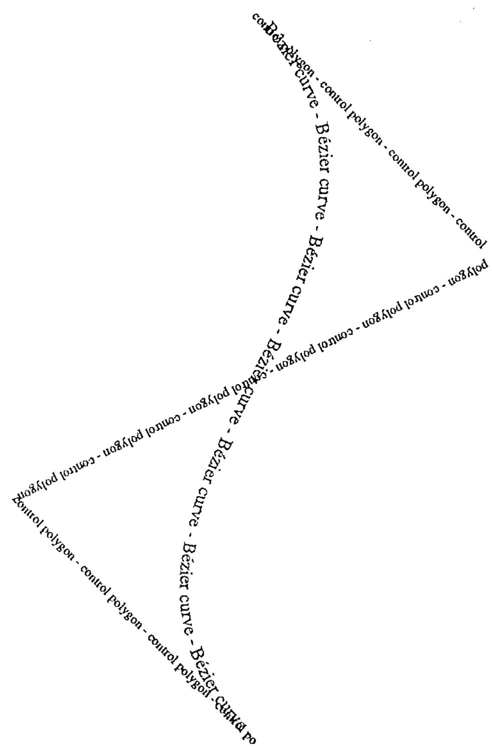
The Analysis Seminar

presents

S. F. Bellenot speaking on

"Bézier Curves"



3:30 pm Wednesday

13 March 1991

102 Love

de Casteljau (1959) construction of Bézier (1966) curves
(Citroën) (Rénault)

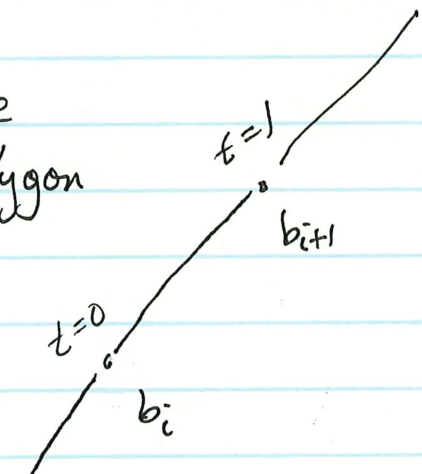
points $b_0, b_1, \ldots b_n$ in $\mathbb{R}^3$ ($\mathbb{R}^2$)
for $t \in \mathbb{R}$

define: $b_i^0(t) \equiv b_i$

$$b_i^r(t) = (1-t)\, b_i^{r-1}(t) + t\, b_{i+1}^{r-1}(t)$$

$$\{ r = 1, \ldots n \qquad i = 0, \ldots, n-r \}$$

$b^n = b^n(t) = b_0^n(t)$ is the Bezier curve
the Polygon $b_0 \ldots b_n$ is the control polygon



$$b_i^1(t) = (1-t)\, b_i + t\, b_{i+1}$$

$$b_i^2(t) = (1-t)^2 b_i + 2t(1-t)\, b_{i+1} + t^2 b_{i+2}$$

C

<u>Cor</u> every poly parametric curve is a Bézier curve

proof: The Bernstein polynomials $B_0^n \dots B_n^n$ are indep polys of degree n.

$$c_0 (1-t)^n + c_1 t^1 (1-t)^{n-1} \dots + c_n t^n \equiv 0$$

$t = 0$ yields $c_0 = 0$
divide by t & $t \to 0$ yields $c_1 = 0$
divide by t^2 & $t \to 0$ yields $c_2 = 0$

- B_j^n are partition of unity.
- quadratic parametric eqns are parabolas

5 Pseudo-local control — max B_j^n near $\frac{j}{n}$ small elsewhere
6 Symmetry $\quad B_j^n(t) = B_{n-j}^n(1-t)$
7 variation diminishing property
the curve crosses any plane no more times than the control polygon.

8 pseudo-local control via <u>Taylor series</u>

$$B_j^n = t^j (1-t)^{n-j} \quad \text{has} \quad \left.\frac{d^r}{dt^r} B_j^n(t)\right|_{t=0} = 0$$

for $r < j$. In particular the first r-derivatives at $t=0$ dependent only on $b_0, \dots b_r$ and not $b_{r+1}, \dots b_n$.

OR $b^n(t) = b_0^n(t)$ and $b_0^r(t)$ are not the same curve but they have the same derivatives up to the r-th at zero.

