NONSTANDARD ANALYSIS

DN 1 The superstructure on a set: For any set A we define the superstructure $\hat{A}$ to be the set $\bigcup_{n=0}^{\infty} A_n$, where $A_0 = A$ and A_n is inductively defined by $A_{n+1} = \mathcal{P}(\bigcup_{i=0}^{n} A_i)$ (Here $\mathcal{P}(X)$ is the power set of X — the set of all subsets of X).

DN 2 The Language: This is perhaps the most important definition to understand; here we inductively define the collections of wff (well-formed formulas) and wfs (well-formed sentences). These will be exactly the statements that transfer between the Models. It consists of

(1) Constants: there are exactly the elements of $\hat{A}$.

(2) Variables: usually denoted $x, y, x_1, x_2, \ldots$ etc.

(3) wff: Defined inductively as follows

(a) Atomic wff: are those of the form $x \in y$ or of the form $(x_1, x_2, \ldots x_k) \in y$ here $x, y, x_1, x_2, \ldots x_k$ can either be constants or variables or any mixture, repeatitions are allowed.

(b) Connectives: if X and Y are wff then so are $(X \text{ and } Y)$, $(X \text{ or } Y)$, $(X \Rightarrow Y)$, $(X \Leftrightarrow Y)$ and $(\text{not } X)$.

(c) Quantifiers: if $A(x)$ is a wff and x is a free variable contained in $A(x)$, then $[\forall x \in C\,(A(x))]$ and $[\exists x \in C\,(A(x))]$ are wff for each constant C (i.e each $C \in \hat{A}$) It is important to note that C cannot be a variable.

Note: The variable x in the wff $A(x)$ is said to be a free variable if it does not already appear under the sign (or scope) of a quantifier (i.e. $A(x)$ does not already contain $(\exists x)$ or $(\forall x)$)

(4) wfs are wff without any free variables

[For those who know what it means, this can be considered as a certain subset of first order logic on $\hat{A}$ via $\forall x \in C \ldots \Leftrightarrow \forall x\,(x \in C) \Rightarrow \ldots$]

DN 3: A binary relation $R \in \hat{A}$ with domain D and range (= image) E is said to be concurrent if for each $e_1, e_2, \ldots e_k \in E$, there is a $d \in D$ so that $(d, e_i) \in R$ for $i = 1, 2, \ldots k$.

DN 4 A Monomorphism (warning this definition is more restrictive then the usual one) If A and B are sets with $A \subset B$ then the 1-1 function $\Phi : \hat{A} \to \hat{B}$ with the following properties is called a Monomorphism

(1) $\forall a \in A \quad \Phi(a) = a$ and $\Phi(A) = B$

[Note: we will write $^*C$ for $\Phi(C)$ if $C \in \hat{A}$]

(2) $\forall x \in \hat{A} \quad ^*\{x\} = \{^*x\}$

(3) $\forall X, Y \in \hat{A} \quad ^*(X \setminus Y) = {^*X} \setminus {^*Y}$ and $^*(X \times Y) = {^*X} \times {^*Y}$

(4) If σ is a grouping or permutation on n elements and $R \in \hat{A}$ is any n-ary relation, then $^*(\sigma R) = \sigma(^*R)$.

(5) If $R \in \hat{A}$ is any binary relation, $^*(D(R)) = D(^*R)$ where $D(R) \equiv$ the domain of the relation R.

(6) If $C \in \hat{A}$ and $R = \{(x,y) \mid x \in y \in C\}$, then $^*R = \{(x,y) \mid x \in y \in {^*C}\}$

(7) If $C \in \hat{A}$ and $R = \{(x,x) \mid x \in C\}$, then $^*R = \{(x,x) \mid x \in {^*C}\}$

DN5 The Φ (or $*$) Transform of a wff. If α is a wff then $^*\alpha$ is a wff (on $\hat{B}$ not $\hat{A}$) which is obtained by $*$-ing all the constants of α (and doing nothing to the variables). Roughly speaking $^*\alpha$ is the same as α except the constants name different objects.

Example: if α is $\exists x \in N \ \forall y \in N \ (x,y) \in R$

then $^*\alpha$ is $\exists x \in {^*N} \ \forall y \in {^*N} \ (x,y) \in {^*R}$

META-THM If Φ is a monomorphism, then

(a) A wfs α is true in $\hat{A}$ if and only if $^*\alpha$ is true in $^*\hat{A}$

[$^*\hat{A} = \bigcup_n {^*A_n} \subset \hat{B}$, I believe this is true with $^*\hat{A}$ replaced with $\hat{B}$]

(b) If $\alpha(x_1, \cdots x_m)$ is a wff and $x_1 \ldots x_m$ is the collection of the free variables occurring in α and if $C \in \hat{A}$ let $E = \{(x_1, \cdots x_m) \in C \mid \alpha(x_1, \ldots x_m) \text{ is true}\}$, then

$^*E = \{(x_1, \ldots x_m) \in {^*C} \mid {^*\alpha}(x_1, \ldots x_m) \text{ is true}\}$.

