Nov 8, 2016

Exam 3

MAC 2312—CALCULUS II, FALL 2016

(NEATLY!) PRINT NAME: KEY

Read all of what follows carefully before starting!

1. This test has **6 problems** (18 parts total), is worth **100 points**, and has **1 bonus problem**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students.
3. No calculators may be used on this exam!
4. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
5. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
6. You **do not** need to simplify results, unless otherwise stated.
7. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
8. You may need the following trig identities:

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\\ &= 1 - 2\sin^2(\theta)\\ \sin(2\theta) &= 2\sin(\theta)\cos(\theta)\end{aligned}$$

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \qquad \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

there is no math on this page

If the Chicago Cubs taught us anything,

it's that

yesterday's failures only serve to magnify today's successes!

1.(a) *(15 pts)* Solve **one** of the following differential equations. **Clearly indicate which one you're solving!**

$$x^2\frac{dy}{dx} = \sqrt{1-y^2} \qquad \textbf{or} \qquad x\ln x = \left(2y\sqrt{3+y^2}\right)y'$$

SOLUTION:

$x^2 \frac{dy}{dx} = \sqrt{1-y^2}$

$\Rightarrow \frac{1}{x^2}\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$

$\Rightarrow \frac{dx}{x^2} = \frac{dy}{\sqrt{1-y^2}}$

$\Rightarrow \int \frac{dx}{x^2} = \int \frac{dy}{\sqrt{1-y^2}}$

$\Rightarrow -\frac{1}{x} + C = \arcsin y$

$\Rightarrow y = \sin\left(\frac{-1}{x} + C\right)$

$x\ln x = \left(2y\sqrt{3+y^2}\right)\frac{dy}{dx}$

$\Rightarrow \underbrace{x\ln x\,dx}_{\text{IBP}} = \underbrace{\left(2y\sqrt{3+y^2}\right)dy}_{u\text{-sub}}$

$\cancel{x}\ln x \quad \frac{1}{2}x^2$

$\frac{1}{x} \quad x$

$u = 3+y^2 \quad du = 2y\,dy \Rightarrow \int \sqrt{u}\,du$

$\Rightarrow \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{2}{3}(3+y^2)^{3/2}$

$\Rightarrow \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C = \frac{2}{3}(3+y^2)^{3/2}$

$\Rightarrow \left[\frac{3}{2}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C\right)\right]^{2/3} = 3+y^2$

$\Rightarrow y = \sqrt{\left[\frac{3}{2}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C\right)\right]^{3/2} - 3}$

Part (b) is on the next page

(b) (*15 pts*) Solve **one** of the following initial value problems. **Clearly indicate which one you're solving!**

$$xy' = e^x - y,\ y(1) = 2 \quad \textbf{or} \quad y' = \frac{y}{x} + x,\ y(2) = 1.$$

SOLUTION:

① $x \frac{dy}{dx} + y = e^x$

$\Rightarrow \frac{d}{dx}(x\,y) = e^x$

Note: integrating factor already there

$= \ x \cdot y = \int e^x dx$

$\Rightarrow x\,y = e^x + C$

$\Rightarrow y = \frac{e^x}{x} + \frac{C}{x}$. Now:

② $2 = \frac{e^1}{1} + \frac{C}{1}$

$\Rightarrow 2 = e + C \Rightarrow C = 2 - e.$

③ So:

$$\boxed{y = \frac{e^x}{x} + \frac{2-e}{x}}$$

① $\frac{dy}{dx} - \frac{1}{x} y = x$

$\Rightarrow I(x) = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

$\Rightarrow \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = 1$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x} \cdot y\right) = 1$

$\Rightarrow \frac{1}{x} y = x + C$

$\Rightarrow y = x^2 + Cx$

② $1 = (2)^2 + 2C$

$\Rightarrow 1 = 4 + 2C$

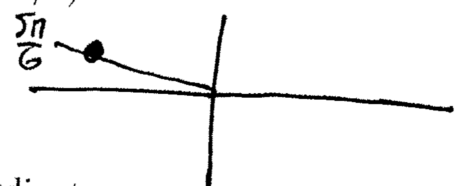
$\Rightarrow -3 = 2C \Longrightarrow C = \frac{-3}{2}$

③ So:

$$\boxed{y = x^2 - \frac{3}{2}x}$$

2. (*3 pts ea.*) Perform each of the indicated conversions below.

(a) Find a pair of polar coordinates with $r < 0$ which represents the same point as the polar coordinates $(3, 5\pi/6)$.



$\rightarrow \boxed{\left(-3, \frac{11\pi}{6}\right)}$

(b) Find a pair of polar coordinates corresponding to the Cartesian coordinates $\left(-1, \sqrt{3}\right)$.

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(-\sqrt{3}\right) = \frac{-\pi}{3}$ $\rightarrow \boxed{\left(2, \frac{2\pi}{3}\right)}$

(c) Eliminate the parameter t to find a Cartesian equation of the curve $x = 1 - t^2$, $y = t^3 + 1$.