$\boxed{F}$ Splines "ducks"

- local control.

minimize "strain" $\int K(s)^2 \, ds$

approx $\int \ddot{s}(u)^2 \, du$

[OK if u is arc length

C^2 cubic interpolatory splines

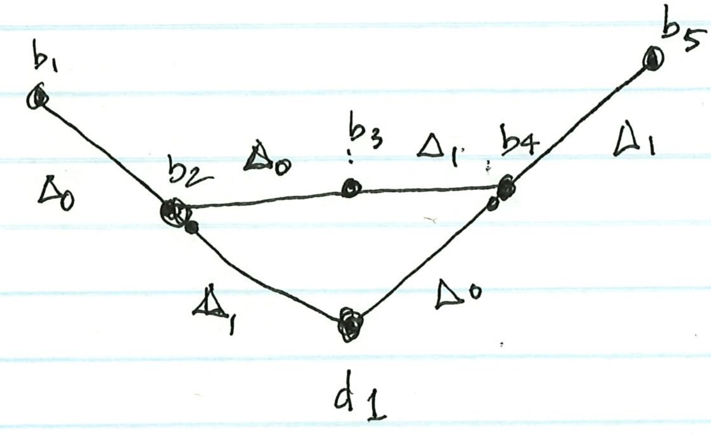
Smoothness

$\underbrace{b_0 \; b_1 \; b_2 \; b_3}_{b_0^3(t)} \cdot \underbrace{b_3 \; b_4 \; b_5 \; b_6}_{b_3^3(t)}$ ← local co-ordinates $t = \dfrac{u - u_i}{u_{i+1} - u_i}$

$u_0 \quad \Delta_0 \quad u_1 \quad \Delta_1 \quad u_2$

C^2-condition at u_1: derivatives^{0,1,2} $b_0^3(t)$ at u_1 agree $b_3^3(t)$ at u_1

local of derivatives – derivatives 0, 1, 2 $b_1^2(t)$ agree with $b_3^2(t)$ but both are degree 2 so they are the same poly! Thus there is d s.t. b_1, d, b_5 $c(t)$ represents $b_1^2(t)$ & $b_3^2(t)$



& hence

$b_1 = c^1(\gamma)$

$b_2 = c^2(\gamma)$

$b_3 = c^3(\gamma)$.

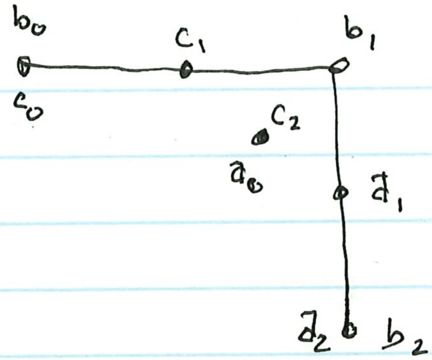
$d_{-1} = b_0 \quad d_0 = b_1 \quad d_i = \cancel{b_{3i}} \qquad d_L = b_{3L-1} \quad d_{L+1} = b_{3L}$

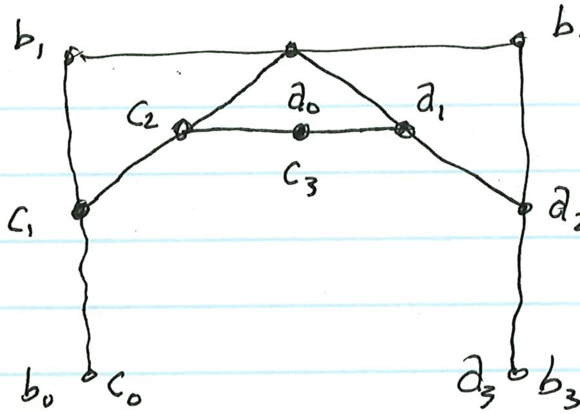
$$b_{3i-1} = \frac{\Delta_i}{\Delta} d_{i-1} + \frac{\Delta_{i-2} + \Delta_{i-1}}{\Delta} d_i$$

$$b_{3i-2} \quad \frac{\Delta_{i-1} + \Delta_i}{\Delta} d_{i-1} + \frac{\Delta_{i-2}}{\Delta} d_i$$

$$b_{3i} = \frac{\Delta_i}{\Delta_{i-1} + \Delta_i} b_{3i-1} + \frac{\Delta_{i-1}}{\Delta_{i-1} + \Delta_i} b_{3i+1}$$