DN6 Standard, Internal and External elements of $\hat{B}$

(A) An element $C \in \hat{B}$ is said to be standard if $\exists E \in \hat{A}$ s.t. $^*E = C$.

(B) An element $C \in \hat{B}$ is said to be internal if $\exists E \in \hat{A}$ s.t. $C \in {^*E}$

(C) Otherwise the element C is said to be external

Remarks If $C \in E \in \hat{B}$ and E is internal then C is internal. And C is internal if and only if $C \in {^*\hat{A}} = \bigcup_n {^*A_n}$.

DNF **Enlargements**: A monomorphism $\Phi: \hat{A} \to \hat{B}$ is said to be an enlargement if for each concurrent relation (see DN3) $R \in \hat{A}$ with domain D and Image E then there is $d \in {}^*D$ so that for each $e \in E$ $(d,e) \in {}^*R$.

THM: For each $\hat{A}$ there is an enlargement.

CONVENTIONS: (A) For each $\hat{A}$ a nonstandard model will be some enlargement

(B) If $a \in A = A_0$, since ${}^*a = \Phi(a) = a$ we will always write a instead of ${}^*a$ in this case.

(C) If $f: C \to D$ is a function $f \in \hat{A}$, It follows that ${}^*f$ is also function this time from ${}^*C \to {}^*D$. Futhermore $f(c) = d$ if and only if ${}^*f({}^*c) = {}^*d$. Thus ${}^*f$ can be considered an extension of f from C to ${}^*C$. For this reason ${}^*f$ is usually written just f

(D) If $R \in \hat{A}$ is an n-ary relation then so is ${}^*R$. Futhermore $(x_1, \dots x_n) \in R$ if and only if $({}^*x_1, \dots {}^*x_n) \in {}^*R$ that is if $R \subset X_1 \times \dots X_n$ then ${}^*R \cap (X_1 \times \dots \times X_n) = R$. It is common to write ${}^*R$ as just R, especially with well known relations examples

$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ (a<b)$ or $(a>b)$ or $(a=b)$ becomes

$\forall a \in {}^*\mathbb{R}\ \forall b \in {}^*\mathbb{R}\ (a<b)$ or $(a>b)$ or $(a=b)$ rather than

$\forall a \in {}^*\mathbb{R}\ \forall b \in {}^*\mathbb{R}\ (a \,{}^*\!< b)$ or $(a \,{}^*\!> b)$ or $(a \,{}^*\!= b)$

Remarks: (1) One must be careful $\Phi(A)$ can be different from the image of A under $\Phi = \Phi[A] = \{{}^*a : a \in A\}$. In fact if Φ is an enlargement $\Phi(X) = \Phi[X]$ if and only if X is a finite set. This follows since $\{(x,y) \mid x \neq y,\ x,y \in X\}$ is concurrent if X is infinite an thus $\exists x \in {}^*X$ with $x \neq y$ for each $y \in X$.

(2) Let $X \in \hat{A}$ and let $\mathcal{F} = \{Y \subset X : Y \text{ is a finite set}\}$ then $R = \{(F,x) \in \mathcal{F} \times X \mid x \in F\}$ is concurrent so there is an $F \in {}^*\mathcal{F}$ s.t. $X \subset F$. That is there are infinite sets which belong to ${}^*\mathcal{F}$, such sets are called <u>star-finite sets</u> or *finite sets.

(3) If A is infinite and Φ is an enlargement, then A considered as a subset of ${}^*A$ is always external otherwise A would be star-finite and each real-valued function on A would have a maximum value which is a contradiction.

(4) Again if A is infinite there are lots of internal elements which are not standard. For instance, any non-zero infinitesimal is internal but not standard.

(5) There other additional properties sometimes added in addition to enlargements (I will not have a use for them) Two of them

(1) K- saturated If $\{A_\alpha : \alpha \in I\}$ is an internal set $\in {}^*\hat{A}$ with card $< K$ so that $A_{\alpha_1} \cap \dots \cap A_{\alpha_n} \neq \emptyset$ for each finite choice of $\alpha_1, \dots \alpha_n \in I$ then $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$

(2) Comprehensive: If ~~$f: C \to D$~~ ~~in~~ $C, D \in \hat{A}$ and $f: C \to {}^*D$ then there is an internal $g: {}^*C \to {}^*D$ so that $f = g|c$.

