TREE BASIS IN BANACH SPACES

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ABSTRACT. Tree basis in a Banach space is a Schauder basis spaces with additional nice "tree projections". It is a property strictly between conditional and unconditional basis, Tree basis are classified. Stronger basis properties like symmetric, and subsymmetric have weaker tree versions as well. These bases are motivated by well known adaptive approximation algorithms.

1. INTRODUCTION

There is a big different between a Banach space having a (conditional) Schauder basis and it having the more restrictive unconditional basis. The existence of subspaces with Schauder basis was known to Banach, while examples of Banach spaces with no unconditional basic sequence [13] are more recent. A tree basis is defined to be an intermediate property, strictly stronger than Schauder and strictly weaker than unconditional. There are many tree based spaces like JT and wavelet bases, like the Haar system, which are tree spaces by construction and have a natural tree basis. The spaces with a tree basis includes most of the interesting Banach spaces that have a basis. Even James quasi-reflexive space J has a tree basis (Proposition 3.10). Most properties of a Banach space with tree basis can be obtained from the classification as the direct sum of a space with unconditional basis and another space with Schauder basis (Proposition 3.8). So tree basis are stronger than Schauder basis as they imply a complemented subspace with an unconditional basis.

We briefly consider *shrub basis* as a generalization of a tree basis and show the existence of a non-trivial shrub basis implies the existence of a tree basis (Proposition 3.11). Even higher dimensional "trees" thus reduce to the usual one-dimensional binary tree.

Most of the tree based constructions have the stronger property that each rooted subtree is isometric to the original space. We call these spaces tree translation invariant. Such spaces have many nice properties, including being isomorphic to their square. (So J's tree basis is not tree transition invariant.) Such spaces are often primary and many rearrangement invariant

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spaces are tree translation invariant. Another classical example of a tree translation invariance is the classic Schauder basis for C.

Although these properties are amusing in their own right, the motivation was originally to abstract the adaptive approximations, like adaptive quadrature methods, [8] pages 220 – 227, commonly used in some numerical algorithms, which subdivide an interval only where the variation is relatively large and do not subdivide the relatively flat spots. For example, consider the next recursive algorithm for approximating a continuous function f on $[0,1]$ by a piecewise linear function. It modifies a list stored in the global variable L. After the procedure adaptive(0,1) is called, then the list $L = \{x_0 = 0, x_1, \dots x_n = 1\}$, contains the intervals of the piecewise linear approximation function, g, that is g is linear on each $[x_{i-1}, x_i]$ and for each i, $g(x_i) = f(x_i)$ hence

$$g(x) = \; f(x_{i-1})\frac{(x-x_i)}{(x_{i-1}-x_i)} + f(x_i)\frac{(x-x_{i-1})}{(x_i-x_{i-1})} \quad x \in [x_{i-1}, x_i]$$

proc *adaptive*(a, b) $\equiv$
  comment: L is a global list whose tail element is currently a
  if f is well approximated on the interval $[a, b]$ by the line through
    $(a, f(a)$ and $(b, f(b))$ **then**
        $L \leftarrow$ the catenation of L and $\{b\}$
      else
        $m \leftarrow (a + b)/2$
        call *adaptive*(a, m)
        call *adaptive*(m, b)
  fi
end

Remark. The midpoint step (ELSE) is the same as deciding we will use a non-zero coefficient for the piecewise linear basis element

$$h(x) = \begin{cases} (x-a)/(m-a) & x \in [a, m] \\ (x-b)/(m-b) & x \in [m, b] \\ 0 & \text{otherwise} \end{cases}$$

Alternately, the non-midpoint step (THEN) is the same as pruning the basis for the subtree rooted at h.

The referee has pointed out other recent work on basis sequences indexed by dyadic trees. These tree basis are different from the tree basis in this paper. For example, any unconditional basis is a tree basis in the sense of this paper, but it is not always a tree basis that could be constructed via Ramsey theory. The Ramsey theory tree basis do not require as many bounded projection as the tree basis in the sense of this paper. Using Ramsey theory for trees and a basic sequence indexed by the dyadic tree, in [4], it is shown that each separable Banach space X with a non-separable dual, the space X^{**} contains an unconditional family of size $|X^{**}|$. Another application of Ramsey theory for trees is in [12], where it is shown that if $T^*(Y^*)$ is non-separable and $T : X \to Y$ is a bounded linear map between separable Banach spaces, then T must fix a sequence indexed by the dyadic tree with properties like that of James Tree. More applications of Ramsey theory are in [19]. None of the results of this paper use Ramsey theory.

