FALL 74

MATH 531

Problems: Due Friday Oct 25

In Text: 2.36, 2.37, 2.46, 3.13, 3.14, 3.28 and:

I. Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets and define:

$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$ and $\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_n$

Prove the following are equivalent:

A) $A = \limsup A_n$

B) $A = \{\xi : \xi \in A_n$ for infinitely many $n \in \mathbb{N}\}$

C) $\chi_A = \overline{\lim_{n\to\infty}} \chi_{A_n}$ (point-wise.)

II. If $\limsup A_n = \liminf A_n$ then call their common value $\lim A_n$.

Prove: A) $m(\liminf A_n) \leq \underline{\lim}\, m(A_n)$

B) If $\bigcup_{n=1}^{\infty} A_n$ has finite measure then $m(\limsup A_n) \leq \overline{\lim}\, m(A_n)$

C) If $\lim A_n$ exists and $m(\bigcup_{n=1}^{\infty} A_n) < \infty$, then $\lim_{n\to\infty} m(A_n)$ exists and $m(\lim A_n) = \lim_{n\to\infty} m(A_n)$.

III. Let $\mathcal{L} = \{(a,b] : b \in \mathbb{R}, a = -\infty$ or $a \in \mathbb{R}$, and $a \leq b\} \cup \{(a,-\infty) ; a \in \mathbb{R}\}$ and $\mathcal{A} = \{A : A$ is a disjoint union of a finite no. of elements of $\mathcal{L}\}$

A) Show that $\mathcal{A}$ is an algebra of sets

A Dynkin system on a set $\mathfrak{X}$, is a collection of subsets of $\mathfrak{X}$, $\mathcal{D}$ such that: 1) $\mathfrak{X} \in \mathcal{D}$

2) $E, F \in \mathcal{D}$ and $E \supset F \Rightarrow E \setminus F \in \mathcal{D}$

3) $(E_n)_{n=1}^{\infty} \subset \mathcal{D}$, $(E_n)_{n=1}^{\infty}$ disjoint $\Rightarrow \bigcup_{n=1}^{\infty} \in \mathcal{D}$.

B) Show that for any $\mathcal{M} \subset \mathcal{P}(\mathfrak{X})$, there is a smallest Dynkin system containg $\mathcal{M}$, call it $\mathcal{D}(\mathcal{M})$

C) Show that $\mathcal{D}(\mathcal{A}) = \mathcal{B}$ (the Borel sets).

IV Let $(\mathfrak{X}, \Sigma)$ and $(\mathfrak{X}', \Sigma')$ be sets with σ-algebras Σ, Σ' resp. Define $f : \mathfrak{X} \to \mathfrak{X}'$ to be Σ-Σ' measurable iff $\forall A \in \Sigma'$ $f^{-1}(A) \in \Sigma'$. Also we define $f^{-1}(\Sigma') = \{f^{-1}(A) : A \in \Sigma'\}$ and $f(\Sigma) = \{A \subset \mathfrak{X}' ; f^{-1}(A) \in \Sigma\}$. Show:

A) For any $f : \mathfrak{X} \to \mathfrak{X}'$, $f^{-1}(\Sigma')$ and $f(\Sigma)$ are σ-algebras

B) f is Σ-Σ' measurable iff $f(\Sigma) \supset \Sigma'$ iff $f^{-1}(\Sigma') \subset \Sigma$.

C) If $\mathcal{M}$ is a collection of subsets of $\mathfrak{X}'$ and $\mathcal{S}(\mathcal{M}) = \Sigma'$ Then $f : \mathfrak{X} \to \mathfrak{X}'$ is Σ-Σ' measurable iff $f^{-1}(\mathcal{M}) \subset \Sigma$.

D) $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}$-$\mathcal{B}$ measurable iff f is measurable in the sense of the text. (Prop 18, p. 65)

Problems: Due Friday Oct 25 MATH 531

In Text: 2.36, 2.37, 2.46, 3.13, 3.14, 3.28 and:

I. Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets and define:

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \quad \text{and} \quad \liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_n$$

Prove the following are equivalent:

A) $A = \limsup A_n$

B) $A = \{\xi : \xi \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$

C) $\chi_A = \overline{\lim_{n\to\infty}} \chi_{A_n}$ (point-wise.)

II. If $\limsup A_n = \liminf A_n$ then call their common value $\lim A_n$.

Prove: A) $m(\liminf A_n) \le \underline{\lim}\, m(A_n)$

B) If $\bigcup_{n=1}^{\infty} A_n$ has finite measure then $m(\limsup A_n) \le \overline{\lim}\, m(A_n)$

C) If $\lim A_n$ exists and $m(\bigcup_{n=1}^{\infty} A_n) < \infty$, then $\lim_{n\to\infty} m(A_n)$ exists and $m(\lim A_n) = \lim_{n\to\infty} m(A_n)$.

