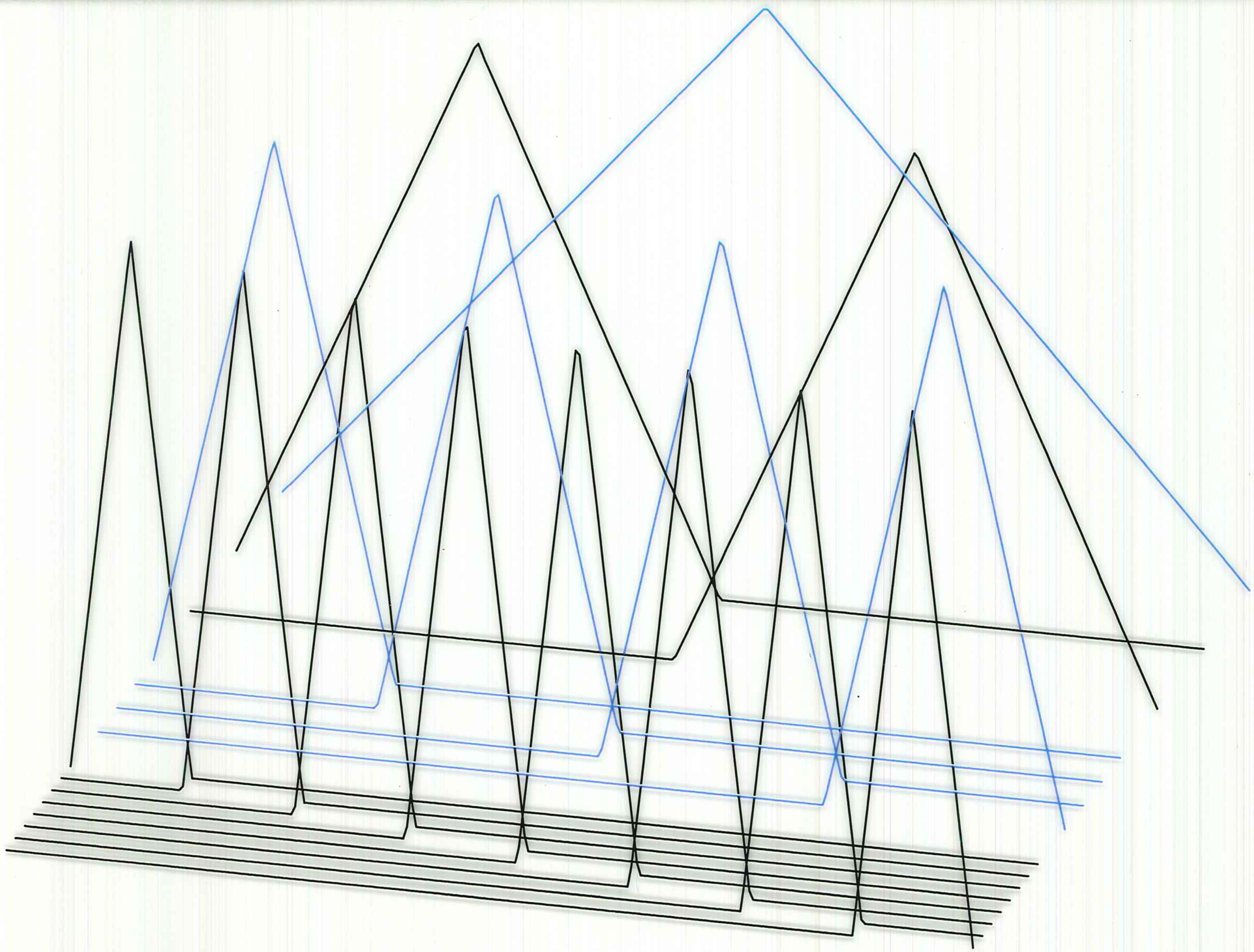
