PROBLEM SET — MATH 282a

Due 10/20/69 — REAL ANALYSIS

Do Problems 1-12, 1-17, 1-19, 2-8, 2-36, 2-37, and 9-11, in text.

I) Do problem 1-24 in text and use the problem to prove that the real numbers are uncountable. binary's

II) If $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets, we define

$$\overline{\lim}\ A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n ; \quad \underline{\lim}\ A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_n .$$

Prove the following theorem, and state and prove an analogue for $\underline{\lim}\ A_n$

THEOREM: The following are equivalent

1) $A = \overline{\lim}\ A_n$

2) $A = \{x : x$ belongs to infinitely many $A_n\}$

3) $\chi_A = \overline{\lim}\ \chi_{A_n}$ (See definition on P. 68)

III) If $\overline{\lim}\ A_n = \underline{\lim}\ A_n$ we call their common value $\lim A_n$. Fill in the blanks in the following theorem and prove it.

THEOREM: If $(X, \mathcal{R}, \mu)$ is a measure space and if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{R}$, then

A) $\mu(\underline{\lim}\ A_n) \leq \underline{\lim}\ \mu(A_n)$;

B) $\overline{\lim}\ \mu(A_n) \leq \mu(\overline{\lim}\ A_n)$ provided $\mu(\quad) < \infty$.

C) If $\lim A_n$ exist and $\mu(\quad) < \infty$, then $\lim \mu(A_n)$ exists and $\mu(\lim A_n) = \lim \mu(A_n)$.

IV) Let $\mathcal{I}$ and $\mathcal{J}$ be semialgebras of sets, and let $\mu : \mathcal{I} \to [0,\infty]$ and $\nu : \mathcal{J} \to [0,\infty]$.

A) Prove that

$$\mathcal{I} \times \mathcal{J} = \{E \times F : E \in \mathcal{I} \text{ and } F \in \mathcal{J}\}$$

~~10:30 tues week~~

PROBLEM SET
Due 11/24/69

Math 282a
REAL ANALYSIS

In these problems, do not use any results not given in class unless they occur on earlier assigned problems. Prove other results you need.

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Text. ~~2-46~~, ~~3-19~~, $\boxed{\text{3-28}}$, ~~4-5~~, $\boxed{\text{4-16}}$, ~~4-20~~, 4-25, ~~11-12~~, ~~11-20~~, ~~11-21b~~.

I. Let $F(x)$ be a normalized increasing function, ν be the Borel measure defined from F, and m be Lebesque measure restricted to the Borel sets.

A) Prove that $F(x)$ is Borel measurable.

?B) If ϕ is m integrable show that $\phi(F(x))$ is ν integrable and that $\int \phi dm = \int \phi(F(x)) d\nu(x)$.
[Hint: First use Problem 12-12 to prove this for characteristic function.]

C) If $\phi(F(x))$ is ν-integrable, then prove $\phi(x)$ is m-integrable
[Hint: Look at ϕ^+ and ϕ^-.]

D) If ϕ is m-integrable and E is a Borel set, prove

$$\int_{F(E)} \phi dm = \int_E \phi(F(x)) d\nu$$

II. We use the notation and results of Problem 11-9. If f is an extended real valued function on X, then $N(f)$ is the set on which f is non-zero. f is said to be $\mathcal{R}$ measurable if $f^{-1}(E) \cap N(f)$ belongs to $\mathcal{R}$ whenever E is a Borel set.

A) Prove that f is $\mathcal{R}$ measurable if, and only if, f is $\mathcal{B}$ measurable and $N(f)$ belongs to $\mathcal{R}$.

B) Define $\int f d\mu$ in the obvious way starting with $\mathcal{R}$ measurable non-negative simple functions. Prove that $\int f d\mu = \int f d\bar{\mu} = \int f d\underline{\mu}$ if the first integral exists.

C) If f is $\bar{\mu}$ integrable, prove that f is μ integrable. What if f is $\underline{\mu}$ integrable?