$x = 1 - t^2 \Rightarrow t = \sqrt{1-x} \Rightarrow \boxed{y = \left(\sqrt{1-x}\right)^3 + 1 = (1-x)^{3/2} + 1}$

(d) Write the parametric equations $x(\theta)$ and $y(\theta)$ corresponding to the polar curve $r(\theta) = \theta^2$.

$\boxed{\begin{aligned} x(\theta) &= \theta^2 \cos\theta \\ y(\theta) &= \theta^2 \sin\theta \end{aligned}}$

(e) Find the Cartesian coordinates corresponding to the polar coordinates $(-4, 7\pi/6)$.

$x = -4\cos(7\pi/6) = 2\sqrt{3}$

$y = -4\sin(7\pi/6) = 2$ $\rightarrow \boxed{\left(2\sqrt{3}, 2\right)}$

(f) Find a pair of polar coordinates with $\theta > 2\pi$ which represents the same point as the polar coordinates $(1, 4\pi/3)$.

$\boxed{\left(1, \frac{10\pi}{3}\right)}$

3. Let C be the curve determined by the parametric equations

$$x(t) = r\sin t \quad y(t) = -r\cos t \quad 0 \leq t \leq 2\pi \quad \text{(where } r > 0 \text{ is a constant).}$$

(a) (*3 pts*) Describe the graph of C.

circle w/ center $(0,0)$ & radius r

(b) (*6 pts*) Find the exact arc length of C.

$$\frac{dx}{dt} = r\cos t \qquad \frac{dy}{dt} = r\sin t$$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{(r\cos t)^2 + (r\sin t)^2}\, dt$$

$$= \int_0^{2\pi} r\, dt = \boxed{2\pi r}$$

Part (c) is on the next page

4. (*10 pts*) Let D be the curve defined by $x(t) = \cos t$ and $y(t) = \sin t$ on the interval $0 \leq t \leq \pi$. Find the surface area of the figure obtained by revolving D around the x-axis.

SOLUTION:

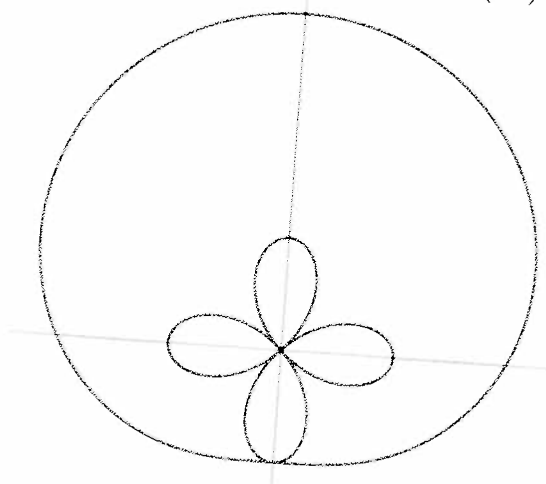
$$A = \int_0^{\pi} 2\pi\, y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} 2\pi \sin t \, dt$$

$$= -2\pi \cos t \Big]_{t=0}^{t=\pi} = -2\pi((-1) - 1)$$

$$= \boxed{4\pi}$$

5. (*15 pts*) Find the area of the shaded region, where the outer curve is given by $r = 2 + \sin\theta$ and the inner curve is given by $r = \cos(2\theta)$.



SOLUTION:

$$A = A_{out} - A_{inn}$$

$$= \frac{1}{2}\int_0^{2\pi} (r_{out}^2 - r_{inn}^2)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (2+\sin\theta)^2 - \cos^2(2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta + \sin^2\theta - \cos^2(2\theta)\, d\theta$$