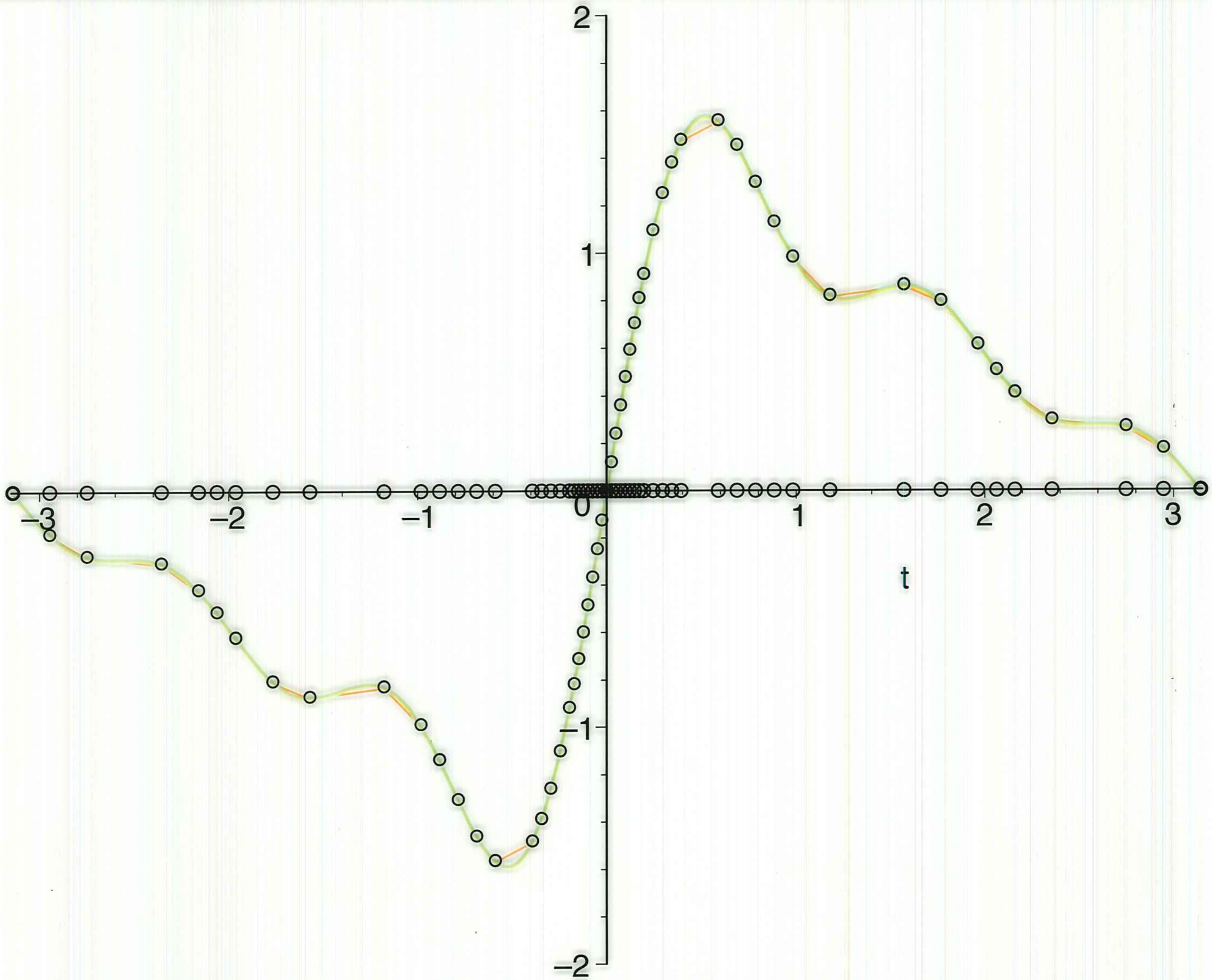
# Schauder's basis for $C[0, 1]$