$A' \cap B' \in \mathcal{R}'$

$f^{-1}(E) \in \mathcal{B} \Rightarrow f^{-1}(E) \in \mathcal{R} \Rightarrow f$

$A, B \in \mathcal{R}'$ $A \sim B = A \sim$

$\Rightarrow \sim(\sim A \sim \sim B) \in \mathcal{R}' \Rightarrow$

suppose $f^{-1}(E) \cap N(f) \in \mathcal{R}'$

$Z(f) \in \mathcal{R}' \Rightarrow Z(f) \cap f^{-1}(E), N(f) \in \mathcal{R}'$

$\Rightarrow \phi \in \mathcal{R}'$
$\Rightarrow X \in \mathcal{R}$

III. Let $(X, \mathcal{B}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measureable functions. We say f_n converges to f almost uniformly if, for all $\varepsilon > 0$, there is a set E of measure less than ε, such that f_n converges to f uniformly on the complement of E.

A) If f_n converges to f almost uniformly, prove that f_n converges to f in measure and also a.e.

B) Give an example for which f_n converges to f a.e, but f_n does not converge in measure or almost uniformly.

C) Prove Egoroff's Theorem: If μ is a finite measure and f_n converges to f a.e., then f_n converges to f almost uniformly.
[Hint: Redo Propositions 3.23 and 3.24 in appropriate language and your own notation, and read the hint in Problem 3-30.]

PROBLEM SET
Due 12/8/69

Math 282a
REAL ANALYSIS

Do problems 1 ~~12-15~~, 2 ~~12-16~~, 3 ~~12-18~~, 4 ~~12-21~~, 5 $\boxed{\text{12-23,}}$ 6 ~~12-25~~, 7 ~~12-26~~, 8 ~~[illegible]~~.
(Show that the given sets are measurable as well; don't do part of problem on p. 274).

~~I.~~ Let $X = Y = [0,1]$ with Lebesque measure. Let $f(x,y) = 1/x^2$ if $x > y$, $f(x,y) = -1/y^2$ if $y > x$ and $f(x,y) = 0$ if $x = y$.

A) Find $\int\int f(x,y)dxdy$ and $\int\int f(x,y)dydx$.

B) What is $\int f^+(x,y)\ d(x,y)$ and $\int f^-(x,y)\ d(x,y)$?

~~II.~~ Assume Tonelli's theorem for finite measure spaces and prove it for σ-finite measure spaces.

III. Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be measure spaces and let E and F be measureable subsets of X and Y, respectively.

~~A)~~ Prove that the restriction measure of $\mu \times \nu$ to $E \times F$ is the same as the product measure of the restrictions of μ and ν to E and F, respectively.

B) Use A) to prove Tonelli's theorem for $f(x,y) \geq 0$ with $\{\langle x,y\rangle : f(x,y) \neq 0\}$ σ-finite (without of course assuming μ and ν σ-finite). How much of Fubini's theorem can you prove for non σ-finite μ and ν [Hint: See Problem 11-20.]

IV. Let $\mathcal{B}_1$ be the Borel sets of $\mathcal{R}_1$ and $\mathcal{B}_2$ be the Borel sets of $\mathcal{R}_2$. Prove that

$$\mathcal{B}_1 \times \mathcal{B}_1 = \mathcal{B}_2$$

MATH 282a
SEMESTER EXAM

January 30, 1970
2:00 p.m.

DO ANY FIVE PROBLEMS

I. A) Suppose that S and T are sets of non-negative extended real numbers. Prove carefully that $\sup(S + T) = \sup(S) + \sup(T)$.

B) Part A) can be used to prove one of the following inequalities for non-negative measureable functions. Which one? Prove it.

$$\int(f+g) \leq \int f + \int g \quad \text{or} \quad \int(f+g) \geq \int f + \int g$$

C) Prove $\int(f+g) = \int f + \int g$ for non-negative measureable functions. State, as unproved lemmas, the results you need about limits of functions.

II. Suppose $(X, \mathfrak{M}, \mu)$ is a measure space and $\alpha = \sup\{\mu(E) : \mu(E) < \infty\}$. Show that there exists an A in $\mathfrak{M}$ with $\mu(A) = \alpha$. If α is finite, what can you say about measureable subsets of $\tilde{A}$? Prove it.

III. Suppose that $(X, \mathfrak{M}, \mu)$ is a measure space and that $\int f$ and $\int g$ exist.

A) If $\int_E f\,d\mu = 0$ for all E in $\mathfrak{M}$, prove that $f = 0$ almost everywhere.

