Meas & $\int$ 2.

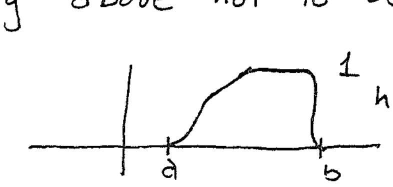
There is yet another way to construct a measure. This construction is like the one done in Rudin in that in produces a measure from a positive linear functional on some space of functions. But it is not limited to the function space $C_c(X)$.

We start with V a normed space of functions $: X \to \mathbb{R}$ with $\|f\| = \sup\{|f(x)| : x \in X\}$. The space V is assumed to have the lattice properties as well as the vector space properties. So if $f, g \in V$ then so are αf, $f+g$, $\max(f,g)$, $\min(f,g)$, $|f|$, f^+ and f^- (these operations are all defined pt-wise). Obviously V could be $C_c(X)$ for some nice top. space X.

The positive linear functional $\Lambda : V \to \mathbb{R}$ is assumed to have an additional property namely

(MCP) If $(f_n) \subset V$, $f_n \le f_{n+1}$, $f_n \to g$ pt-wise
<u>and</u> $g \in V$ then $\Lambda(g) = \lim \Lambda(f_n) = \sup \Lambda(f_n)$.

Note that it is possible for the g above not to be in V for example if $f_n = h^{1/n}$ then $g = \chi_{(a,b)} \notin C_c(X)$.



The first steps in this construction shows how to extend V and Λ to larger collections of functions.
Let $V_\sigma = \{g : g$ can be written as pt-wise limit of f_n with $f_n \le f_{n+1}$ and $(f_n) \subset V\}$
and $V_\delta = \{g :$ same thing but with $f_n \ge f_{n+1}\}$
Note that if $V = C_c(X)$ X loc compact & metric then $(V_\sigma)_\delta$ already contains χ_E for every open E and every compact E. This is enuff to approx any integrable function.

Your job is to fill in some of the details

(A) Show (MCP) implies if $(f_n) \subset V$, $f_n \geq f_{n+1}$, $f_n \to g$ pt-wise <u>and</u> $g \in V$ then $\Lambda(g) = \lim \Lambda(f_n) = \inf \Lambda(f_n)$

(B) If $(f_n) \subset V$, $f_n \leq f_{n+1}$, $f_n \to h$ and $g \in V$ satisfies $g \leq h$ then $\Lambda(g) \leq \lim \Lambda(f_n)$

(C) If $(f_n), (g_n) \subset V$, $f_n \leq f_{n+1}$, $g_n \leq g_{n+1}$, $f_n \to h$ pt-wise $g_n \to k$ pt-wise $h \leq k$ then $\lim \Lambda(f_n) \leq \lim \Lambda(g_n)$

(D) Extend Λ to V_σ by if $f_n \to g$ pt-wise $f_n \leq f_{n+1}$ $(f_n) \subset V$ then $\Lambda(g) = \lim \Lambda(f_n)$

(D) Show Λ is well-defined.

(E) Show Λ is still positive & linear (where not $= \infty$)

(F) Show Λ still satisfies (MCP) with resp to V_σ

(G) Suppose $V = C_c(X)$, Λ is a bdd positive linear fnt'l without using the existence of the measure show that Λ satisfies (MCP). (Hint: pt-wise conv of cont fcns which monotonely converge to zero implies uniform conv)