Oct 13, 2016

Exam 2

MAC 2312—Calculus II, Fall 2016

(Neatly!) Print Name: KEY

Read all of what follows carefully before starting!

1. This test has **6 problems** (15 parts total), is worth **91 points**, and has **2 bonus problems**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students.
3. No calculators may be used on this exam!
4. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
5. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
6. You **do not** need to simplify results, unless otherwise stated.
7. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
8. **Note:** Throughout, L_n, M_n, R_n, T_n, and S_n denote left endpoint, midpoint, right endpoint, trapezoidal, and Simpson's rule integral approximations with n subdivisions, respectively.

there is no math on this page

$$\int_0^1 x^{-p}\,dx = \int_0^1 \left(\frac{1}{x}\right)^p dx$$

let $y=\frac{1}{x}$ $\Rightarrow$ $dy = \frac{-1}{x^2}\,dx = -\left(\frac{1}{x}\right)^2 dx = -y^2\,dx$

$\Rightarrow$ $\frac{-dy}{y^2} = dx$. Also: $x=1 \Rightarrow y=1$

$x=0 \Rightarrow y=\infty$

$$\Rightarrow \int_0^1 x^{-p}\,dx = \int_\infty^1 y^p\left(\frac{-dy}{y^2}\right) = \int_1^\infty y^{p-2}\,dy$$

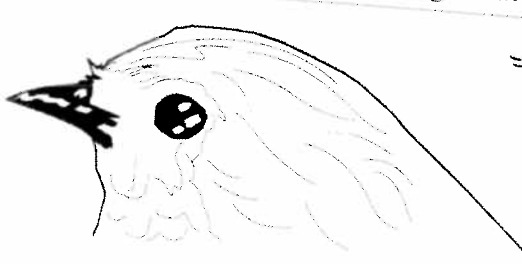
You're amazing

and

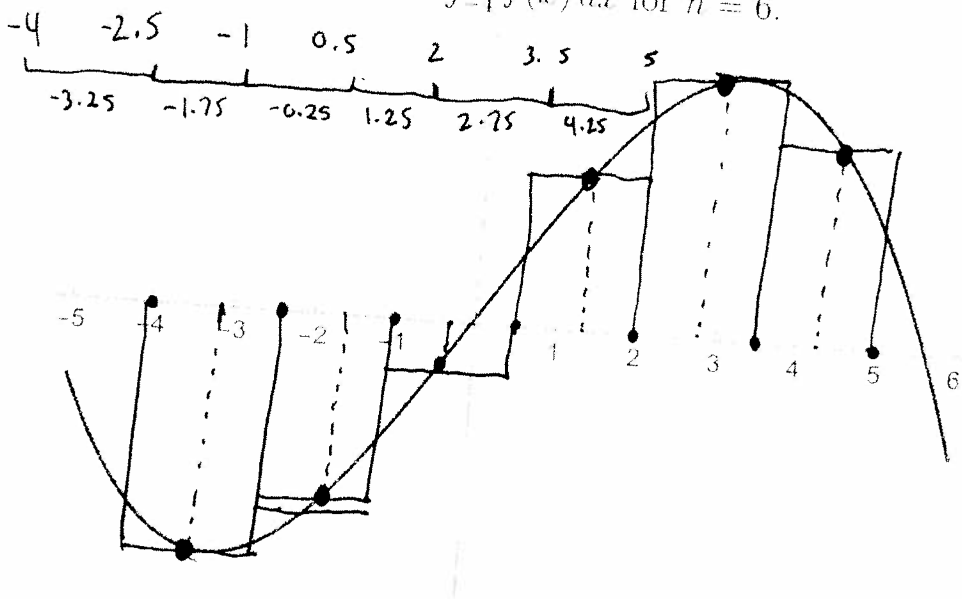
$$= \int_1^\infty \frac{1}{y^{2-p}}\,dy$$ diverges iff $2-p \le 1$

nothing you write on this exam will change that!

$\Rightarrow p \ge 1$.



