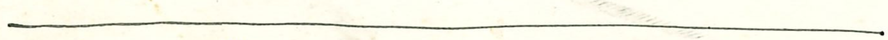
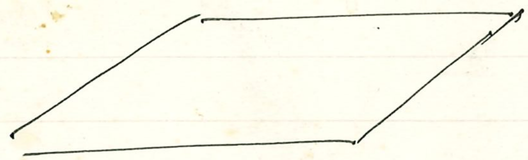
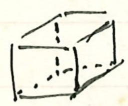
There is room in n-Space 28 MARCH 90

object: To explore higher dimensional geometry at an elementary level. A long the way we will see some surprizes. Obviously $\mathbb{R}^{n+1}$ is bigger than $\mathbb{R}^n$

$$\mathbb{R}^n = \{ (x_1, \ldots x_n) : x_i \in \mathbb{R} \}$$

$\mathbb{R}^1 = \mathbb{R}$ 

$\mathbb{R}^2$ 

$\mathbb{R}^3$  $\mathbb{R}^0 = \{0\}$

for $n \geq 4$ it is a little harder to draw

Example 1. $\mathbb{R}^1 \setminus \{x\}$ is discont. $\mathbb{R}^n \setminus \{x\}$ is ~~not n≥2~~ connected
$\mathbb{R}^n \setminus \{x\}$ $n \geq 2$ is connected. $x \in \mathbb{R}^n$

Hyperplanes $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x_1, x_2, \ldots x_n) = a_1 x_1 + a_2 x_2 + \ldots a_n x_n$$

(linear functional — cont linear)

$V = \{x : f(x) = 0\}$ (null space or kernel of f)

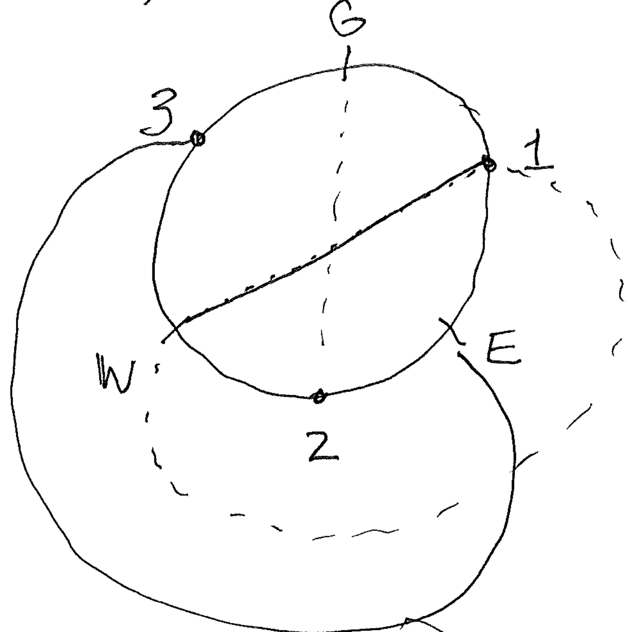
is a vector space with $\dim V = n-1$ (of co-dim 1)