E

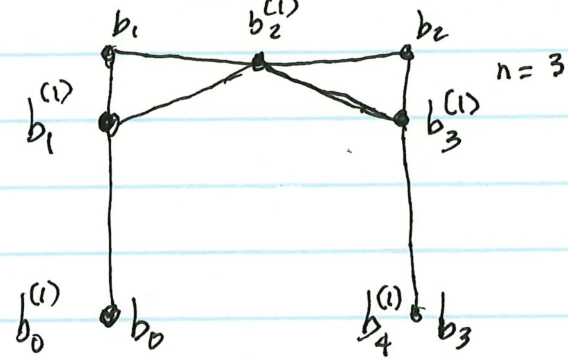




Degree Elevation given $b_0^n(t)$ it is a poly of degree n hence a poly of degree $n+1$. Thus it has a control polygon $b_0^{(1)}\ b_1^{(1)} \ldots\ b_{n+1}^{(1)}$

claim: $$b_j^{(1)} = \frac{j}{n+1} b_{j-1} + \left(1 - \frac{j}{n+1}\right) b_j \qquad 0 \le j \le n+1$$

$b_0^{(1)} = b_0 \qquad b_{n+1}^{(1)} = b_n$



Comment the control polys converge to $b_{n+r}^{(r)}(t) = b_0^n(t)$ $= t^j(1-t)^{n-j}$

In general $$b_i^{(r)} = \sum_{j=0}^{n} b_j \binom{n}{j} \boxed{\frac{\binom{r}{i-j}}{\binom{n+r}{i}}}$$

$\lim_{r\to\infty} \frac{i}{n+r} \to t$

proof $$(t + (1-t)) \sum_{j=0}^{n} b_j \binom{n}{j} t^j (1-t)^{n-j} = \sum_{j=0}^{n+1} b_j^{(1)} \binom{n+1}{j} t^j (1-t)^{n+1-j}$$

eventually $$b_j \binom{n}{j} + b_{j-1}\binom{n}{j-1} = b_j^{(1)} \binom{n+1}{j}$$

$$\frac{n!}{j!\,(n-j)!} \cdot \frac{j!\,(n+1-j)!}{(n+1)!} = \frac{n+1-j}{n+1}$$

$\boxed{D}$

corr $b^n(t)$ is tangent to $b_0 b_1$ at ~~zero~~ $t=0$ & $b_{n-1} b_n$ at $t=1$.

comment curve $f(t)$ then $x_n(t) = \sum_{i=0}^{n} f(\frac{i}{n}) B_i^n(t)$

and $x_n(t) \longrightarrow f(t)$ Stone-Weierstrass

comment $b^n(t)$ = the curve $b_0, \dots b_n$

$b^{n\prime}(t)$ = the curve $n(b_1-b_0), n(b_2-b_1), \dots n(b_n-b_{n-1})$

$$\binom{n}{j} t^j (1-t)^{n-j} \Rightarrow j\binom{n}{j} t^{j-1}(1-t)^{n-j} - (n-j)\binom{n}{j} t^j (1-t)^{n-j-1}$$

$$B_j^{n\prime}(t) = n\left(B_{j-1}^{n-1}(t) - B_j^{n-1}(t)\right)$$

~~• Subdivision: $b^n(t)$ $t \in [0,\gamma]$ is the Bézier curve $c^n(s)$ where $c_i = b_0^i(\gamma)$~~

~~$c^n(s) = b^n(cs)$ so $c_0^n(s)$ & $b_0^n(cs)$ are the same!~~

~~$c^n(t/c) = b^n(t)$~~

Subdivision: given $b_0 \dots b_n$ and $\gamma \in [0,1]$

consider $c(t) = b^n(\gamma t)$ $c(t)$ is an n-degree poly & has a control polygon $c_0 \dots c_n$ s.t. $c(t) = c_0^n(t)$

CLAIM: $c_i = b_0^i(\gamma)$.

pf: $c(t)$ & $b^n(\gamma t)$ are the same curve so all derivatives at $t=0$ agree. Let $0 \le r \le n$ $c_0^r(t)$ (resp $b_0^r(\gamma t)$) has the same derivatives $(0 \le i \le r)$ at zero as $c(t)$ (resp $b_0^n(\gamma t)$) but both are r-deg polys so $c_0^r(t) = b_0^r(\gamma t)$ $c_r = c_0^r(1) = b_0^r(\gamma)$.