2. Preliminaries

Our notation about bases in Banach spaces follows [17] and [18] or [14]. The sequence (e_n) is a Schauder (respectively unconditional) basis for its closed linear span $X = [e_n]$ if there is a constant M to that $\|\sum_{n\in F}\alpha_n e_n\| \le M\|\sum \alpha_n e_n\|$ for all $x = \sum \alpha_n e_n \in X$ and all initial finite subsets $F = \{1, 2, 3 \ldots k\} \subset \mathbb{N}$ (respectively all finite subsets $F \subset \mathbb{N}$).

A Banach space X is isomorphic to its hyperplanes (respectively its square) if $X \approx X \oplus \mathbb{K}$ where $\mathbb{K}$ is the scalar field (respectively if $X \approx X \oplus X$). A space X is said to be primary if $X \approx Y \oplus Z$ implies that either $X \approx Y$ or $X \approx Z$.

There are many common ways of describing binary trees in analysis. We use the (binary tree) predecessor function $\phi : \mathbb{N}\backslash\{1\} \to \mathbb{N}$ given by $\phi(n) = \lfloor n/2 \rfloor$, Where $\lfloor\cdot\rfloor$ is the floor or greatest integer function. If $\phi(n) = m$, then we say m is the *parent* of n and n is a *child* of m. Each integer m has two children $2m$ and $2m + 1$. A finite or infinite sequence of integers $\{n_i\}$ is an initial branch if $n_1 = 1$ and $\phi(n_{i+1}) = n_i$. Another common notation is for a binary tree uses the Cantor set $\Gamma = 2^\omega$.

The subtree rooted at m, S_m is the collection of integers that are descendants of m under ϕ. This has the same structure as the complete tree under a similarly function $\Phi = \Phi_m$ defined inductively by $\Phi(1) = m$ and $\Phi(2n) = 2\Phi(n)$, $\Phi(2n + 1) = 2\Phi(n) + 1$. In the Cantor set view, this is a dilation followed by a translation. We will call such Φ a tree translation.

The level of integer n, $\ell(n) = \lfloor\log_2 n\rfloor$ is the number of generations to the integer 1, the root of the tree. A branch permutation β is a permutation on $\mathbb{N}$ that preserves 1 and parenthood. A branch permutation clearly preserves levels while permuting the branches.

3. Tree basis

Definition 3.1. *A finite subset $F \subset \mathbb{N}$ is a* tree-subset *if $n \in F\backslash\{1\}$ implies its predecesssor $\phi(n) \in F$. A basis (e_n) for X is a* tree basis *if there is a constant $M < \infty$ for that for all finite tree subsets F and $x = \sum \alpha_n e_n \in X$*

$$\|\sum_{n\in F}\alpha_n e_n\| \le M\|\sum \alpha_n e_n\|.$$

Remark. The space X can be renormed so that the constant M is one. The existence of a tree basis condition is strictly stronger than the existence of a basis (Proposition 3.8) and strictly weaker than an unconditional basis (Proposition 3.10). However, our first chore is to show that (in some sense) a tree basis is the only intermediate notion between Schauder and unconditional basis.

Definition 3.2. *A function* $\psi : \mathbb{N}\backslash\{1\} \to \mathbb{N}$ *is called a (shrub)* predecessor function *if* $\psi(n) < n$ *for all integers* n.

Remark. The condition $\psi(n) < n$ has two effects. Most importantly it requires well-foundedness, that is there are no sequences $(n_i)_{i=1}^{\infty}$ so that $\psi(n_i) = n_{i+1}$. Secondly, it implies there is a single initial integer, 1, that has no predecessor, and it is the root. This second condition is unimportant as multiple roots (even infinitely many roots) are easily handled.