1. (a) (*3 pts*) Given the graph of $f(x)$ below, sketch the shapes corresponding to a midpoint approximation of $I = \int_{-4}^{5} f(x)\,dx$ for $n = 6$.



(b) (*5 pts*) Given that $|f''(x)| \leq 3900$ on $[-4, 5]$, how many subintervals should be used to ensure that the trapezoidal approximation of I is accurate to within 10^{-16}?

$$\Delta x = \frac{5-(-4)}{6} = \frac{9}{6} = 1.5$$

SOLUTION:

$$|E_T| \leq \frac{K_1(b-a)^3}{12n^2} = \frac{3900(9)^3}{12n^2}$$

Want: $\dfrac{3900(9)^3}{12n^2} \leq 10^{-16} \Rightarrow n^2 \geq \dfrac{3900(9)^3}{12 \cdot 10^{-16}}$

$$\Rightarrow \boxed{n \geq \sqrt{\frac{3900(9)^3}{12 \cdot 10^{-16}}}}$$

2. (a) (*2 pts ea.*) In the blank provided, write whether each of the following integrals is Type I improper, Type II improper, or neither. *Do not integrate!*

i.) $\displaystyle\int_1^5 \frac{x^2-9}{(x+3)(x^2+9)}\,dx$ Neither

ii.) $\displaystyle\int_{-\infty}^0 e^{-e^x}\,dx$ Type I

iii.) $\displaystyle\int_0^\pi \sec x\,dx$ Type II

(b) (*8 pts*) Determine whether the integral $\displaystyle\int_3^\infty xe^{-x}\,dx$ converges or diverges, and if it converges, find its value.

SOLUTION: Note: $\int xe^{-x}dx$

$u=x \quad v=-e^{-x}$

$u'=1 \quad v'=e^{-x}$

$= -xe^{-x} - \int -e^{-x}dx = -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x}$

So: $\displaystyle\int_3^\infty xe^{-x}dx = \lim_{t\to\infty}\left[\left[-xe^{-x}-e^{-x}\right]_{x=3}^{x=t}\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+3e^{-3}+e^{-3}\right]$

$= \displaystyle\lim_{t\to\infty}\left[-te^{-t}-0+4e^{-3}\right]$. Now, using L'Hopital:

Part (c) is on the next page

$\displaystyle\lim_{t\to\infty} -te^{-t} = \lim_{t\to\infty}\frac{-t}{e^t} = \lim_{t\to\infty}\frac{-1}{e^t}$ $= 0$. So:

$\displaystyle\int_3^\infty xe^{-x}dx = \lim_{t\to\infty}\left[-te^{-t}-0+4e^{-3}\right] = 0-0+4e^{-3}$ $\boxed{= 4e^{-3}}$

(c) *(8 pts)* Use the comparison test to determine whether

$$\int_0^1 \frac{\sec^2(x)}{x^{5/2}}\,dx$$

converges or diverges. *Do not attempt to integrate!*

SOLUTION: For $0 \le x \le 1$

$0 \le \cos^2 x \le 1 \implies \sec^2 x \ge 1.$

$x^{5/2} < x$

$\frac{1}{x^{5/2}} > \frac{1}{x}$

$\sec > 1 \implies \sec^2 x > \sec x >$

$\implies \frac{\sec^2 x}{x^{5/2}} > \frac{1}{x}$

Hence,

$$\frac{\sec^2 x}{x^{5/2}} \ge \frac{1}{x^{5/2}},$$

and $\int_0^1 \frac{1}{x^{5/2}}\,dx$ diverges:

$$\int_0^1 \frac{1}{x^{5/2}}\,dx = \lim_{t \to 0^-} \int_t^1 \frac{1}{x^{5/2}}\,dx = \lim_{t \to 0^-} \int_t^1 x^{-5/2}\,dx$$

$$= \lim_{t \to 0^-} \left[\frac{-2}{3} x^{-3/2} \right]_{x=t}^{x=1} = \lim_{t \to 0^-} \left[\frac{-2}{3x^{3/2}} \right]_{x=t}^{x=1}$$

$$= \lim_{t \to 0^-} \left(\frac{-2}{3} + \frac{2}{3t^{3/2}} \right) = \infty.$$

So by Comparison test,

$$\boxed{\int_0^1 \frac{\sec^2 x}{x^{5/2}}\,dx \text{ DIVERGES!}}$$

3. (a) *(5 pts)* Let $f(x) = e^{-x^2}$. Set up the integral for the length of the arc $y = f(x)$ from the point $(-3, f(-3))$ to the point $(3, f(3))$ *but do not attempt to integrate.*