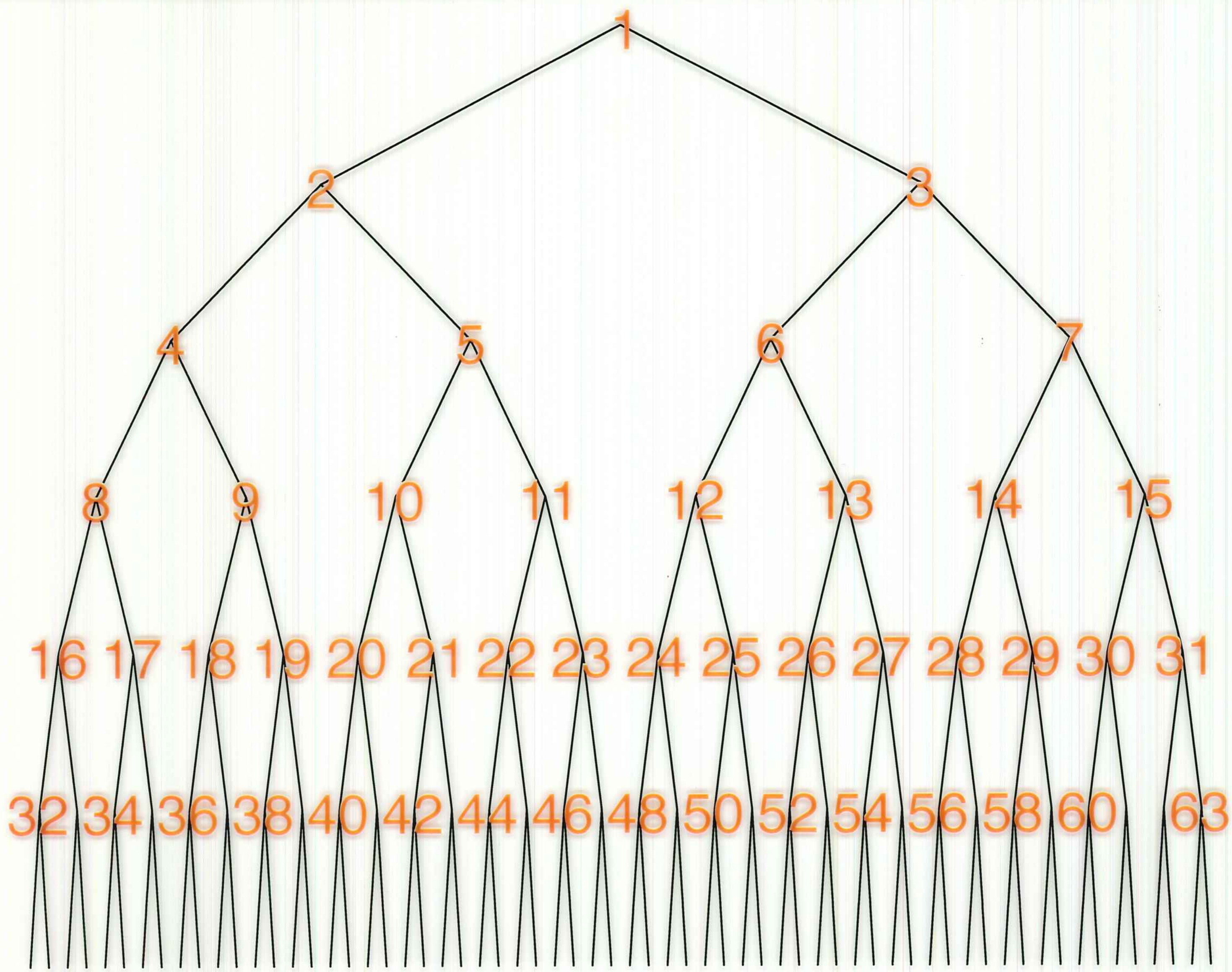
Actually for codim 2 subspace of $\{f : f(0) = f(1) = 0\}$.