Definition 3.3. *Given a* predecessor function ψ *we define a* fork or branching node *to be an integer* n *with more than one solution to* $\psi(m) = n$. *Again* $\psi(m) = n$ *implies* n *is the* parent with respect to ψ *of* m *and* m *is a* child with respect to ψ *of* n. *Inductively the notions of ancestor and descendent with respect to* ψ *are also defined as they are for trees. A set* $\{n_i\} \subset \mathbb{N}$ *is* independent *or an* anti-chain *if* $i \neq j$ *implies* n_i *and* n_j *are* unrelated, *neither is a descendent of the other. Independent sequences are mutually incomparable.*

Definition 3.4. *Given a predecessor function* ψ, *we define a finite or infinite subset* $F \subset \mathbb{N}$ *to be a* shrub subset *if* $n \in F\backslash\{1\}$ *implies* $\psi(n) \in F$. *A basis* (e_n) *for* X *is a shrub basis of there is a constant* $M < \infty$ *so that for all* finite *shrub subsets* F *and* $x = \sum \alpha_n e_n \in X$.

$$\|\sum_{n\in F} \alpha_n e_n\| \leq M\|\sum \alpha_n e_n\|.$$

Lemma 3.5. *If* ϕ *is a predecessor function,* F *an infinite shrub subset and* (e_n) *a shrub basis, then the projection* $P(\sum \alpha_n e_n) = \sum_{n\in F} \alpha_n e_n$ *is bounded by the shrub basis constant* M.

Proof: This is almost automatic, we need to only show $\sum_{n\in F} \alpha_n e_n$ converges. Let $F_k = F \cap \{1, \ldots k\}$. Since each F_k is a shrub subset of $\mathbb{N}$, the projections $P_k(\sum \alpha_n e_n) = \sum_{n\in F_k} \alpha_n e_n$ are uniformly bounded in norm by some M. If $\sum \alpha_n e_n$ has converges, then

$$\|\sum_{\substack{n\in F\\ p\leq n\leq q}} \alpha_n e_n\| = P_q(\sum_{n=p}^{q} \alpha_n e_n) \leq M\|\sum_{n=p}^{q} \alpha_n e_n\| \to 0 \text{ as } p, q \to \infty,$$

and hence $P_k(\sum \alpha_n e_n) \to P(\sum \alpha_n e_n)$. ■

Proposition 3.6. *If there is an infinite anti-chain* M *for the predecessor function psi, then any shrub basis* (e_n), *with respect to psi, has an unconditional basic sequence* $(e_n)_{n\in M}$ *which is naturally complemented by*

$$P(\sum \alpha_n e_n) = \sum_{n\in M} \alpha_n e_n.$$

Proof: Let M be the infinite anti-chain, let G be the collection of the ancestors of M and let $F = G \cap M$. Both G and F are shrub sets and so is any H with $G \subset H\ subset F$. Note $P = P_F - P_G$ where P_F and P_G are projections given by Lemma 3.5.

To see $(e_n)_{n \in M}$ is unconditional. let K be any finite subset $K \subset M$. The projection $P_K(\sum_{n\in M} \alpha_n e_n) = \sum_{n\in K} \alpha_n e_n$ is $P_{G\cup K} - P_G$ which is bounded by twice the shrub basis constant. ■

Proposition 3.7. *If the basis (e_n) has unconditional subsequence $(e_n)_{n\in M}$, which is naturally complemented, then there is a permutation π so that $(e_{\pi(n)})$is a tree basis.*

Proof: Let $N = \mathbb{N}\backslash M$ Let $N = (n_i)$ and $M = (m_j)$ be listing of these sets as increasing subsequences of $\mathbb{N}$. Define $\pi(n_i) = 2^{i-1}$ so that $(e_{\pi(n_i)})$ is left most branch of the tree and $(e_{\pi(m_j)})$ is the rest. If F is a finite tree subset then so is $F \cap \pi(n_i) = F_L$ and $F_R = F - F_L$. The projection onto F is the sum of the projection on F_L, which is an initial segment of $\pi(n_i)$, and F_R, one of the unconditional projections. Thus $(e_{\pi(n)})$ is a tree basis. ■

Corollary 3.8. *X has a tree basis if and only if $X \approx U \oplus Y$, where Y has a basis and where U is infinite dimensional and has an unconditional basis.*

James's Theorem on spaces with unconditional basis gives the next result.

