$$\frac{d}{dx}\left(\#^x\right) = \frac{1}{\ln(\#)} \cdot \#^x$$

Dec 1, 2016

Exam 4

MAC 2312 Calculus II, Fall 2016

(Neatly!) Print Name: KEY

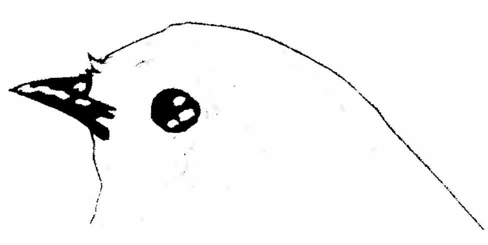
Read <u>all</u> of what follows <u>carefully</u> before starting!

1. This test has **5 problems** (15 parts total), is worth **115 points**, and has **1 bonus problem**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students.
3. No calculators may be used on this exam!
4. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
5. You may use appropriate results from class and/or from the textbook <u>as long as you fully and correctly state the result and where it came from.</u>
6. You **do not** need to simplify results, unless otherwise stated.
7. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
8. $\{a_n\}$ <u>always</u> means $\{a_n\}_{n=1}^{\infty}$, $\sum a_n$ <u>always</u> means $\sum_{n=1}^{\infty} a_n$, and "$n!$" <u>always</u> means "n factorial."

there is no math on this page

What if Kanye wrote a song about Kanye
called *I miss the old Kanye*?

Man, that'd be **SO** Kanye!



1. (5 *pts ea.*) Provide examples of sequences satisfying the below criteria or state that no such example exists. For each example, describe why the criteria are met; if no example exists, explain why. **No credit will be given without justification!**

(a) A sequence $\{a_n\}$ which is increasing, bounded below, and not bounded above.

$\{n\}$, $\{n^2\}$, etc.

(b) A sequence $\{b_n\}$ which is decreasing, bounded, and converges to 4.

$\left\{4+\frac{1}{n}\right\}$, $\left\{4+\frac{1}{n^2}\right\}$, etc.

2. Consider the sequence $\left\{\frac{25}{9}, -\frac{5}{3}, 1, -\frac{3}{5}, \frac{9}{25}, -\frac{27}{125}, \dots\right\}$.

(a) (5 *pts*) Find a formula for the general term a_n.

$$a_n = \frac{25}{9}\left(\frac{-3}{5}\right)^{n-1}$$

(b) (5 *pts*) Compute $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} a_n = 0$$

(c) (5 *pts*) Use part (a) to write the series $a_1 + a_2 + a_3 + \cdots$ in summation notation.

$$\sum_{n=1}^{\infty} \left(\frac{25}{9}\right)\left(\frac{-3}{5}\right)^{n-1}$$

(d) $\frac{68}{45}$ $\frac{10}{9}$

(e) $\frac{125}{72}$

Part (e) is on the next page

$$\sum_{n=1}^{\infty} \frac{25}{9}\left(\frac{-3}{5}\right)^{n-1}$$

(d) (5 *pts*) Write the partial sums s_1, s_2, and s_4 as fractions in reduced form. Simplify fully!

$$S_1 = \frac{25}{9}$$

$$S_2 = \frac{25}{9} + \frac{-5}{3} = \frac{25-15}{9} = \frac{10}{9}$$

$$S_4 = \frac{10}{9} + 1 - \frac{3}{5} = \frac{50+45-27}{45} = \frac{68}{45}$$

(e) (10 *pts*) Does the series from part (c) converge? Why or why not? If it converges, find its value. **No credit will be given without justification!**

- yes!
- geometric series w/ $|r| = \left|\frac{-3}{5}\right| = \frac{3}{5} < 1$!
- sum $= \frac{a}{1-r} = \frac{25/9}{1-(-3/5)} = \frac{25/9}{8/5} = \frac{25}{9} \cdot \frac{5}{8}$

$$\boxed{= \frac{125}{72}}$$

3. (a) (5 *pts ea.*) Explain why you *can't* use the integral test to determine the convergence of each of the following series. **There may be more than one reason why the integral test cannot be used!**

i.) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$ not positive, not decreasing

ii.) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos^2(k)}{1+k^2}$ not decreasing

Part (b) is on the next page

(b) *(10 pts)* Use the integral test to determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges or diverges.