A

goal is to describe cubic B-splines

Inductive facts:

(1) $b_i^r(t)$ is in the convex hull of $b_i, b_{i+1}, \dots b_{i+r}$

(2) $A: \mathbb{R}^n \to \mathbb{R}^n$ is affine $\iff$ $Ax - A0$ is linear

If $c_i = A(b_i)$ then $c_i^r(t) = A\, b_i^r(t)$

[comment: draw & affine maps commute]

Affine invariance [not projectively invariant]

(3) Endpoint interpolation $b_i^r(t)$ * $b_i^r(0) = b_i$ $b_i^r(1) = b_{i+r}$

(4) Linear precision if $b_i = (\frac{i}{n}, y_i, z_i)$

then $b_0^n(t) = (t, y(t), z(t))$

$$b_i^r(t) = \left(\frac{i}{n} + \frac{r}{n}t,\ y_i^r(t),\ z_i^r(t)\right)$$

$$(1-t)\left[\frac{i}{n} + \frac{r}{n}t\right] + t\left[\frac{i+1}{n} + \frac{r}{n}t\right]$$

$$= \frac{i}{n} - \frac{i}{n}t + \frac{r}{n}t - \cancel{\frac{r}{n}t^2} + \frac{i}{n}t + \frac{1}{n}t + \cancel{\frac{r}{n}t^2}$$

$$= \frac{i}{n} + \frac{r+1}{n}t.$$

Corollary $b_i^2(t)$ is a parabola:

Bernstein Polynomials (1912)

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

(*) $$B_i^n(t) = (1-t) B_i^{n-1}(t) + t B_{i-1}^{n-1}(t)$$

$$B_0^0(t) \equiv 1 \qquad B_j^n(t) \equiv 0 \quad j \notin \{0, \dots n\}$$

(**) $$b_i^r(t) = \sum_{j=0}^{r} b_{i+j} \; B_j^r(t), \quad b_0^n = b_0^n(t) = \sum_{j=0}^{n} b_j B_j^n(t)$$

t.

Lagrange Polynomials $\qquad t_0, t_1, \dots, t_n$ domain

$$L_i^n(t) = \frac{\prod_{\substack{j=0 \\ j \neq i}}^{n} (t - t_j)}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (t_i - t_j)}$$

interpolation.

~~drawing algorithm~~
~~find the midpoint~~
~~if small enuff draw-line~~
~~else draw left half~~
~~if small enuff draw-line~~
~~else draw right half.~~

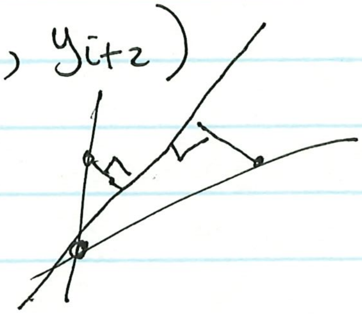
~~Draw (control)~~
~~find midpoint~~
~~find left control~~
~~find right control~~
~~if $|\text{leftpt} - \text{midpt}| < \varepsilon$~~
~~draw line~~
~~else~~
~~draw (left control)~~
if $|\text{midpt} - \text{leftpt}| < \varepsilon$
draw line
else draw (right control.

$\boxed{B}$

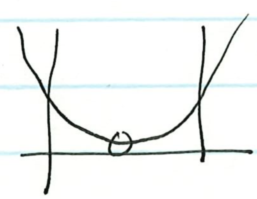
proof: Use an affine map to map $b_i \; b_{i+1} \; b_{i+2}$

to $(0, y_i) \quad (\frac{1}{2}, y_{i+1}) \quad (1, y_{i+2})$

~~$\perp$ bisector~~



$b_n(t) = (t, \text{quadratic})$
parabola.



translate, rotate, ~~scale~~, translate

~~$y = x^2$~~

~~Affine Change of parameter~~

(*) $(1-t)B_i^{n-1}(t) + t\, B_{i-1}^{n-1}(t)$

$= \binom{n-1}{i} t^i (1-t)^{n-1-i+1} + \binom{n-1}{i-1} t^{i-1+1} (1-t)^{n-1-i+1}$

$= t^i (1-t)^{n-i} \left[\binom{n-1}{i} + \binom{n-1}{i-1} \right]$

$= B_i^n(t)$

$B^r_{r+1}(t) = B^r_{-1}(t) = 0$

(**) $(1-t)\, b_i^r(t) + t\, b_{i+1}^r(t)$

$= (1-t) \sum_{j=0}^{r\,(+1)} b_{i+j} B_j^r(t) + t \sum_{j=0}^{r\,+1} b_{i+1+j} B_j^r(t)$

$i+j \quad B^r_{j-1}(t)$

$= \sum_{j=0}^{r+1} b_{i+j} \left[(1-t) B_j^r(t) + t\, B_{j-1}^r(t) \right]$