OUTLINE OF PROOF OF META-THM

The idea is simple and the proof is a straightforward one common to logic. First we prove (a) for atomic sentences then by induction (on the number of brackets or connectives) we prove (a) for wfs without quantifers. Next we prove (b) for atomic wff and by induction, we prove (b) for wff without quantifers. Finally we complete the proof of (a) and (b) (together) by ~~the~~ induction on the number of quantifers.

First we derive some conclusions about monomorphisms. Since $\emptyset = X \setminus X$ by (3) $^*\emptyset = {}^*(X \setminus X) = {}^*X \setminus {}^*X = \emptyset$ so that

(A) $^*\emptyset = \emptyset$

Since $X \subseteq Y$ is equivalent to $X \setminus Y = \emptyset$, by (3) and (A) we have

(B) $X \subset Y$ if and only if $^*X \subset {}^*Y$ for $X, Y \in \hat{A}$

Since $x \in Y$ is equivalent to $\{x\} \subset Y$, by (2) and (B) we have

(C) if $x, Y \in \hat{A}$ then $x \in Y$ if and only if $^*x \in {}^*Y$

[at this point the atomic sentences of the form $p \in q$ (where p, q are constants in $\hat{A}$) is ~~are~~ true in $\hat{A} \Leftrightarrow {}^*p \in {}^*q$ is true in $\hat{B}$)

Since $X \cap Y = X \setminus (X \setminus Y)$ and by (3) we have

(D) if $X, Y \in \hat{A}$, then $^*(X \cap Y) = {}^*X \cap {}^*Y$

Let $Z = X \cup Y$ then $Z = Z \setminus [(Z \setminus X) \cap (Z \setminus Y)]$ so by (D) and (3) we have

(E) if $X, Y \in \hat{A}$, then $^*(X \cup Y) = {}^*X \cup {}^*Y$

Now by induction using (2) and (E) we have

(F) $^*\{x_1, \ldots x_n\} = \{{}^*x_1, \ldots, {}^*x_n\}$

Similarly using (E) and (F) we have

(G) $^*(x_1, \ldots x_n) = ({}^*x_1, \ldots {}^*x_n)$

Combining (G) and (C) we have

(H) $(x_1, \ldots x_n) \in R$ if and only if $({}^*x_1, \ldots {}^*x_n) \in {}^*R$

[at this time the atomic sentence $(p_1, \cdots p_n) \in q$ with $p_1, \cdots q_n, q$ constants is true in $\hat{A}$ if and only if $({}^*p_1, \ldots {}^*p_n) \in {}^*q$ is true in $\hat{B}$.
This is extended to all ~~ate~~ wfs without quantifers by the obvious (α and β) is true $\Leftrightarrow$ α is true, and β is true $\Leftrightarrow$ $^*\alpha$ is true and $^*\beta$ is true $\Leftrightarrow$ ($^*\alpha$ and $^*\beta$) = $^*(\alpha$ and $\beta)$ is true; and etc.]

Since $^*(X \times Y) = {}^*X \times {}^*Y$ by induction we have

(I) $^*(X_1 \times \ldots \times X_n) = {}^*X_1 \times \ldots \times {}^*X_n$

Now R is an n-ary relation $\in \hat{A}$ if and only if $R \subset (A_m)^n$ for some m thus by (I) and (B) we have

(J) $R \in \hat{A}$ is an n-ary relation if and only if $^*R$ is an n-ary relation,

[So far we have only use (2) & (3), which is fortunate since conditions (4) and (5) implicitly require (J) to be true.]

If $R \in \hat{A}$ is a binary relation and letting $\sigma(x,y) = (y,x)$ by (4) we have

(K) $^*(R^{-1}) = ({}^*R)^{-1}$ and thus with (5) we have

(L) $^*(\text{Image} R) = \text{Image}({}^*R)$ if $R \in \hat{A}$ is a binary relation

Now $R \in \hat{A}$ is a binary relation on $X \in \hat{A}$, then $R \cap (X \times \text{Image} R)$ is

α is the statement $\exists_{x \in C} \beta(x)$ [prenex normal & $\forall$ or not $\exists$]
where $C \in \hat{A}$. α is true iff $\{x \in C : \beta(x)\} \neq \emptyset$ but
$\beta(x)$ is wff with $\leq n$ quantifers so by (b)
$^*\{x \in C : \beta(x)\} = \{x \in {}^*C : {}^*\beta(x)\} \neq \emptyset$ iff $\{x \in C : \beta(x)\} \neq \emptyset$
but $^*\alpha$ is true iff $\{x \in {}^*C : {}^*\beta(x)\} \neq \emptyset$