Let $a_n = \left\{ \frac{1}{n^2+4} \right\}$

- This is positive b/c $n^2+4 > 0$
- This is decreasing b/c

$$(n+1)^2+4 > n^2+4$$

$$\Rightarrow \frac{1}{(n+1)^2+4} < \frac{1}{n^2+4}$$

$$\Rightarrow a_{n+1} < a_n$$

- By the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges iff $\int_1^{\infty} \frac{1}{x^2+4}\,dx$ converges.

- $\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\left(\frac{1}{2}\arctan\left(\frac{x}{2}\right) \Big]_1^t \right)$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\arctan\left(\frac{t}{2}\right)\right) - \frac{1}{2}\arctan\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - (\text{some number}) = \text{some number}.$$

- Hence, $\int \cdots$ converges $\Longrightarrow$ (by integral test) $\sum \cdots$ converges

1. All parts of this question relate to the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.

(a) (5 pts) Show that the ratio test is inconclusive for this series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n}\right| = \left|\frac{n}{n+1}\right| \to 1 \text{ as } n \to \infty.$$

(b) (10 pts) Show that this series is (conditionally) convergent.

Let $b_k = \frac{1}{k}$. Then b_k is decreasing & $b_k \to 0$ as $k \to \infty$, so by alternating series test, $\sum (-1)^k b_k$ converges.

(c) (10 pts) Is this series absolutely convergent? Why or why not? **No credit will be given without justification!**

No!

$$\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$$ diverges by p-test (it's the harmonic series).

5. (*10 pts ea.*) Using any of the methods we discussed in class, determine whether each of following series is convergent or divergent.

(a) $\sum_{n=1}^{\infty}\left(\frac{2^n}{n+1}\right)^n$

SOLUTION:

Root test:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{2^n}{n+1}\right|^n} = \frac{2^n}{n+1} \to \infty \text{ as } n \to \infty.$$

Hence, $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ & the series

DIVERGES by root test!

Part (b) is on the next page

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ **Hint:** $\displaystyle\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = \frac{1}{e}$

SOLUTION: <u>Ratio test</u>

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \left|\frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}\right|$$

$$= \left|(n+1) \cdot \frac{n^n}{(n+1)^{n+1}}\right| = \left|\frac{n^n}{(n+1)^n}\right| = \left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{n+1}{n+1} - \frac{1}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n = \frac{1}{e} < 1.$$

So, by ratio test, series converges <u>absolutely</u>.

Part (c) is on the next page

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2)}{1+4^n}$

SOLUTION: Note: ① $1+4^n > 4^n$

$$\Rightarrow \frac{1}{1+4^n} < \frac{1}{4^n} . \Rightarrow \frac{|\sin(n^2)|}{1+4^n} < \frac{|\sin}{4^n}$$

② $|\sin(n^2)| \leq 1 \Rightarrow \dfrac{|\sin(n^2)|}{1+4^n} \leq \dfrac{1}{4^n}$.

③ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$ is geometric w/ $|r| = \left|\frac{1}{4}\right| < 1$

& hence converges,

So, by comparison test,

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2)}{1+4^n}$ converges.

Bonus *(10 pts)*: Recall "the ε-definition" of sequence convergence:

> **The ε-definition:** A sequence $\{a_n\}$ *converges* to a number L if, for all $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

Prove that the sequence $\{1 - 1/n\}_{n=1}^{\infty}$ converges to $L = 1$ using the ε-definition by (a) letting ε be arbitrary, (b) considering the inequality $|a_N - 1| < \varepsilon$, and (c) solving for N.

Hint: Your first sentence should be: "Let $\varepsilon > 0$ be arbitrary and consider the inequality $|a_N - 1| < \varepsilon$."

SOLUTION:

Scratch Paper