SOLUTION:

$$f'(x) = e^{-x^2} \cdot -2x$$

$$\ell(\gamma) = \int_{-3}^{3} \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_{-3}^{3} \sqrt{1 + (-2xe^{-x^2})^2}\, dx$$

$$= \int_{-3}^{3} \sqrt{1 + 4x^2 e^{-2x^2}}\, dx.$$

Part (b) is on the next page

(b) (*8 pts*) Use Simpson's rule with $n = 6$ to estimate the arc length integral obtained in (b).

SOLUTION:

Want to approximate $\int_{-3}^{3} \sqrt{1+4x^2e^{-2x^2}}\,dx$

So $\Delta x = \frac{3-(-3)}{6} = \frac{6}{6} = 1$, so

$$S_6 = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(\right.$$

where $x_0 = -3$

$x_1 = -2$

$x_2 = -1$

$x_3 = 0$

$x_4 = 1$

$x_5 = 2$

$x_6 = 3$

$$\Rightarrow S_6 = \frac{1}{3}\left(\sqrt{1+36e^{-18}} + 4\sqrt{1+16e^{-8}} + 2\sqrt{1+4e^{-2}} + 4\sqrt{1} + 2\sqrt{1+4e^{-2}} + 4\sqrt{1+16e^{-8}} + \sqrt{1+36e^{-[illegible]}}\right).$$

4. (a) (*3 pts*) Write the formula for the surface area A of the solid obtained by revolving the graph of the function $x = g(y)$, $c \le y \le d$, around the y-axis:

$$A = \int_c^d 2\pi x \sqrt{1+[g'(y)]^2}\, dy$$

(b) (*5 pts*) In your own words, describe how to derive/obtain/"prove" the formula you got in part (a).

1 or 3 or 5

SOLUTION:

- Revolve $x=g(y)$ around y-axis to get "vase".
- Approximate $x=g(y)$ as piecewise linear w/ finite pieces.
- Notice each finite piece of vase surface is a cylinder w/ surface area $\approx 2\pi \cancel{y} \sqrt{1+[g'(y)]^2}$.
- $\Rightarrow$ total vase is sum of these finite pieces.
- Take limit as #pieces $\to \infty$, so sum $\to \int$.

Part (c) is on the next page

(c) (8 pts) Find the area of the surface of revolution formed by revolving the curve $9y = x^2 + 18$, $2 \le y \le 6$, about the y-axis.

SOLUTION:

$$x^2 = 9y - 18 \implies x = \sqrt{9y-18}$$
$$\implies x' = \frac{9}{2\sqrt{9y-18}}$$

Formula = 3

+ Algebra

integrate =

plug in: 1

$$A = \int_2^6 2\pi \cancel{x} \sqrt{1 + \frac{81}{4(9y-18)}}\, dy$$

$$= \int_2^6 2\pi \sqrt{9y-18}\sqrt{1 + \frac{81}{4(9y-18)}}\, dy$$

$$= \frac{2\pi}{2}\int_2^6 \sqrt{4(9y-18)+81}\, dy$$

$$= \frac{2\pi}{2}\int_2^6 \sqrt{36y+9}\, dy$$

$u = 36y+9 \quad du = 36\,dy$

$\implies \frac{1}{36}\,du = dy$

$$\int \sqrt{36y+9}\,dy = \frac{1}{36}\int \sqrt{u}\,du = \frac{1}{36}\,\frac{2}{3}u^{3/2} = \frac{1}{54}(36y+9)^{3/2}$$

$$= \frac{\cancel{2}\pi}{54}(36y+9)^{3/2}\Big]_{y=2}^{y=6}$$

$$\implies \boxed{A = \frac{\cancel{2}\pi}{54}\left[(36(6)+9)^{3/2} - (36(2)+9)^{3/2}\right]}$$

5. (a) (*4 pts*) Let $f(x)$ be the function defined as follows:

$$f(x) = \begin{cases} \frac{c}{1+x^2} & \text{if } 0 \le x \le \frac{1}{\sqrt{3}} \\ 0 & \text{otherwise} \end{cases}.$$

For what value of c is f a probability density function?

