REALS

DO ALL 5 PROBLEMS

1. Either prove or give a counterexample:

If $f : [0, +\infty) \longrightarrow \mathbb{R}$ is continuous and $\lim_{x \to \infty} f(x) = 0$, then f is uniformly continuous on $[0, +\infty)$.

2. Give counterexamples:

Assume $\{ x_{m,n} : m = 1, 2 \ldots ; n = 1, 2 \ldots \}$ is a double sequence of real numbers so that both iterated limits exist:

$$\text{A} = \lim_{m \to \infty} \lim_{n \to \infty} x_{m,n} \quad \text{and}$$

$$\text{B} = \lim_{n \to \infty} \lim_{m \to \infty} x_{m,n} .$$

A. Show $\text{A} \neq \text{B}$ is possible.

B. Suppose $\text{A} = \text{B} = 0$, show that $\lim x_{mn}$ can still fail to exist. (i.e. It is false that there is an L such that for each $\varepsilon > 0$ there is an N so that $m, n \geq N$ implies $|x_{m,n} - L| < \varepsilon$).

3. Let $\mathcal{U}$ be a collection of subsets of X so that $\emptyset, X \in \mathcal{U}$. Let $f: \mathcal{U} \to [0, +\infty]$ so that $f(\emptyset) = 0$.

A. Show $\mu^*(E) = \inf\{\sum f(U_n) : (U_n) \subset \mathcal{U},\ E \subset \bigcup U_n\}$ defines an outer measure on X.

B. If $\mathcal{U}$ is a σ-algebra and f is a positive measure, then show

$$\mu^*(E) = \min\{f(U) : U \in \mathcal{U},\ E \subset U\}.$$

4. Assume there is $\phi \in C[0,1]$ such that

$$(*) \quad \int_0^1 \int_0^1 f(rs)\, dr\, ds = \int_0^1 f(t)\phi(t)\, dt$$

for all simple functions $f(t)$ on $[0,1]$.

A. Show $(*)$ is true for any $f(t) \in C[0,1]$.

B. Show $\phi(t) = \ln\left(\frac{1}{t}\right)$.

5. Let g be a real-valued function on $\mathbb{R}$ so that g is continuous at 0, $g(x) \neq 0$ for some x and $g(x+y) = g(x)g(y)$.

A. Show g is continuous everywhere.

B. Show $g(x) = a^x$, where $a = g(1)$.