¿Octonions?

A non-associative geometric algebra

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Scalar Product Spaces

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Scalar Product Spaces

Clifford Algebras

Composistion Algebras

Hurwitz' Theorem

Summary

Let K be a field with $1 \neq -1$
Let V be a vector space over K.
Let $\langle \cdot, \cdot \rangle : V \times V \to K$.

Definition

V is a **scalar product space** if:

- **Symmetry**
  - $\langle x, y \rangle = \langle y, x \rangle$
- **Linearity**
  - $\langle ax, y \rangle = a \langle x, y \rangle$
  - $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

for all $x, y, z \in V$ and $a, b \in K$

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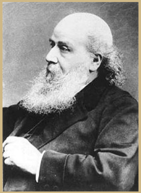
Let $N(x) = -\langle x, x\rangle$ be a **modulus** on V.

$$N(x+y) = -\langle x, x\rangle - 2\langle x, y\rangle - \langle y, y\rangle$$
$$\langle x, y\rangle = (N(x) + N(y) - N(x+y))/2$$
$$N(ax) = -\langle ax, ax\rangle = a^2 N(x)$$

$\langle x, y\rangle$ can be recovered from $N(x)$.
$N(x)$ is homogeneous of degree 2.
Thus $\langle x, y\rangle$ is a quadratic form.

Scalar Product Spaces

(James Joseph) Sylvester's Rigidity Theorem: (1852)



A scalar product space V over $\mathbb{R}$, by an appropriate change of basis, can be made diagonal, with each term in $\{-1, 0, 1\}$. Further, the count of each sign is an invariant of V.

Scalar Product Spaces

The **signature** of V, (p, n, z), is the number of 1's, -1's and 0's in such a basis.

The proof uses a modified Gram-Shmidt process to find an orthogonal basis. Any change of basis preserving orthogonality preserves the signs of the resulting basis.
This basis is then scaled by $1/\sqrt{|N(x)|}$.

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This can be generalized to any field K.
Let $a \sim b$ if $a = k^2 b$ for some $k \in K$.
This forms an equivalence relation.

The signature of V over K is then unique, up to an ordering of equivalence classes.

A standard basis for V is an orthogonal basis ordered by signature.

Scalar Product Spaces

Over $\mathbb{Q}$ the signature is a multi-set of products of finite subsets of prime numbers, plus -1.

Any other $q \in \mathbb{Q}$ can be put in reduced form then multiplied by the square of its denominator to get a number of this form.

Scalar Product Spaces

- For any finite field there are exactly 3 classes.
- Over $\mathbb{F}_5$, for example, $-1 \sim 1$ so we need another element for the third class. The choices are ± 2. Thus the signature uses $\{-1, 2, 0\}$.
- For quadratically closed fields, like $\mathbb{C}$, the signature is entirely 1's and 0's.

Scalar Product Spaces

A scalar product space is **degenerate** if its signature contains 0s.

$x \in V$ is degenerate if $\langle x, y \rangle = 0$ for all $y \in V$.
V is degenerate iff it contains a degenerate vector.

Scaler product spaces will be assumed to be non-degenerate, unless stated otherwise.

Inner Product Spaces

V is an **inner product space** if additionally:

- **Positive definite**
  - $N(x) \geq 0$
  - $N(x) = 0$ iff $x = 0$

In particular this restricts K to ordered fields.

Note: If K is a subset of $\mathbb{C}$ symmetry is typically replaced by conjugate symmetry. This forces $N(x) \in \mathbb{R}$.

Clifford Algebras

William Kingdon Clifford

Let $V, N(x)$ be a scalar product space.
Let $T(V)$ be the tensor space over V.

Let I be the ideal $\langle x^2 + N(x) \rangle$,
for all $x \in V$.

Then the **Clifford algebra** over V is
$Cl(V, N) = T(V)/I$

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This shows Clifford algebras are initial, or the freest, among algebras containing V with $x^2 = -N(x)$. Thus they satisfy a universal property.

Further, if $N(x) = 0$ for all x this reduces to the exterior algebra over V.

Scalar product spaces over $\mathbb{R}$ relate to geometry. Thus if V is over $\mathbb{R}$ or $\mathbb{C}$ these are **geometric algebras**.

Clifford Algebras

Let e_i be a standard basis for V.
Let E be an ordered subset of basis vectors.
Since I removes all squares of V,
$\prod_E e_i$ forms a basis for $Cl(V, N)$
Let the product over the empty set be identified with 1.

Thus $\dim(Cl(V)) = 2^d$, where $d = \dim(V)$.

Clifford Algebras

By construction:
$e_i^2 = \langle e_i, e_i \rangle = -N(e_i)$.

Since this basis is orthogonal:
$\langle e_i, e_j \rangle = 0$ for $i \neq j$, and $e_i e_j = -e_j e_i$.

Since Clifford algebras are associative,
this uniquely defines the product.

Clifford Algebras

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Consider a vector space over $\mathbb{Q}$ with $N(ae_1 + be_2 + ce_3) = a^2 - 2b^2 - 3c^2$.
Thus $e_1^2 = -1$, $e_2^2 = 2$ and $e_3^2 = 3$.

Let $p = e_1e_2 + e_2e_3$ and $q = e_2 + e_1e_2e_3$.

$$\begin{aligned} pq &= e_1e_2e_2 + e_2e_3e_2 + e_1e_2e_1e_2e_3 + e_2e_3e_1e_2e_3 \\ &= 2e_1 - e_2e_2e_3 - e_1e_1e_2e_2e_3 + e_2e_1e_3e_3e_2 \\ &= 2e_1 - 2e_3 + 2e_3 + 3e_2e_1e_2 \\ &= 2e_1 - 3e_1e_2e_2 = 2e_1 - 6e_1 = -4e_1 \end{aligned}$$

Clifford Algebras

Let $Cl_{p,n}$ denote the geometric algebra with signature $(p, n, 0)$. Let Cl_n represent $Cl_{0,n}$.

$Cl_1 = \mathbb{C}$, $Cl_2 = \mathbb{H}$
$Cl_{1,0} = \mathbb{C}^- \cong \mathbb{R} \oplus \mathbb{R}$, $Cl_{2,0} = Cl_{1,1} = \mathbb{H}^- \cong M_2(\mathbb{R})$.
The algebras in second line, and all larger Clifford algebras have zero divisors.

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$Cl_{3,1}$ and $Cl_{1,3}$ are used to create algebras over Minkowski space-time. $Cl_{0,3}$ is sometimes used, with the scalar treated as a time-coordinate.

Physicists like to use differential operators. This motivates active research in Clifford analysis. Clifford analysis gives us well behaved differential forms on the underlying space.

One difficulty in Clifford analysis is the lack of the composition property.

Unital Algebras

Definition

A **unital algebra** A is a vector space with:

- **bilinear product** $A \times A \to A$
  - $(x+y)z = xz + yz$
  - $x(y+z) = xy + xz$
  - $(ax)(by) = (ab)(xy)$
- **Multiplicative identity** $1 \in A$
  - $1x = x1 = x$

For all $x, y, z \in A$ and $a, b \in K$.

In particular, associativity is not required.

$a \in K$ is associated with a1 in A, in particular 1.

Composition Algebras

Definition

A **composition algebra** is a unital algebra A that is a scalar product space with:

- **Multiplicative modulus**
  - $N(xy) = N(x)N(y)$

Clifford algebras only give $N(xy) \leq CN(x)N(y)$.
$\mathbb{C}$, $\mathbb{H}$, $\mathbb{C}^{-}$, and $\mathbb{H}^{-}$ do have this property.
Any associative algebra over a scalar product space will be a Clifford algebra.

Can we get more by relaxing associativity?

Hurwitz' Theorem

Hurwitz Theorem (1923)

The only positive definite composition algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

Adolf Hurwitz

Yes! We get precisely the octonions, $\mathbb{O}$.

This can be generalized as follows:
A non-degenerate composition algebra over any field ($1 \neq -1$) must have dimension 1,2,4 or 8.
Over $\mathbb{R}$ this adds only $\mathbb{C}^-$, $\mathbb{H}^-$ and $\mathbb{O}^-$.

Hurwitz' Theorem

John H. Conway

We will follow the proof of Conway.

The identities on the next slide follow from the definitions.
Conway exhibits 2 line proofs of each from the prior, using non-degeneracy for the last two.

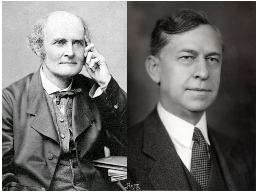
(i.e. $\langle x, t\rangle = \langle y, t\rangle$ for all t iff $x = y$)

Hurwitz' Theorem

Let $\overline{x} = 2\langle x, 1\rangle - x$.

- (Scaling) $\langle xy, xz\rangle = N(x)\langle y, z\rangle$ and $\langle xy, zy\rangle = \langle x, z\rangle N(y)$
- (Exchange) $\langle xy, uz\rangle = 2\langle x, u\rangle\langle y, z\rangle - \langle xz, uy\rangle$
- (Braid) $\langle xy, z\rangle = \langle y, \overline{x}z\rangle = \langle x, z\overline{y}\rangle$
- (Biconjugation) $\overline{\overline{x}} = x$
- (Product Conjugation) $\overline{xy} = \overline{y}\,\overline{x}$

Hurwitz' Theorem



Arthur Cayley and Leonard Eugene Dickson

Now, if A contains a proper unital sub-algebra, H, we construct a Cayley-Dickson double, $H + iH$, within A.

Let H be a proper unital sub-algebra of A.
Let i be a unit vector of A orthogonal to H.
Let a, b, c, d and t be typical elements of H.

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Inner-product doubling:

$$\langle a+ib, c+id\rangle = \langle a,c\rangle + N(i)\langle b,d\rangle$$
$$\langle a, id\rangle = \langle a\bar{d}, i\rangle = \langle ib, c\rangle = \langle i, c\bar{b}\rangle = 0$$
$$\langle ib, id\rangle = N(i)\langle b,d\rangle$$

Conjugation doubling: (so $ib = -\overline{ib} = -\bar{b}\,\bar{i} = \bar{b}i$)

$$\overline{a+ib} = \bar{a} - ib$$
$$\overline{ib} = 2\langle ib, 1\rangle - ib = -ib$$

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Product doubling:

$$(a+ib)(c+id) = (ac - N(i)d\bar{b}) + i(cb + \bar{a}d)$$

$$\langle a \cdot id, t\rangle = \langle id, \bar{a}t\rangle = 0 - \langle it, \bar{a}d\rangle = \langle t, i \cdot \bar{a}d\rangle$$

$$\langle ib \cdot c, t\rangle = \langle ib, t\bar{c}\rangle = \left\langle \bar{b}i, t\bar{c}\right\rangle = 0 - \left\langle \bar{b}\bar{c}, ti\right\rangle$$

$$= \left\langle \bar{b}\bar{c} \cdot i, t\right\rangle = \langle i \cdot cb, t\rangle$$

$$\langle ib \cdot id, t\rangle = -\langle ib, t \cdot id\rangle = 0 + \langle i \cdot id, tb\rangle$$

$$= -\langle id, i \cdot tb\rangle = -N(i)\langle d, tb\rangle$$

$$= N(i)\left\langle -d\bar{b}, t\right\rangle$$

Hurwitz' Theorem

Theorem (Lemma 1)

$K = J + iJ$ is a composition algebra iff J is an associative composition algebra.

Theorem (Lemma 2)

$J = I + iI$ is an associative composition algebra iff I is also, plus commutative.

Theorem (Lemma 3)

$I = H + iH$ is a commutative, associative composition algebra iff H is also, plus trivial conjugation.

Hurwitz' Theorem

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Summary

Now A is unital, so A contains a copy of $\mathbb{R}$.
Thus A contains a double of $\mathbb{R}$, introducing conjugation.
This must be equivalent to $\mathbb{C}$; since $N(i) = 1$.
Now A contains a double of $\mathbb{C}$, breaking commutativity.
This must be equivalent to $\mathbb{H}$.
Now A contains a double of $\mathbb{H}$, breaking associativity.
This must by equivalent to $\mathbb{O}$.

If this is not A, then A would contain a double of $\mathbb{O}$.
But this would not be a composition algebra.
Thus neither is A. $\square$

Indefinite Hurwitz' Theorem

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The only accommodation this proof needs for indefinite cases is to allow $N(i) \neq 1$.
This allows 3 choices of sign for 8 total possibilities.

However, any two basis vectors multiply to a third, such that $(e_1 e_2)^2 = -e_1^2 e_2^2$.
Choosing the positive definite roots first gives us an isomorphism between the seven indefinite cases.
Similarly for the quaternionic case.

Thus over $\mathbb{R}$ we get only $\mathbb{C}^-$, $\mathbb{H}^-$ and $\mathbb{O}^-$. $\square$

Some Other Fields

- Over $\mathbb{C}$ there is a unique octonionic algebra.
- Over $\mathbb{Q}$ there is an 8 dimensional algebra for each choice of three square free integers.
- Over a finite field there are two choices. Like $\mathbb{F}_5$ we may need to select a representative other than 1 for the split class.

Characteristic 2

If $1 = -1$ everything but Lemmas 1 and 2 break.
In particular, doubling does not produce a non-trivial conjugation. There exist composition algebras of dimension 2^n for any n over any such field.

It is possible to construct a non-trivial conjugation. The resulting algebra will then allow just two Dickson doubles.

Eureka!

- Scalar products give a geometric flavor to vector spaces over $\mathbb{R}$ and $\mathbb{C}$.
- Clifford algebras are a natural algebra over scalar product spaces.
- The composition identity fails for $Cl(V, N(V))$ when $\dim(V) > 2$.
- The octonions allow us to extend this to $\dim(V) = 3$, at the cost of associativity.
- The octonions are the unique positive definite non-associative geometric composition algebra.