Split Octonions

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Group Algebras

Let R be a ring (with unity).
Let G be a group.

Definition

An element of **group algebra** $R[G]$ is the formal sum:

$$\sum_{g_n \in G} r_n g_n$$

Addition is component wise (as a free module).
Multiplication follows from the products in R and G, distributivity and commutativity between R and G.

Note: If G is infinite only finitely many r_i are non-zero.

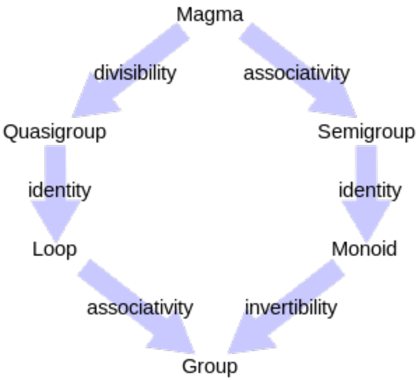
Group Algebras

A group algebra is itself a ring.
In particular, a group under multiplication.

A set M with a binary operation is called a magma.

This process can be generalized to magmas.
The resulting algebra often inherit the properties of M.
Divisibility is a notable exception.

Magmas


Magma
divisibility
associativity
Quasigroup
Semigroup
identity
identity
Loop
Monoid
associativity
invertibility
Group

Loops

In particular, loops are not associative.
It is useful to define some weaker properties.

- **power associative**: the sub-algebra generated by any one element is associative.
- **diassociative**: the sub-algebra generated by any two elements is associative.
- **associative**: the sub-algebra generated by any three elements is associative.

Loops

Power associativity gives us $(xx)x = x(xx)$.
This allows x^n to be well defined.

Diassociative loops have two sided inverses, x^{-1}.
Diassociative gives us $(xy)x^{-1} = x(yx^{-1})$.
This allows conjugation to be well defined.

Loop Algebras

Let L be a loop.

Definition

An element of **loop algebra** $R[L]$ is the formal sum:

$$\sum_{l_n \in L} r_n l_n$$

Addition is component wise (as a free module). Multiplication follows from the products in R and L, distributivity and commutativity between R and L.

Note: If L is infinite only finitely many r_n are non-zero.

Octonion loop

Let O be the loop defined by this table.

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

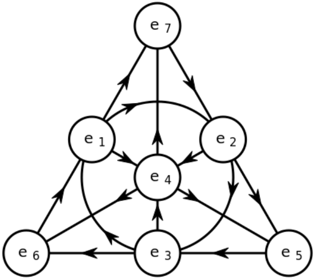
Note: There are really 16 elements, $\pm$ for each shown.
Formally, $-e_n$ is really ϵe_n, where $\epsilon^2 = 1$ and commutes.

Other Representations

Let e_0 be the identity and i_n be indexed mod 7.
Let $i_n^2 = -e_0$ and $i_n i_m = -i_m i_n$ $(m \neq n)$.
Let $i_n i_{n+1} = i_{n+3}$. This is isomorphic to O.

One such mapping is: $i_1 = e_1$, $i_2 = e_2$, $i_3 = e_4$.
This forces $i_4 = e_3$, $i_5 = e_6$, $i_6 = -e_7$, $i_7 = e_5$.
Also $i_{n+1} i_{n+3} = i_n$ and $i_{n+3} i_n = i_{n+1}$.

This representation exhibits a seven fold symmetry in O.



Fano Plane

7 lines and 7 points.
Central circle is a line.

Three points on each line.
Three lines at each point.

Points labeled with roots.
Lines labeled to indicate positive products.

Other Representations

John H. Conway

Gino Fano

Octonion Algebras

$(e_1 e_2)e_4 = e_3 e_4 = e_7 = -e_1 e_6 = -e_1(e_2 e_4)$.
Thus O is not associative.

Also note that any two elements generate the associative quaternion group.
Thus O is diassociative.

The anti-automorphism, $\overline{q}$, toggles the sign of e_1 to e_7.

Octonion Algebras

Let $I = \langle 1 + \epsilon \rangle$ if $1 \neq -1$ or $\langle 1 \rangle$ otherwise.
This formally associates ϵ with -1 in R,
unless $char(R) = 2$.
$\mathbb{O}_R$ is the loop algebra $R[O]/I$. $\mathbb{O}$ is $\mathbb{O}_\mathbb{R}$.