$H = \{x \in \mathbb{R}^n : f(x) = c\}$ is a hyperplane a trans V

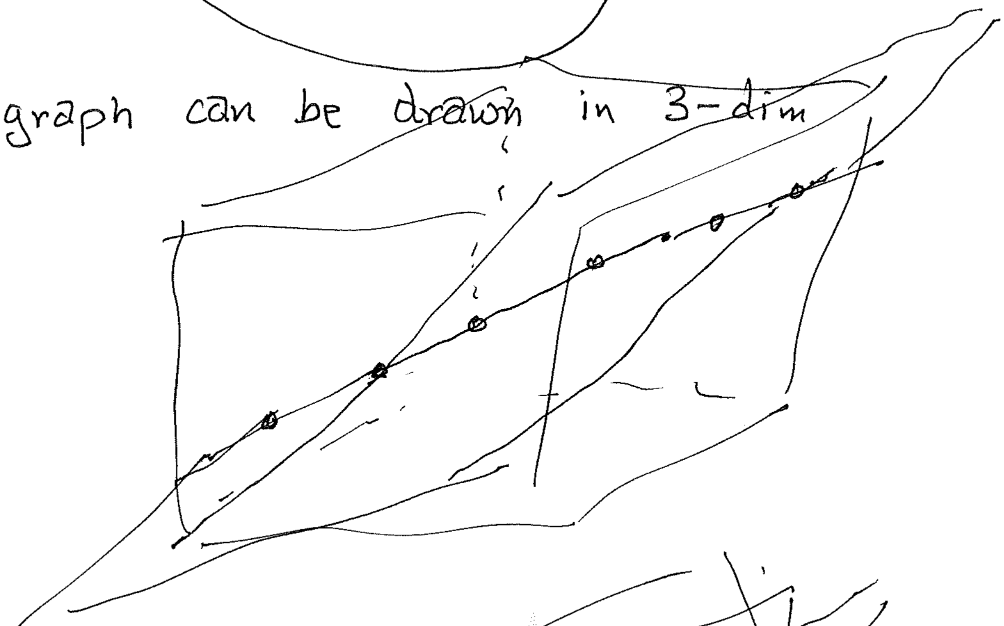
$\mathbb{R}^n \setminus H$ is disconnected — proof IVT

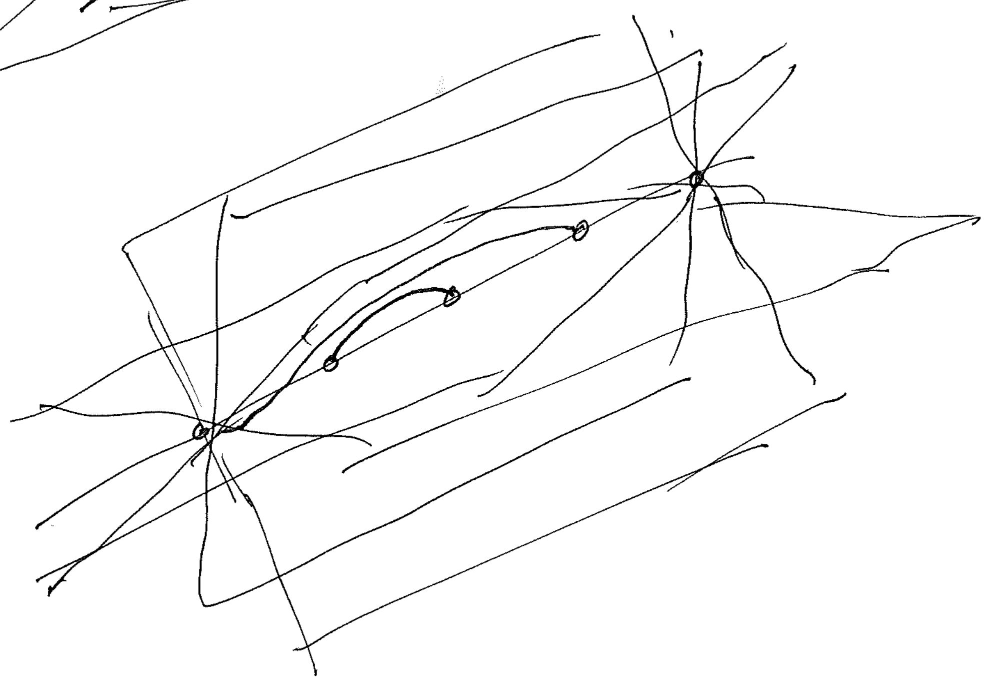
if K has co-dim 2 ~~the~~ $\mathbb{R}^n$ then $\mathbb{R}^n \setminus K$ is connected.