SOLUTION: $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{1/\sqrt{3}} \frac{c}{1+x^2}\,dx = c\arctan x \Big]_{x=0}^{x=\frac{1}{\sqrt{3}}}$

$= c\left(\arctan \frac{1}{\sqrt{3}} - \arctan 0\right)$

$= c\,\frac{\pi}{6}$. So $\boxed{c = \frac{6}{\pi}}$ makes f a PDF

(b) (*4 pts*) For that value of c, find $P(-1 < X < 1)$.

SOLUTION:

$P(-1<X<1) \overset{\text{DEF}}{=} \int_{-1}^{1} \frac{6/\pi}{1+x^2}\,dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{6/\pi}{1+x^2}\,dx = \boxed{1.}$

Part (c) is on the next page

(c) (4 pts) For that value of c, find the mean of $f(x)$.

SOLUTION:

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{1/\sqrt{3}} \frac{6/\pi\ x}{1+x^2}\,dx = \frac{6}{\pi}\int_0^{1/\sqrt{3}} \frac{x}{1+x^2}\,dx \quad (\star)$$

Let $u = 1+x^2$ $\quad du = 2x\,dx \Rightarrow \frac{du}{2} = dx$, so:

$$(\star) = \frac{6}{2\pi} \ln|1+x^2| \Big]_{x=0}^{x=\frac{1}{\sqrt{3}}} = \boxed{\frac{3}{\pi}\left(\ln(4/3) + \ln(1)\right)}$$

(d) (2 pts) Let m denote the median of f for that value of c. Write down the integral formula defining m *but do not integrate.*

SOLUTION:

$$\int_m^{\infty} f(x)\,dx = 0.5.$$

6. (2 pts ea.) Indicate whether each statement true by writing either *true* or *false* in the blanks provided.

(a) $M_n = (L_n + R_n)/2$ for all n. False

(b) For $f(x) = e^{-x}$, the left approximation L_n is an underestimate of $\int_a^b f(x)\,dx$ for all n. False

(c) If m is the median of a probability distribution function $f(x)$, then $\int_m^\infty f(x)\,dx = 0.5$. True

(d) The surface area of a right circular cylinder with no top/bottom, radius r, and height h is $\pi r^2 h$. False

(e) $S_{2n} = 2M_n/3 + T_n/3$ for all n. True

(f) If $f'(x)$ and $g'(x)$ exist, then $\lim\limits_{x\to\infty}(f(x)/g(x)) = \lim\limits_{x\to\infty}(f'(x)/g'(x))$ by L'Hôpital's rule. False

(g) If $f(x) \le g(x)$ for all $x \ge a$ and if $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges as well. False

(h) $\int_0^1 x^{-p}\,dx$ diverges for $p \le 1$. False

(i) $\int_2^\infty e^{-x^2}\,dx$ diverges. False

(i)

$x^2 > x$ for $2 \le x < \infty$

$\Rightarrow -x^2 < -x$

$\Rightarrow e^{-x^2} < e^{-x}$ and $\int_2^\infty e^{-x}\,dx$ converges

(h) $x^{-p} = \frac{1}{x^p} = \left(\frac{1}{x}\right)^p$

let $y = \frac{1}{x}$. Then

$x^{-p} = y^p$, &

$x = 0 \Rightarrow y = \infty$

$x = \infty \Rightarrow$

Bonus:

(a) *(5 pts)* Using arc length, prove that the circumference of the circle $x^2+y^2=r^2$ is $2\pi r$.

SOLUTION: