = 3$, $K_1 = 2$, and $K_2 = 12$ still hold):
$$\frac{2(6)^3}{12n^2} \leq 10^{-6} \implies 2(6)^3 \leq 10^{-6}(12n^2) \implies \frac{2(6)^3}{12 \cdot 10^{-6}} \leq n^2.$$
Because the square root is a monotone function (*don't worry about this fact*) means we can take the square root of both sides without changing the inequality:
$$\sqrt{\frac{2(6)^3}{12 \cdot 10^{-6}}} \leq n.$$

Thus, for any n larger than

$$\sqrt{\frac{2(6)^3}{12 \cdot 10^{-6}}} = 6000,$$

$|E_T| \leq 10^{-6}$ will be satisfied.

$\square$

True or False

While this section is largely computational, our classroom discussion hinted at the fact that there are also tons of good true/false questions!

Example 2. Here are some facts to know which make really good true/false questions!

Solution:

1. For f strictly increasing, L_n is an underestimate and R_n is an overestimate for the area under the curve.
2. For f strictly decreasing, L_n is an overestimate and R_n is an underestimate for the area under the curve.
3. In general, $L_n < M_n < R_n$ is false.
4. $M_n \neq (L_n + R_n)/2$.
5. $T_n = (L_n + R_n)/2$ for all n.
6. $S_{2n} = 2/3M_n + 1/3T_n$ for all n.

$\square$