# Recursive Adaptation



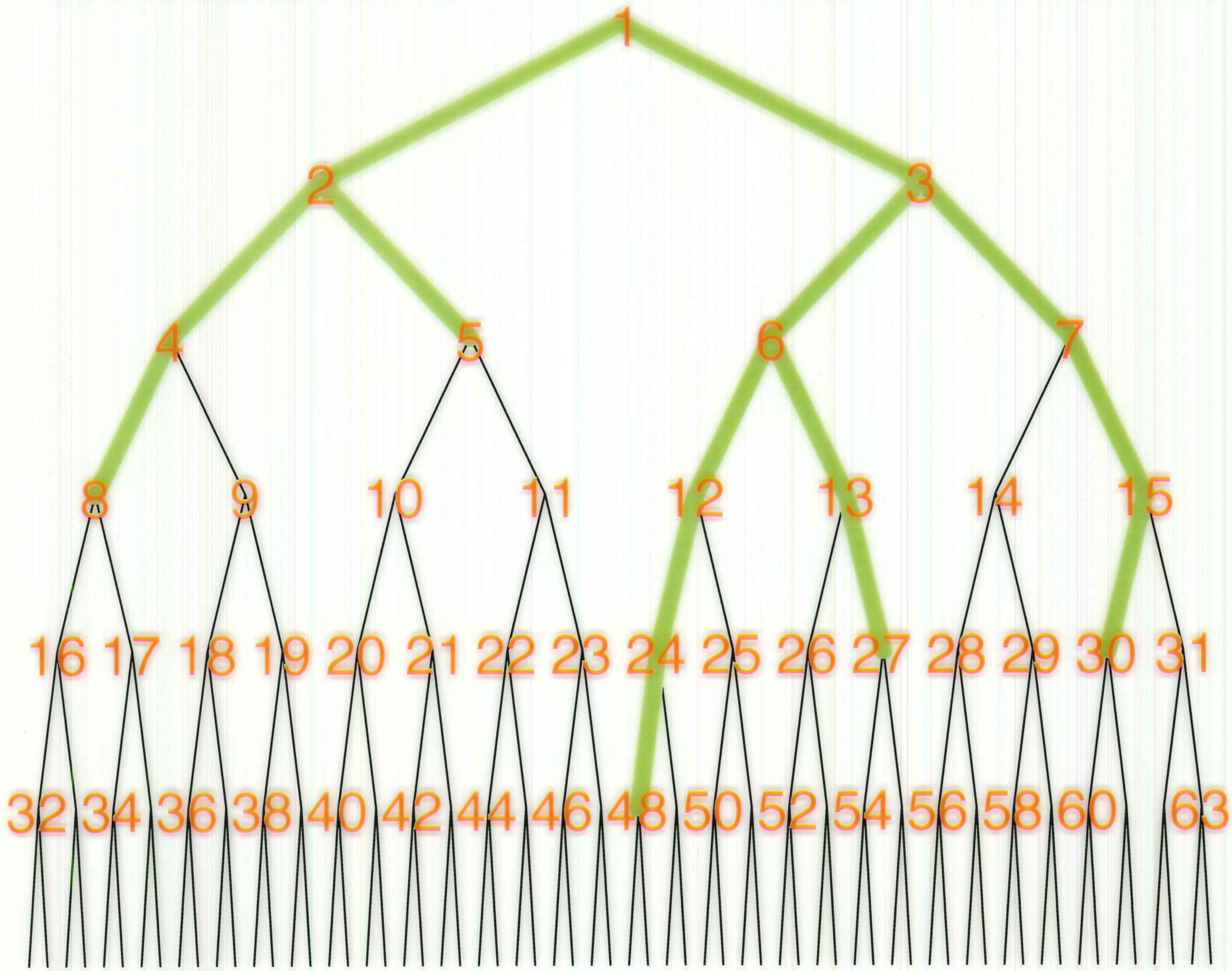

Function in Green, Approximation (connecting circles) in Red

# Tree Definition




Predecessor function $\phi(n) = \lfloor n/2 \rfloor$