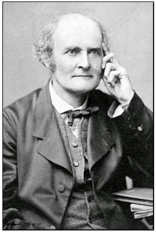
$N(q) = q\bar{q} = \bar{q}q$
These are composition algebras, $N(xy) = N(x)N(y)$.

Over $\mathbb{R}$ only $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are positive definite composition algebras.

Discovery of the Octonions

William R. Hamilton

John T. Graves



Aurther Cayley

Theorem (Moufang's theorem)

Ruth Moufang

Let an Loop L satisfy

- $xy \cdot zx = x \cdot yz \cdot x$
- $x \cdot y \cdot xz = xyx \cdot z$
- $xy \cdot z \cdot y = x \cdot yzy$

Let x, y and z be elements of L. If $x(yz) = (xy)z$, then x, y and z generate an associative sub-loop (i.e a group).

Loops with these properties are Moufang loops.

Moufang Loops

All Moufang loops are diassociative.
Thus the RHS are well defined.

In particular, the units of $\mathbb{O}_R$ are Moufang loops.

If K is a field we can define
$q^{-1} = \overline{q}/N(q)$.
This makes $\mathbb{O}_K$ a division algebra.

Projective spaces

Theorem (Desargues' theorem)

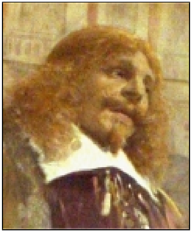
Two triangles are in perspective axially iff they are in perspective centrally.

This was proved for Euclidean spaces.
It applies to all projective spaces, if $dim \neq 2$.

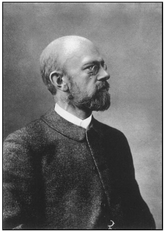
Hilbert found counter examples when $dim = 2$.
Moufang showed that if an algebra is non-associative it's projective space is non-Desargues.
In particular, she was interested in “Cayley Planes”.

Projective spaces

Thus there is a non-Desarguesian octonionic projective plane, but no higher octonionic projective spaces.



Girard Desargues



David Hilbert

Parker Loop

The Golay code is a 12 dimensional linear code over $\mathbb{F}_2$ with a natural representation in $\mathbb{F}_2^{24}$.
They can be viewed as sets of bits.
The Parker Loop is a Moufang loop created as a double of the Golay Code, by adding a sign bit.

Let $HW(x)$ be the cardinatity of code point x.
$x^2 = (-1)^{HW(x)/4}$
$xy = (-1)^{HW(x \cap y)/2} yx$
$x(yz) = (-1)^{HW(x \cap y \cap z)} (xy)z$
Note: The various sign choices here are isomorphic.

Parker Loop

Parker showed that this is indeed a Moufang Loop.

Conway used this loop to simplify Griess' construction of the Fischer-Griess Monster group.
(Griess's constructed it as the automorphism group of a commutative, non-associative, algebra).

The monster group plays a significant role in the classification of finite simple groups.
So Moufang loops play a role in one of the biggest, if not most important, theorems of modern mathematics.

Parker Loop

Marcel J. E. Golay

Richard A. Parker

Robert Griess

Paige Loops

Moufang loop are diassociative.
Thus conjugation is well defined.

This allows us to define a normal sub-loop,
and simple Moufang loops in the usual way.

Paige showed that the units of split-octonion
algebras over a finite field are simple.
Liebeck showed that their are no other cases.

Finite simple Moufang loops are called Paige loops.

Paige Loops

Lowell Paige

Martin Liebeck

Consider $\mathbb{O}_\mathbb{C}$.
Clearly the real parts of this algebra form $\mathbb{O}$.
However, it contains another algebra over $\mathbb{R}$.
Specifically, multiply each e_n, $n \geq 4$ by $\sqrt{-1}$.

The resulting algebra has a split signature.
Thus they are called the split-octonions, $\mathbb{O}^-$.

Rename e_n as i_n for $0 < n < 4$, since $i_n^2 = -1$.
Also, $\sqrt{-1}e_n$ as j_n for $n \geq 4$, since $j_n^2 = 1$.

Split-Octonions

$N(q) = q\overline{q} = r_0^2 + r_1^2 + r_2^2 + r_3^2 - r_4^2 - r_5^2 - r_6^2 - r_7^2$
$\mathbb{O}^-$ not positive definite, as $N(j_4) = -1$.