Corollary 3.9. *A space with a tree-basis contains a subspace isomorphic to c_o, ℓ_1 or a complemented infinite dimensional reflexive space.*

Proposition 3.10. *The space J, James quasi-reflexive space has a tree basis.*

Proof: One common basis for J is the shrinking basis e_n with norm

$$\|\sum \alpha_n e_n\| = \sup(\sum_{i=1}^{k}(\alpha_{n(i+1)} - \alpha_{n(i)})^2)^{\frac{1}{2}}$$

where the sup is over finite sequences $n(1) < n(2) < \dots n(k) < n(k+1)$. the projection $P(\sum \alpha_n e_n) \rightarrow \sum_{n=1}^{\infty}(\alpha_{2n} + \alpha_{2n+1})(e_{2n} + e_{2n+1})/2$ is a norm one projection with range isometric to J. The projection $Q = I - P$ has range $[(e_{2n} - e_{2n+1})]$, and $(e_{2n} - e_{2n+1})_{n=1}^{\infty}$ is equivalent to the usual basis of Hilbert space. Thus the basis which alternates between these two basic sequences $e_1 + e_2$, $e_1 - e_2$, $e_3 + e_4$, $e_3 - e_4$, ... is a basis which satisfies the hypothesis of Proposition 3.8. ■

Remark. We have reproved the known fact that $J \approx J \oplus \ell_2$. This known fact and Corollary 3.8 is another way to prove J has a tree basis.

Remark. Is is well known that J cannot have an unconditional basis and hence having a tree basis is strictly weaker than having an unconditional basis.

Proposition 3.11. *If ψ is a non-trivial predecessor function and X has a ψ shrub basis e_n, then X has a tree basis.*

Proof: Suppose for ψ there is an integer n whose set of children M is infinite. This is an anti-chain, so $(e_n)_{n\in M}$ is unconditional by Proposition 3.6 and X has a tree basis(b_n) Proposition 3.7. Otherwise ψ has infinitely many forks. By the Infinity Lemma, there is an infinite branch (n_i) which contains infinitely many forks, $(n(s(i)))_i$. For each i, there must be $m(i) \neq n(s(i)+1)$ but $\psi(m(i)) = n(s(i))$. It follows that $M = (m(i))$ is an anti-chain. Thus, as in the first case X has a tree basis. ■

Remark. The "minimal" shrub ψ is given by $\psi(n+1) = n$ which only requires the same projections as those for a Schauder basis. There spaces X with a basis with no non-trivial decomposition into $Y \oplus Z$. These must be spaces without tree basis.

The next most "minimal" example would be the space $X \oplus X$, which has a ψ shrub basis for ψ given by $\psi(n) = \max\{1, n-2\}$. Which has exactly one fork, namely 1. Clearly $X \oplus X$ is not isomorphic to X. There is an infinite family of trivial predecessor functions realizable by the finite sums $X \oplus ... \oplus X$ which do not have tree basis.

4. Tree Translation Invariance

Given a tree basis (e_n) for X we will say X is *tree translation equivalent* (respectively *tree translation invariant*) if each transformation T of the form $T = T_m$

$$T(\sum \alpha_n e_n) = \sum \alpha_n e_{\Phi(n)}$$

where $\Phi = \Phi_m$ is a tree translation, is an isomorphism (respectively an isometry).

Example 4.1. *Any subsymmetric basis is tree translation invariant.*

Example 4.2. *Let $C = C[0,1]$, the continuous functions on the unit interval with the sup norm. The usual Schauder system for $\{f \in C : f(0) = f(1) = 0\}$ is tree translation invariant, but not unconditional.*

Example 4.3. *The Haar system in a rearrangement invariant function space X on $[0,1]$ (actually the co-dimension one subspace of functions f so that $\int f = 0$) is tree translation equivalent.*

Example 4.4. *Tsirelson space T is an example of a space that is tree translation equivalent but not tree translation invariant. Tree translation equivalence follows since the growth rate of the function $\Phi_m(n)$ function is bounded* [6]. *Attempts to renorm the space T to make it tree translation invariant using the usual construction fail as this will generate a norm equivalent to ℓ_1-norm.*

Example 4.5. *In* [7], *a superspace S of a Tsirelson space is constructed that is not isomorphic to its square. By the theorem below, S is not even tree* *translation equivalent. However one side of the equation holds as* $\|\sum \alpha_n e_n\| \leq \|T_m(\sum \alpha_n e_n)\|$.

Tree spaces with bases satisfying similar one sided dominance conditions were also constructed in [5].