III. Let $\mathcal{L} = \{(a,b] : b \in \mathbb{R},\ a = -\infty \text{ or } a \in \mathbb{R}, \text{ and } a \le b\} \cup \{(a,-\infty) : a \in \mathbb{R}\}$ and $\mathcal{A} = \{A : A$ is a disjoint union of a finite no. of elements of $\mathcal{L}\}$

A) Show that $\mathcal{A}$ is an algebra of sets

A Dynkin system on a set $\mathfrak{X}$, is a collection of subsets of $\mathfrak{X}$, $\mathcal{D}$ such that: 1) $\mathfrak{X} \in \mathcal{D}$

2) $E, F \in \mathcal{D}$ and $E \supset F \Rightarrow E \setminus F \in \mathcal{D}$

3) $(E_n)_{n=1}^{\infty} \subset \mathcal{D}$, $(E_n)_{n=1}^{\infty}$ disjoint $\Rightarrow \bigcup_{n=1}^{\infty} \in \mathcal{D}$.

B) Show that for any $\mathcal{M} \subset \mathcal{P}(\mathfrak{X})$, there is a smallest Dynkin system containg $\mathcal{M}$, call it $\mathcal{D}(\mathcal{M})$

C) Show that $\mathcal{D}(\mathcal{A}) = \mathcal{B}$ (the Borel sets).

IV Let $(\mathfrak{X}, \Sigma)$ and $(\mathfrak{X}', \Sigma')$ be sets with σ-algebras Σ, Σ' resp. Define $f : \mathfrak{X} \to \mathfrak{X}'$ to be Σ-Σ' measurable iff $\forall A \in \Sigma'$ $f^{-1}(A) \in \Sigma'$. Also we define $f^{-1}(\Sigma') = \{f^{-1}(A) : A \in \Sigma'\}$ and $f(\Sigma) = \{A \subset \mathfrak{X}' : f^{-1}(A) \in \Sigma\}$. Show:

A) For any $f : \mathfrak{X} \to \mathfrak{X}'$, $f^{-1}(\Sigma')$ and $f(\Sigma)$ are σ-algebras

B) f is Σ-Σ' measurable iff $f(\Sigma) \supset \Sigma'$ iff $f^{-1}(\Sigma') \subset \Sigma$.

C) If $\mathcal{M}$ is a collection of subsets of $\mathfrak{X}'$ and $\mathcal{S}(\mathcal{M}) = \Sigma'$ Then $f : \mathfrak{X} \to \mathfrak{X}'$ is Σ-Σ' measurable iff $f^{-1}(\mathcal{M}) \subset \Sigma$.

D) $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}$-$\mathcal{B}$ measurable iff f is measurable in the sense of the text. (Prop 18, p. 65)

V. Let (Ω, Σ, P) be a probability space. $\mathcal{I} \subset \Sigma$ is said to be independent if for all $n = 1, 2, \ldots$ and all choices of $A_1, A_2, \ldots, A_n \in \mathcal{I}$ we have:

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$$

A) show that $T = \{A \in \Sigma : P(A) = 0 \text{ or } P(A) = 1\}$ is independent.

B) Let $\Omega = \{1, 2, 3, 4\}$, $\Sigma = \mathcal{P}(\Omega)$, and define P s.t. $P(A) = 1/4$ the number of elements in A. Give an example of three sets that are pairwise independent but are not independent as a threesome.

c) Construct an ~~a~~ example of an infinite independent set $\mathcal{I}$ such that $\mathcal{I} \cap T = \emptyset$ and $(\Omega, \Sigma, P) = ([0,1], \mathcal{L}, m)$.

VI. Let Λ be an index set. Suppose: that for each $\alpha \in \Lambda$ $\emptyset \neq \mathcal{M}_\alpha \subset \Sigma$ (contining notation of V) we say $\{\mathcal{M}_\alpha : \alpha \in \Lambda\}$ is independent if for all finite subsets F of Λ and for all choices of $A_\alpha \in \mathcal{M}_\alpha$ $(\alpha \in F)$ we have $P\left(\bigcap_{\alpha \in F} A_\alpha\right) = \prod_{\alpha \in F} P(A_\alpha)$.

If x_α, $\alpha \in \Lambda$ are real valued random varaibles on Ω we say $\{x_\alpha : \alpha \in \Lambda\}$ are an independent set of random varaibles if $\{x_\alpha^{-1}(\mathcal{B})$ [notation of IV] $: \alpha \in \Lambda\}$ is independent.

A) Show $\{x_\alpha : \alpha \in \Lambda\}$ are independent iff the set $\{\mathcal{M}_\alpha : \alpha \in \Lambda\}$ is independent. where the $\mathcal{M}_\alpha$'s are defined by $\mathcal{M}_\alpha = x_\alpha^{-1}(\mathcal{J})$, $\mathcal{J} = \{(a, \infty) : a \in \mathbb{R}\}$.

[HINT: USE PROBLEM III C]

B) Define $r_n : [0,1] \longrightarrow \mathbb{R}$ by $r_n(t) = sg\left[\sin \frac{t}{2^n \pi}\right]$ where $sg(t) = \begin{cases} 0 & * \; t = 0 \\ \frac{|t|}{t} & t \neq 0 \end{cases}$. Show $\{r_n : n \in \mathbb{N}\}$ are independent random varaibles.