Infact, $\mathbb{O}^-$ has zero divisors, $N(1 + j_4) = 0$.
The zero divisors all have $N(q) = 0$.

$\mathbb{O}^-$ has the composition property.
The elements with $N(q) \neq 0$ are invertible.
The invertible elements form a loop.

Split Sub-Algebras

q is a split root if $N(q) = -1$ (proper if $N(q) = 1$).

Any split root defines a split-complex algebra $\mathbb{C}^-$.
Any two orthogonal split roots define a split-quaternion algebra, $\mathbb{H}^-$.

Any sub-algebra is either one of these, $\mathbb{C}$ or $\mathbb{H}$.
In particular, the algebra generated by a proper root and an orthogonal split root is also $\mathbb{H}^-$.

Lorentz Transforms

$\mathbb{C}^- \cong \mathbb{R} \times \mathbb{R}$. This algebra describes the Lorentz transformations in one dimension.

$\mathbb{H}^- \cong M_2(\mathbb{R})$. This algebra describes the Lorentz transformations in two dimensions. The third basis, $N(i_3) = 1$, models Thomas precession.

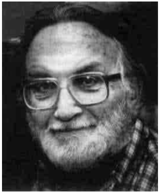
Can we use $\mathbb{O}^-$ to model Lorentz transformations in three dimensions? $Cl(0,3)$ is more standard.

Vector Representation

Let $\vec{J}$ be three split generators, interpreted as an axial vector.
Let $\vec{I}$ represent a polar vector.
Let k represent a pseudo scalar.
Up to a sign convention on k we get the following table:

e	$\vec{J}$	$\vec{I}$	k
$\vec{J}$	$\cdot : e$ $\times : \vec{I}$	$\cdot : -k$ $\times : -\vec{J}$	$-\vec{I}$
$\vec{I}$	$\cdot : k$ $\times : -\vec{J}$	$\cdot : -e$ $\times : \vec{I}$	$-\vec{J}$
k	$\vec{I}$	$\vec{J}$	e

Zorn Matrix Representation



Max Zorn
Known for
his lemma

These representations of $\mathbb{C}^-$ and $\mathbb{H}^-$ use idempotents as a basis.
This halves the cost of multiplication.

Zorn found a similar representation for $\mathbb{O}^-$.
Let $a = e_0 + j_4$, $b = e_0 - j_4$.
Let $I = a\langle i_1, i_2, i_3\rangle$ and $J = -b\langle i_1, i_2, i_3\rangle$.

$$q = \begin{pmatrix} a & I \\ J & b \end{pmatrix}$$

$$N(q) = \det(q) = ab - J \cdot I$$

This product is then $\mathbb{O}^-$.

$$
\begin{aligned}
q'' &= qq' \\
a'' &= aa' + I \cdot J' \\
I'' &= aI' + Ib' + J \times J \\
J'' &= Ja' + bJ' - I \times I \\
b'' &= J \cdot I' + bb'
\end{aligned}
$$

This is just matrix multiplication,
plus two cross products.
Note: There are several sign conventions possible, impacting only the cross product terms.

Fueter Analysis

In 1936 Fueter developed the theory of analysis on the quaternions.

The difference operator is too strict,
only linear functions have a derivative.

Taylor series are too loose.
Functions exist that converge on domains,
but are not what we intuitively feel are smooth.

Generalizing the concept of analytic is the just right.

Fueter Analysis

The quaternions are not commutative,
so we have to define left and right analytic.

This uses The following Dirac operators:
$D = \sum_i e_i \frac{\partial}{\partial r_i} \qquad \overline{D} = \sum_i \overline{e_i} \frac{\partial}{\partial r_i}$

A smooth function from a domain in $\mathbb{R}^8$
to the octonions is left (right) analytic if:
$Df = \sum_i e_i \frac{\partial f}{\partial r_i} = 0 \ (fD = \sum_i \frac{\partial f}{\partial r_i} e_i) = 0$

Note that $D\overline{D} = \overline{D}D$ is the Laplacian operator.
Thus analytic functions are harmonic functions.

Fueter Analysis

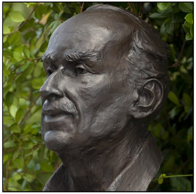
Next we view the quaternions as $\mathbb{R}^4$,
and generate a Clifford algebra over $\mathbb{R}^4$.