$f \geq 0$

B) Show by example that it is possible for $\int_E f\,d\mu = \int_E g\,d\mu$ for all measureable E, without $f = g$ almost everywhere.

C) Prove: if f and g belong to $L^1(\mu)$ or if μ is finite, and if $\int_E f\,d\mu = \int_E g\,d\mu$ for all E in $\mathfrak{M}$, then $f = g$ almost everywhere.

$\int f^+ - \int f^- = \int g^+ - \int g^-$

$\int f^+ + g^- = \int g^+ + f^-$

IV. Let $(X, \mathfrak{M})$ be a measureable space and let $\mathcal{V}$ be the collection of all finite signed measures on $\mathfrak{M}$. For ν in $\mathcal{V}$, define $\|\nu\| = |\nu|(X)$. Prove that $\mathcal{V}$ is a vector space on which $\|\nu\|$ is a complete norm (i.e., $\mathcal{V}$ is a Banach space).

V. A) Let m be Lebesgue measure on the reals, R. If $m(E)$ is finite and $\varepsilon > 0$, prove that there is a set J which is a union of a finite number of finite intervals, such that $m(J \,\Delta\, E) < \varepsilon$.

B) State and prove a theorem characterizing all positive linear functionals on $C_c(R)$ in terms of normalized increasing functions. In your proof, you may state, as unproved lemmas, any results about general abstract measures or measures on locally compact spaces; but prove any properties special to the real numbers.

$f =$

$g =$

VI. Suppose that μ_1 and ν_1 are finite measures on $(X, \mathcal{R})$ and that μ_2 and ν_2 are finite measures on $(Y, \mathcal{B})$. Using any results from class, prove:

A) If $\mu_1 \perp \nu_1$ then $\mu_1 \times \mu_2 \perp \nu_1 \times \nu_2$

B) If $\nu_1 << \mu_1$ and $\nu_2 << \mu_2$, then $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$

VII. Suppose that $\mathcal{C}$ is a collection of subsets of X and that $\mathcal{C}$ contains X and ϕ. Suppose also that $\mu : \mathcal{C} \to [0,\infty]$ and $\mu(\phi) = 0$. Define

$$\mu^*(E) = \inf \{ \sum_1^\infty \mu(C_n) : \bigcup_1^\infty C_n \supseteq E; \ \{C_n\} \subseteq \mathcal{C}\}.$$

A) Prove that μ^* is an outer measure.

B) State and prove a necessary and sufficient condition that μ and μ^* agree on $\mathcal{C}$.

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C) Suppose that $\mathcal{C}$ is an algebra and that μ is a measure on $\mathcal{C}$. Prove that E is measureable if

$$\mu^*(C) \geq \mu^*(C \cap E) + \mu^*(C - E), \quad \text{for all } C \text{ in } \mathcal{C}.$$ Prove that all sets in $\mathcal{C}$ are measureable.

VIII. Let $(X, \mathcal{M})$ be a measureable space and let $\mathcal{F}^+$ be the set of non-negative extended real valued $\mathcal{M}$- measureable functions. Do two of the following:

A) Suppose $I : \mathcal{F}^+ \to [0,\infty]$ satisfies:

1) $I(f+g) = I(f) + I(g)$ 2) $I(cf) = c\, I(f)$ if $c \geq 0$

3) If $f_n \uparrow f$, then $I(f_n) \to I(f)$. Prove that $\lambda(A) = I(\chi_A)$ is a measure and that $I(f) = \int f \, d\lambda$ for all $f \in \mathcal{F}^+$.

B) Use Part A) to prove $\int f \, d\nu = \int fg \, d\mu$ for all f in $\mathcal{F}^+$, if ν and μ are measures on $(X, \mathcal{M})$ and g is the Radon-Nikodym derivative of ν with respect to μ.

C) Assume that Lebesgue measure is translation-invariant, and use Part A) to prove

$$\int_E f(x+a) \, dx = \int_{a+E} f(x) \, dx$$

D) State Tonelli's Theorem. Assume that it is true for characteristic functions and use Part A) to prove it in general.