# Tree Subset




$$n \in F \implies \phi(n) \in F$$

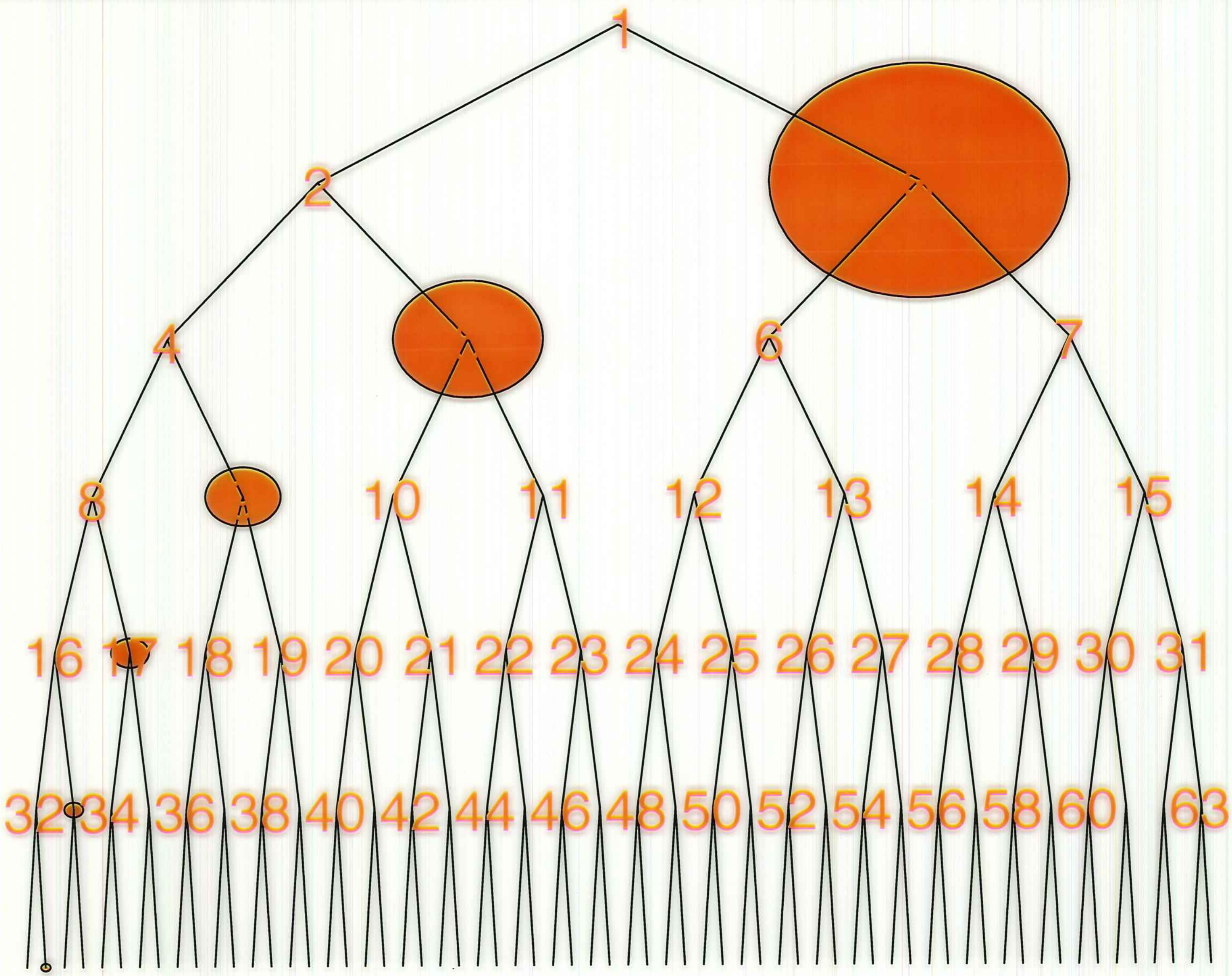
# Conditional, Tree, Unconditional

The sequence $\{e_n\}$ is basic when

$$\|\sum_{n\in F} a_n e_n\| \leq M \|\sum_n a_n e_n\|$$

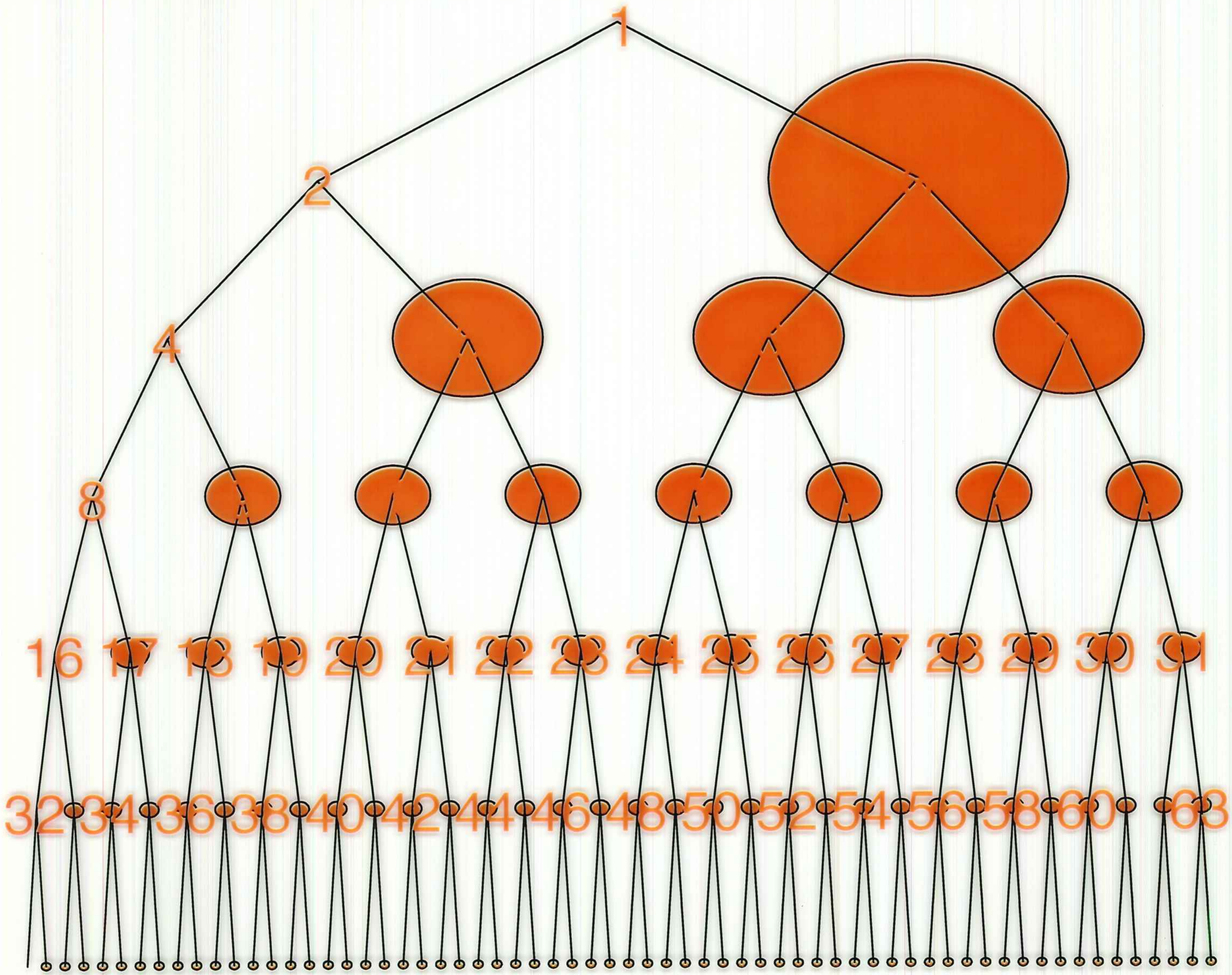
- Conditional for all initial $F = \{1, \ldots N\}$
- Tree basis: for all tree subsets $F$.
- Unconditional for all finite $F$

# Tree = $B \oplus U$ part I




Red dots are an unconditional basis sequence.

