MATH 531 FINAL DO 4 PROBLEMS WITH AT LEAST ONE FROM PAGE TWO.

1) LET $f \geq 0$ on $[0, \infty) = A$. Show that

$$\int_A f = \lim_{n \to \infty} \int_0^n f.$$

2 A) Suppose that S and T are sets of non-negative extended real numbers. Prove carefully that

$$\sup(S + T) = \sup S + \sup T.$$

B) Part A can be used to prove one of the following inequalites for non-negative measurable functions. Which one? Prove it.

$$\int (f+g) \leq \int f + \int g \quad \text{or} \quad \int (f+g) \geq \int f + \int g.$$

3) On c, the vector space of all real sequences (x_n) such that $\lim_{n \to \infty} x_n \in \mathbb{R}$, define $\| \cdot \|$ by $\| (x_n) \| = \sup_n |x_n|$.

A). Show that $\| \cdot \|$ turns c into a Banach space.

B). Define $T : c \to \mathbb{R}$ by $T((x_n)) = \lim_{n \to \infty} x_n$. Show that T is a bounded linear functional.

4) Show that if $\{f_n\}$ is a sequence of measurable functions, then the set $\{x : f_n(x)$ converges (to an extended real) as $n \to \infty\}$ is measurable.

5) Suppose $E \subset \mathbb{R}$ with $m(E) < \infty$ and $\lambda \in [0,1]$. Show that there is an $S \subset E$ such that

$$m(S) = \lambda\, m(E).$$

6) Show that for $\infty > p > q \geq 1$:

A) $L_p[0,1] \subset L_q[0,1]$

B) $\exists f \in L_p(\mathbb{R}) \setminus L_q(\mathbb{R})$ and $\exists g \in L_q(\mathbb{R}) \setminus L_p(\mathbb{R})$

7) Show that there are no σ-algebras with a countably infinite number of elements.

8) If $f \in L_1(\mathbb{R})$ then $\lim_{n \to \infty} \int_{\mathbb{R}} f(x) \sin nx \, dx = 0$