Theorem 4.6. *If* (e_n) *is a tree translation equivalent basis for* X*, then*

(1) X *is isomorphic to its hyperplanes*
(2) X *is isomorphic to its square* $X \oplus X$
(3) X *is isomorphic to an unconditional decomposition* (X_n) *with each* X_n *naturally isomorphic to* X

Proof: Let $\mathbb{K}$ be the scalar field.

(1) The isomorphism T_2 maps the complemented subspace $W = [e_2, e_4, e_8, ...]$ so W is isomorphic to $W \oplus \mathbb{K}$. Hence $X \oplus \mathbb{K} \approx W \oplus Z \oplus \mathbb{K} \approx W \oplus Z \approx X$.
(2) Let the isomorphisms T_2 and T_3 have ranges X_2 and X_3 respectively. Clearly $X \approx X_2 \oplus X_3 \oplus \mathbb{K} \approx X \oplus X \oplus \mathbb{K} \approx X \oplus X$ by part (1).
(3) Let $W_1 = X_2 \cup e_1$ and $W_{n+1} = T_3(W_n)$. The (W_n) form a decomposition of X and T_3 provides a translation for this decomposition. Obviously $W_n \approx X \oplus \mathbb{K} \approx X$ by part (1). ■

Remark. Most of the known primary spaces (with exception of J [9]) are tree translation equivalent. However Tsilerson space T is tree translation equivalent but is known not to be primary [11] (page 58). It is not known if all symmetric sequence spaces are primary. The usual conditions to imply primary can be modeled after [10], [2], [1] and [3]. To apply the Pelcynski decomposition method one needs two facts in addition to the theorem above. First we need a condition that says $X \approx Y \oplus Z$ implies either Y or Z has a complemented copy of X. For tree spaces, this could be done with the following complemented subtree condition below. Second we need a way to shift Y into the the unconditional decomposition $(X_n) \approx (Y_n \oplus Z_n)$ while holding the Z_n fixed. The usual proves require additional information on the unconditional decomposition. For example, the fact it is a ℓ_2 sum in the JT case.

Definition 4.7. *A subtree* S *of* T *is a subset of the integers so that the order inherited from* T *is order isomorphic to the order of a binary tree. A tree basis is said to have the complemented subtree condition if for each subtree* S*, the basis* $(e_n)_{n \in S}$ *is equivalent to* (e_n) *and the projection* $P_S(\sum \alpha_n e_n) = \sum_{n \in S} \alpha_n e_n$ *is bounded.*

Example 4.8. *The Tsirelson space* T *fails the complemented subtree condition as we can pick a subtree* $S = (i(n))_n$ *so that the rate of growth is too large for* $(e_n)_{n \in S}$ *to be equivalent to* (e_n) [6].

5. Branch invariant tree spaces

Definition 5.1. *A tree basis is branch invariant if for every branch permutation β the operator*

$$T_\beta(\sum \alpha_n e_n) = \sum \alpha_n e_{\beta(n)}$$

is an isometry.

If $\delta, \gamma \subset \mathbb{N}$ are a branches, then the basis $\{(e_n) : n \in \delta\}$ and $\{(e_n) : n \in \gamma\}$ are isometrically equivalent in a branch invariant space.

Example 5.2. *Rearrangement invariant spaces and symmetric spaces are examples of branch invariant spaces as is* JT. *Since the basis of JT is conditional, branch invariance doesn't imply unconditionality.*

Example 5.3. *The space C is tree translation invariant but not branch invariant. Indeed, if $n(i) = 2^{i-1}$, then $(e_{n(i)})$ is equivalent to usual basis of C, while if $m(i)$ is inductively defined by $m(i) = 1$, $m(2n+1) = 2m(2n)$ and $m(2n+2) = 2(m(2n+1)) + 1$ then $(e_{m(i)})$ is equivalent to the summing basis.*

Proposition 5.4. *In a space with a branch invariant tree basis (e_n), the projection*

$$P(\sum \alpha_i e_i) = \sum_{n=0}^{\infty} (\sum_{l=0}^{2^n-1} \alpha_{2^n+i})(\sum_{l=0}^{2^n-1} e_{x^n+i})/2^n$$

has norm one.

Proof: If α_i is non-zero only when it's level, $\ell(i) \leq n$, then P is the average of 2^n branch permutations generated by the permutations on level n integers. ∎

Remark. If the basis is the standard Haar basis, the range of this projection is the closed linear span of the Rademacher functions.

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