Now let $\alpha(x_1, \dots x_m) = \exists y \in Q\ \beta(x_1, \dots x_m, y)$ and let $C \in \hat{A}$ where β has n-quantifers, Let $R = \{((x_1, \dots x_n), y) \in C \times Q : \beta(x_1, \dots x_m, y)\}$ then by inductive hypothesis $^*R = \{((x_1, \dots x_n), y) \in {}^*C \times {}^*Q : \beta(x_1, \dots x_m, y)\}$
Now $E = D(R) = \{(x_1, \dots x_m) \in C \mid \exists y \in Q\ \beta(x_1, \dots x_m, y)\}$ hence
$^*E = D(^*R) = \{(x_1, \dots x_m) \in {}^*C \mid \alpha(x_1, \dots x_n)\}$ and this completes the proof of the Meta-Thm.

<u>ONE WAY TO PROVE THE EXISTENCE of an enlargement.</u>

This is just an idea of the proof. If I is an index set which is large enough (I think it needs to have cardinality $\geq$ card $\mathcal{P}(\mathcal{P}(\hat{A}))$) And $\mathcal{U}$ is a ultrafilter on I (this means: (1) $\emptyset \notin \mathcal{U}$, (2) $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$, (3) $A \in \mathcal{U}$ & $B \supset A \Rightarrow B \in \mathcal{U}$, (4) $\forall A \subset I$ $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$) with the property that $\cap \mathcal{U} = \emptyset$ (that is $\mathcal{U}$ is free). Then the ultrapower is an enlargement.

The ultrapower is constructed as follows: Let $\mathcal{D} = \{f : I \to \hat{A}\}$ and define an equivalence relation $=_{\mathcal{U}}$ on $\mathcal{D}$ by
$f =_{\mathcal{U}} g$ if and only if $\{i \in I \mid f(i) = g(i)\} \in \mathcal{U}$, and define
$f \in_{\mathcal{U}} g$ if and only if $\{i \in I \mid f(i) \in g(i)\} \in \mathcal{U}$.
The ultrapower is the set of equivalence classes of $\mathcal{D}$ under $=_{\mathcal{U}}$ which we will write $\mathcal{D}/\mathcal{U}$. $\hat{A}$ can be embedded in $\mathcal{D}/\mathcal{U}$ by
$a \in \hat{A}$ $\quad a \longmapsto f$ where $f : I \to \hat{A}$ is the fcn identically equal to a
If $X \in \hat{A}$ $\quad f \in_{\mathcal{U}} {}^*X \Leftrightarrow \{i \in I \mid f(i) \in X\} \in \mathcal{U}$. The conditions of monomorphism are straightforward to check (with a couple of exceptions) for instance
(2) $\quad f \in_{\mathcal{U}} {}^*\{x\} \Leftrightarrow \{i \in I \mid f(i) \in \{x\}\} = \{i \in I \mid f(i) = x\} \in \mathcal{U}$.
ie $\Leftrightarrow f =_{\mathcal{U}} x$.
or
(5) $\quad f \in_{\mathcal{U}} {}^*(D(R)) \Leftrightarrow C = \{i \in I \mid f(i) \in D(R)\} \in \mathcal{U}$ define
$g : C \to \hat{A}$ by $g(i)$ is some element s.t. $(f(i), g(i)) \in R$, extend g arbitrary to I then $(f, g) \in_{\mathcal{U}} {}^*R$ and $f \in_{\mathcal{U}} D(^*R)$
the converse is easier.
OF COURSE (1) and (7) are false, but there are ways around this.

To see that $\mathcal{D}/\mathcal{U}$ is an enlargement, let R be concurrent with domain D & range E. We need $f \in_{\mathcal{U}} D$ with $(f, e) \in_{\mathcal{U}} R$ for each $e \in E$. It suffices to produce a function $T : I \to D$ so that the filter generated by $\{T(U) \mid U \in \mathcal{U}\}$ contains each $\{d \in D \mid (d, e) \in R$ for $e \in F\}$ for each finite set $F \subset E$. (Well its only an idea of proof)

References pp 25–28 & pp 109–122 (Two Articles) in Applications of Model Theory to Algebra, Analysis and Probability Ed. by WAJ Luxemburg Holt, Rinehart and Winston (New York) 1969.