Example 2 $K_{3,3}$ is non-planar (can't be drawn in $\mathbb{R}^2$)



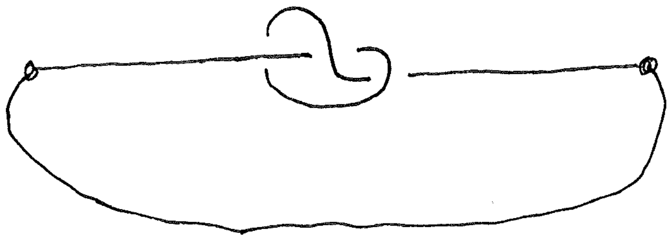
Any graph can be drawn in 3-dim





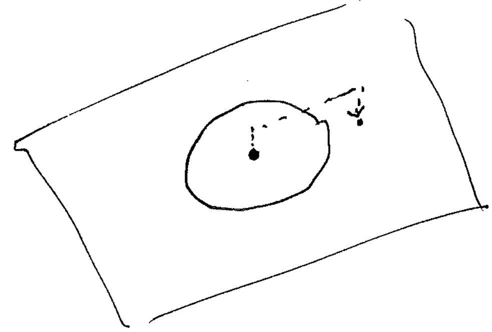
Example 3

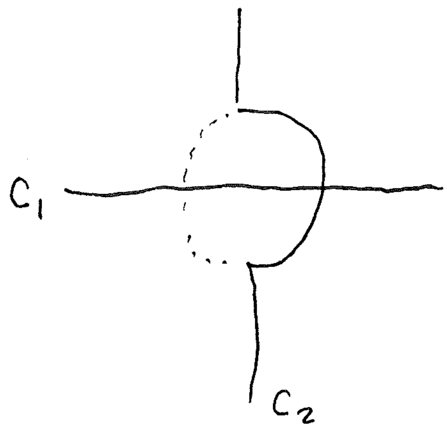
Knot $\subset \mathbb{R}^3$



Unknotted in $\mathbb{R}^4$!

Knots are co-dim 2





C_1 : $(s, 0, 0)$ $s \in (-\infty, \infty)$

C_2 : $(0, t, g(t))$ $t \in (-\infty, \infty)$

C_3 $(0, t, -g(t))$

$$g(t) = \begin{cases} \cos t & -\frac{\pi}{2} \le t \le \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

$C_\lambda = (0, t, \lambda g(t))$ $\lambda = 1$ C_2

$\lambda = -1$ ~~C_2~~ C_3

cross at $\underline{\lambda = 0, t = 0}$ $\underline{s = 0}$

$C^1_\mu = (0, t, g(t), \mu g(t))$ $\mu = 0$ C_2

$(0, t, g(t), g(t))$ $\mu = 1$

C^2_λ $(0, t, \lambda g(t), g(t))$ $\lambda = 1$

$\downarrow$

$(0, t, -g(t), g(t))$ $\lambda = -1$, $\mu = 1$

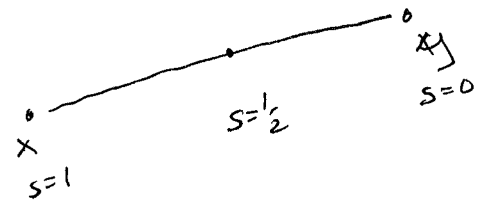
$(0, t, -g(t), \mu g(t))$

$\downarrow$

$\mu = 0$ C_3

Example 4.

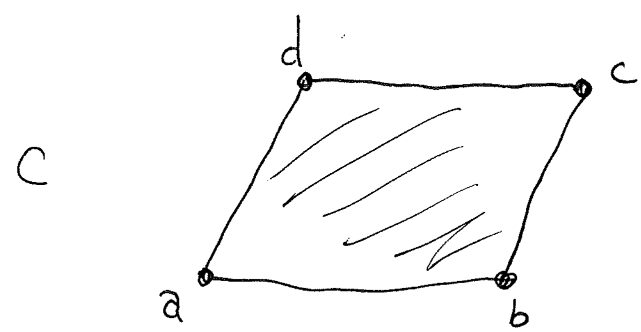
$x, y \in \mathbb{R}^n \qquad [x,y] = \{ sx + (1-s)y : 0 \le s \le 1 \}$