A quaterion cooresponds to a 1-form, and its exterior derivative cooresponds to a 3 form – which has precisely a quaternionic value!

Denote these dr and $\star dr$, where $\star$ is the Hodge duel.

Karl Rudolf Fueter

Brook Taylor



Paul Dirac

Fueter Analysis

Generalizing the Cauchy Kernel, $\Phi(r) = \frac{\bar{r}}{N(r)^2}$,
plus a bit of integral gymnastics
gives us a version of the Cauchy integral formula.
$f(r_0) = \frac{1}{2\pi^2} \int_{\partial U} \Phi(r - r_o) \cdot \star dr \cdot f(r)$,
with the usual domain restrictions.

From here it is down hill to generalize many of the theorems of complex analysis, in particular:
The mean value, maximum modulus and uniform convergence theorems.

Octonionic Analysis

In 2002 Li and Peng generalized this process to the octonions. Here associativity comes into play.

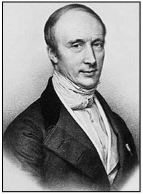
$$\Phi(r) = \frac{\bar{r}}{N(r)^4},$$

$$f(r_0) = \frac{3}{\pi^4} \int_{\partial U} \Phi(r - r_o) \cdot (\star dr \cdot f(r)),$$

In particular,

$$\left(\frac{3}{\pi^4} \int_{\partial U} (\Phi(r - r_o) \cdot \star dr) \cdot f(r)\right) - f(r_0)$$
$$= \int_U \sum_i [\Phi, Df_i, e_i] dV \neq 0.$$

Octonionic Analysis



Augustin-Louis Cauchy

Xingmin Li

Lizhong Peng

In 2011 Libine generalized this to the split-quaternion.
At zero divisors the Cauchy Kernel becomes zero.
The solution is to complexify the algebra and perturbe it.
$\Phi(r) = \frac{\bar{r}}{(N(r)+i\epsilon|r|^2)^2}$
If the boundary of our domain crosses the null cone transversely, the following limit converges.

$$f(r_0) = \lim_{\epsilon\to 0} \frac{-1}{2\pi^2} \int_{\partial U} \Phi(r - r_o) \cdot \star dr \cdot f(r)$$

Any convex domain with the point in the interior will satisfy this. In particular cubes and spheres.

Octonionic Analysis

Matvei Libine

Split-Octonionic Analysis?

Using these two outlines, this very same process should work for the split octonions!
Deriving the appropriate Cauchy integral formula is the first step in this investigation.

Perhaps something like:
$\Phi(r) = \frac{\bar{r}}{(N(r)+i\epsilon|r|^2)^4}$

$$f(r_0) = \lim_{\epsilon \to 0} \frac{-3}{\pi^4} \int_{\partial U} \Phi(r - r_o) \cdot (\star dr \cdot f(r))$$

Malcev Algebras

Malcev studies whether one can generalize Lie algebras in group theory to similar algebras over Moufang loops.

For example, the tangent space of the identity of a smooth Moufang loop forms a Malcev algebra.

S^7, interpreted as the unit octonions in $\mathbb{R}^8$
is a quintisential example of a smooth Moufang loop.
The resulting Malcev algebra is equivelent to
the pure octonions with $[x, y] = xy - yx$.

Malcev Algebras

Using the split octonions with $N(x) = 1$ we get a smooth Moufang loop on a hyperboloid.
The resulting Malcev algebra is equivelent to the pure split-octonions with the same product.

If a Moufang loop is a connected, simply connected and real-analytic, it can be recovered from its Malcev algebra.

Some formula need to be modified.

Malcev Algebras

$J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \neq 0$ but $[J(x, y, z), x] = J(x, y, [x, z])$.

Let L_x and R_x be the action of left and right multiplication by x.
$[Lx, Ly] - L_{[x,y]} = -2[L_x, R_y] = [R_x, R_y] + R_{[x,y]} \neq 0$

In 2006 Eugen Paal demonstrated that one can define Noether currents and charges whithin this framework.

Malcev Algebras

Sophus Lie

Anatoly Ivanovich Malcev

Eugen Paal

Summary

Octonions are neat!
Split octonions might be neater!
They are related to many exceptional objects.

Much of abstract algebra uses associativity
to get to a well defined conjugation.
This machinery applies to Moufang loops.
These tools then apply to create exceptional objects.

They break when pushed too far,
when associativity is strictly needed.