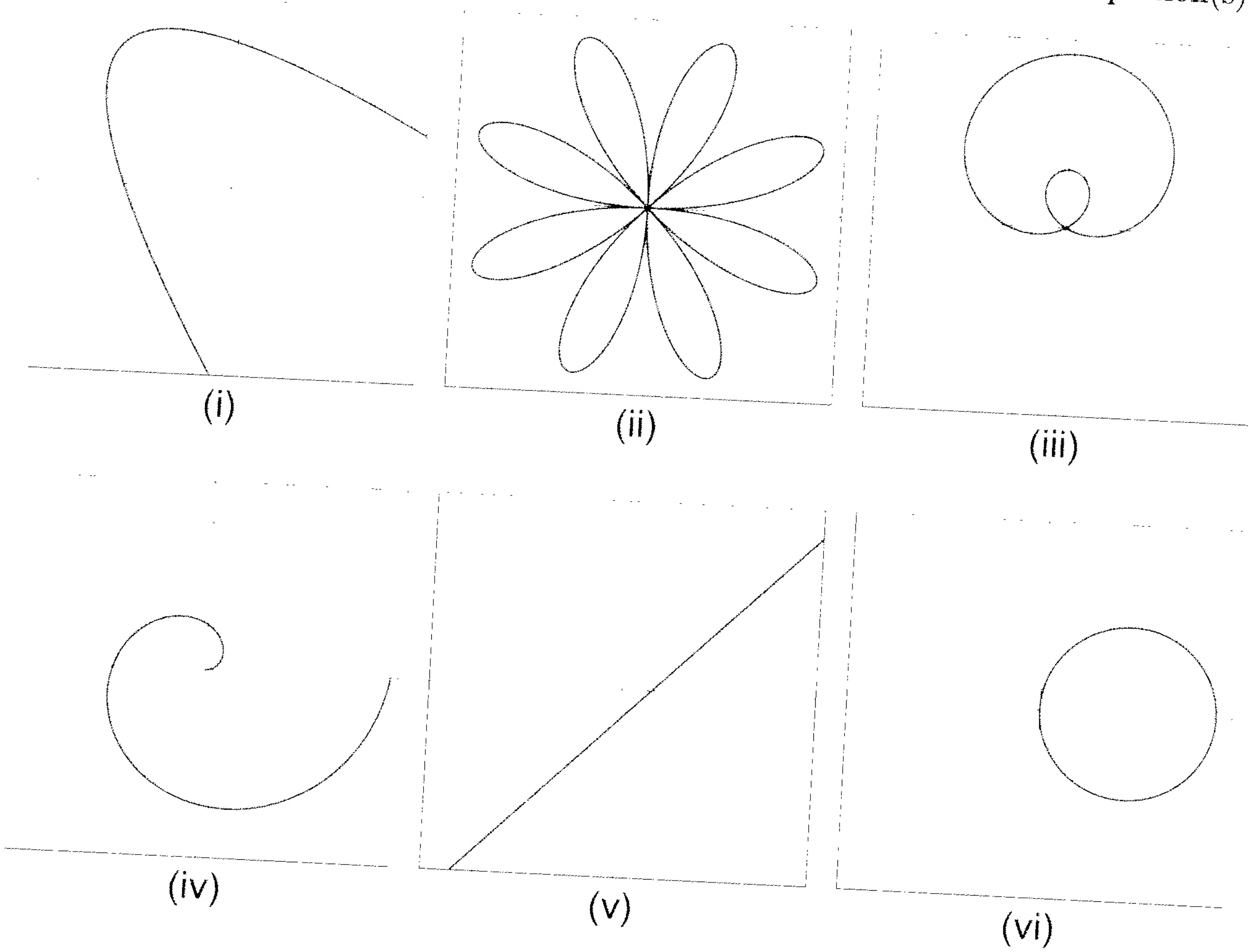
$$= \frac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta + \frac{1}{2}(1 - \cos(2\theta)) - \frac{1}{2}(1 + \cos 4\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta - \frac{1}{2}\cos(2\theta) - \frac{1}{2}\cos(4\theta)\, d\theta$$

$$= \frac{1}{2}\left[\left[4\theta - 4\cos\theta - \frac{1}{4}\sin(2\theta) - \frac{1}{8}\sin(4\theta)\right]_{\theta=0}^{\theta=2\pi}\right]$$

$$= \frac{1}{2}[8\pi - 4 - 0 - 0 - 0 + 4 + 0 + 0] = \boxed{4\pi.}$$

6. (*3 pts ea.*) Match the following graphs with the Cartesian/parametric/polar equations which define them by writing (i)—(vi) beneath the appropriate equation(s).



(a) $r = 3\sin(4\theta)$

(ii)

$r^2 = \theta^2 \Rightarrow r = \theta$

(c) $x^2 + y^2 = \arctan^2\left(\frac{y}{x}\right)$

(iv)

(e) $x = t^2 - 2t - 1,$
$y = 2.5 - t^2$

(i)

(b) $x = (1 + 2\sin t)\cos t,$
$y = (1 + 2\sin t)\sin t$

(iii)

(d) $x = 6\cos 2t \cos 2t,$
$y = 6\cos 2t \sin 2t$

(vi)

(f) $r = \dfrac{-1}{\sin\theta - \cos\theta}$

(v)

Bonus (*10 pts*): Prove that the arc length of a polar curve $r = r(\theta)$, $\alpha \le \theta \le \beta$, is given by

$$L = \int_\alpha^\beta \sqrt{[r(\theta)]^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

by converting r into parametric equations and using the parametric arc length formula.

SOLUTION: $x = r(\theta)\cos\theta \Rightarrow \frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$

$y = r(\theta)\sin\theta \Rightarrow \frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$

$$\Rightarrow L = \int_\alpha^\beta \sqrt{\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)^2 + \left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{r^2\sin^2\theta - 2r\frac{dr}{d\theta}\sin\theta\cos\theta + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta + r^2\cos^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{r^2(\sin^2\theta + \cos^2\theta) + \left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{[r(\theta)]^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$