$C \subset \mathbb{R}^n$ is convex if $x, y \in C \Rightarrow [x,y] \subset C$

$D \subset C$ is extremal in if $x, y \in C \quad [x,y] \cap D \neq \emptyset$
$\Rightarrow [x,y] \subset D$

if $\{x\} = D$ is extremal in C then $\{x\}$ is an extreme point of C



a, b, c, d are extreme pts
$[a,b]$, $[b,c]$, $[c,d]$ and $[d,a]$ are extremal but $[a,c]$ is not.

$$C = \text{convex hull } \{a,b,c,d\} = \left\{ as + bt + cu + dv : \begin{array}{l} s,t,u,v \ge 0 \\ s+t+u+v = 1 \end{array} \right\}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear functional

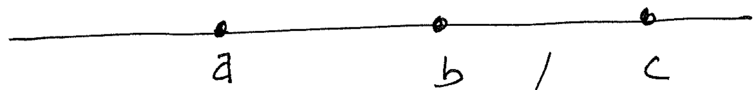
Suppose C convex $\quad m = \min\{f(c) : c \in C\}$ exists
then $\quad D = C \cap \{x : f(x) = m\}$ is extrema in C

if $[x,y] \cap D \neq \emptyset$ then $m = f(sx + (1-s)y)$ some s
$m = sf(x) + (1-s)f(y) \geq sm + (1-s)m = m.$
hence $x, y \in D$

Question

Find $p=p(n)$ the maximum number s.t. $\exists\, x_1, \dots, x_p \in \mathbb{R}^n$
w $C =$ convex hull $\{x_1, \dots, x_p\}$ s.t. $\forall i \neq j$ $[x_i, x_j]$
is extremal in C.

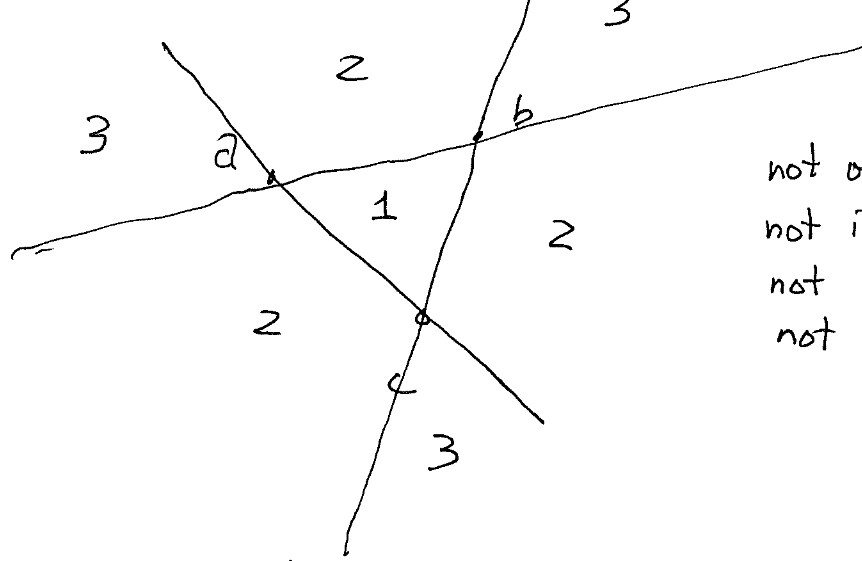
$n=1$ $\mathbb{R}^1$



$p=2$

$\mathbb{R}^2$

$p=3$



not on lines
not in 1
not in 2
not in 3

$\mathbb{R}^n$ $p \geq n+1$ $x_1 = (1, 0 \dots 0)$ $x_2 = (0, 1, 0 \dots)$

simplex $\dots$ $x_n = (0, 0 \dots 1)$ $x_{n+1} = (0, \dots 0)$

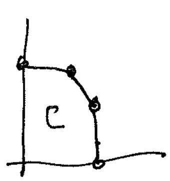
$[x_{n+1}, x_i]$ f $(+1, +1 \dots +1 \underset{i\text{th}}{0} +1 \dots +1)$