# Tree = $B \oplus U$ part II




From $B \oplus U$ to Tree basis.

# Tree Basis Examples/Properties

- Any unconditional basis is a tree basis.
- Tree like spaces $C[0,1]$, $JT$, Haar basis in rearrangement invariant.
- The quasi-reflexive space $J \approx J \oplus \ell_2$.
- Cor: There are spaces with tree basis that don't have an unconditional basis
- Cor: Tree basis space has $c_0$, $\ell_1$ or reflexive subspace.
- Cor: There are spaces with basis that don't have a tree basis.

# Tree Translation – limited subsymmetry

The tree translation $\Phi_M$ is defined recursively. It moves the whole tree to the subtree rooted at $M$.

- $\Phi_M(1) = M$
- $\Phi_M(2n) = 2\Phi_M(n)$
- $\Phi_M(2n+1) = 2\Phi_M(n) + 1$
- $\Phi_M(\sum a_n e_n) = \sum a_n e_{\Phi_M(n)}$

If $\Phi_M$ is always an isometry, then the space is *Tree Translation Invariant,* and if $\Phi_M$ is always an isomorphism, the the space is *Tree Translation Equivalent.*

# Invariance vs Equivalence

- $C[0, 1]$, $JT$ are tree translation invariant
- Tsirelson's space $T$ is tree translation equivalent, not invariant – even when re-normed
- rearrangement invariant spaces are (in general only) tree translation equivalent
- There are Tsirelson superspaces which are not even tree translation equivalent

# Properties of Tree Translation Equivalent

- A Tree Translation Equivalent space $X$ is
- $\approx$ hyperplanes
- $\approx X \oplus X$
- $\approx$ unconditional sum $(X_n)$ with each $X_n \approx X$

# Primary

The space $X$ is primary, if $X \approx Y \oplus Z$ implies $X \approx Y$ or $X \approx Z$.

- Most Primary spaces are Tree Translation Equivalent ($JT$, $C[0,1]$, certain Rearrangement Invariant spaces)
- Tsirelson's $T$ is not primary
- Does Tree Translation Invariance imply primary?
- Subsymmetric bases are Tree Translation Invariant.

# Branch Invariant Tree Spaces

A generalization of symmetric bases.

- Permutation $\pi$ is *Branch Invariant* if
- $\ell(i) = \ell(\pi(i))$ preserves level and
- $\phi(\pi(i)) = \pi(\phi(i))$ preserves branches
- A tree basis $\{e_n\}$ is *Branch Invariant* if for branch invariant permutations $\pi$

$$\|\sum a_n e_n\| = \|\sum a_n e_{\pi(n)}\|$$

# Examples Branch Invariant

- $JT$, rearrangement invariant spaces
- Not $C[0,1]$, some branches $c_0$, some the summing basis.
- $T$ can be renormed to be Branch Invariant

# Rademachers complemented

In a branch invariant tree basis, the projection

$$P(\sum a_i e_i) = \sum_{n=0}^{\infty}(\sum_{i=0}^{2^n-1} a_{2^n+i})(\sum_{i=0}^{2^n-1} e_{2^n+i})/2^n$$

has norm one.
For the Haar basis, this is the projection onto the Rademachers.

# Summary

- The Tree basis is equivalent to $B \oplus U$
- The Tree Translation Equivalence $\approx$ hyperplanes, squares, $\mathrm{UD}(X_n \approx X)$; but not always primary, nor Tree Translaton Invariant.
- Question, is there a tree translation invariant space that is not primary?