$[x_i, x_j]$ $(0 \quad \underset{i}{-1} \quad \underset{j}{-1} \quad 0)$

$\mathbb{R}^3$

$p=4$

Simplex Method



max linear F on C
occurs at extreme points

Thm Let $t_1, \dots t_p \in \mathbb{R}$ distiact

$z_i = (t_i, t_i^2, t_i^3, \dots t_i^n) \in \mathbb{R}^n$

$C =$ convex hull $\{z_1, \dots, z_p\}$

Let $y_1, \dots y_d$ be d points from $\{x_1 \dots x_p\}$ $d \le \frac{n}{2}$

$D =$ convex hull $\{y_1 \dots y_d\}$ is extrema in C

pf: ~~RE~~ Suppose $(s_i, \dots s_i^n) = y_i$

$$P(t) = (t-s_1)^2 (t-s_2)^2 \dots (t-s_d)^2 \quad \deg 2d \le n$$

$$= a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$$

$P(t) = 0$ at exactly $s_1, s_2, \dots s_d$

$P(t) > 0$ for all other t

$$f(x) = a_1 x_1 + \dots + a_n x_n$$

$$f(y_i) = a_1 s_i + a_2 s_i^2 + \dots a_n s_i^n = P(s_i) - a_0 = -a_0$$

$$f(z_j) = a_1 t_j + a_2 t_j^2 + \dots a_n t_j^n = P(t_j) - a_0 > -a_0$$

Thm On higher dimensional planets almost everyone lives in the tropics

Cor: $\forall \varepsilon > 0$ $\forall k$ $\exists n$ Each symmetric body in $\mathbb{R}^n$ has a k-dim subspace that is an ellipsoid within ε $\quad E \subset C \cap V \subset (1+\varepsilon) E$

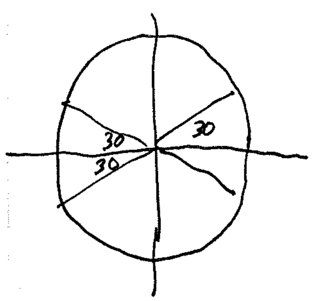
fix θ.

$$S^{n-1} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_n^2 = 1 \}$$

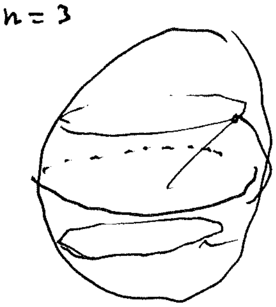
$$\text{Tropics}^{n-1} = \{ x \in S^{n-1} : |x_n| \leq \cos \theta \}$$

$$\lim_{n \to \infty} \frac{\text{area}(\text{Tropics}^{n-1})}{\text{area}(S^{n-1})} = 1.$$

$\theta = \frac{\pi}{6}$ $\quad n = 2$ $\quad n = 3$



$\frac{1}{3}$



$\frac{1}{2}$

$$V_n(R) = \text{volume of } B_n(R) \{ (x_1, \dots x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 \leq R^2 \}$$

$$V_n^\theta(R) = \text{volume of } \{ (x_1 \dots x_n) \in B_n(R) : \frac{|x_n|}{\|x\|^2} \leq \cancel{R} \cos \theta \}$$

$$\frac{d V_n^\theta(R)}{dR} = A_n^\theta(R) \qquad \text{plug in } R = 1$$

$x_n = \sin t \, dt$

$$V_n(R) = \int_{-R}^{R} V_{n-1}(\sqrt{R - x_n}) \, dx_n$$