FINAL EXAMINATION

Real Analysis January 28, 1974

A. MULTIPLE CHOICE. Each problem is worth 10 points [-5 for first error; -2 for each of the others]. Circle all true cases.

1. Let $f_{x_0}(y) = f(x_0, y)$ and $f^{y_0}(x) = f(x, y_0)$. Given that f maps R^2 into R^1 and f_{x_0} is Borel measurable for all x_0, when is f Lebesgue measurable?
   (a) For all y_0, f^{y_0} is Borel measurable and $f^{y_0}(x) = 0$ for almost all x.
   (b) For all y_0, f^{y_0} is continuous.
   (c) For almost all y_0, f^{y_0} is Lebesgue measurable and $f^{y_0}(x) = 0$ for all x.
   (d) For all y_0, f^{y_0} is Borel measurable.

2. Indicate each case for which there is a "finitely additive positive measure μ" such that μ is defined on all subsets of X, $\mu(E) = m(E)$ whenever $m(E)$ is defined, and $\mu(A) = \mu(B)$ if A and B are congruent.
   (a) $X = R^1$ and m is Lebesgue measure.
   (b) $X = R^2$ and m is Lebesgue measure.
   (c) $X = R^3$ and m is Lebesgue measure.
   (d) X is the surface of the unit ball in R^3 [i.e., $X = \{(x,y,z): x^2+y^2+z^2 = 1\}$] and m is the Borel measure for which the measure of each spherical rectangle is its area.

3. Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and let f, g and h belong to $L^2(\mu)$. Which of the following are true?
   (a) $fg \in L^1(\mu)$.
   (b) $fg \in L^2(\mu)$.
   (c) $|fg|^{\frac{1}{2}}h \in L^1(\mu)$.
   (d) $\phi \in L^{3/2}(\mu)$ if $\theta\phi \in L^1(\mu)$ for all θ in $L^3(\mu)$.

4. Which of the following are true?
   (a) The set of rational numbers is a G_δ-subset of R^1.
   (b) If $\lim_{n\to\infty} \|f - f_n\| = 0$, then $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all x.
   (c) Each compact subset of R^1 is the support of a continuous function.
   (d) There is a measure space $(X, \mathfrak{M}, \mu)$, an f in $L^1(\mu)$, and a sequence of measurable sets $\{E_n\}$ (not necessarily disjoint) such that, for all n,
   $$\int_{E_n} f\,d\mu > 1, \quad \mu(E_n) < 1/n.$$

B. Do exactly 6 of the following. State explicitly which problem is to be omitted, unless work is done on 6 or less.

5. Assume $(X, \mathfrak{M}, \mu)$ is a measure space, f is μ-measurable, and $\int_E f\,d\mu \geq 0$ for all $E \in \mathfrak{M}$. Prove that $f(x) \geq 0$ for almost all x.

6. Suppose the measure μ on a σ-algebra in R^2 is positive, translation-invariant, complete, and

$$\mu(\{(x,y): \ |x| \leq 1 \ \text{and} \ |y| \leq 1\}) = 4.$$

Explain why μ is Lebesgue measure or why it might not be.

7. Assume $(X, \mathfrak{M}, \mu)$ is a measure space, μ is a positive measure, $\mu(X) < \infty$, and $\mathfrak{N}$ is a σ-algebra contained in $\mathfrak{M}$. Prove or disprove:

"For any $f \in L^1(X, \mathfrak{M}, \mu)$, there is a $g \in L^1(X, \mathfrak{N}, \mu)$ for which $\int_E f\,d\mu = \int_E g\,d\mu$ if $E \in \mathfrak{N}$."

8. Let S be a σ-algebra and μ a "finitely additive measure" on S. Prove that μ is countably additive if

$$\mu(\cup_1^\infty E_n) = \lim_{n\to\infty} \mu(E_n)$$

whenever $\{E_n\}$ is an increasing sequence of measurable sets.

9. Let E be a subset of R^1 whose Lebesgue measure is finite. Show that there is a subset S of E for which

$$m(S) = \tfrac{1}{2}m(E).$$

10. Define "measurable function" and prove that if $\{f_n\}$ is a sequence of measurable functions, then the set of all x for which $\{f_n(x)\}$ converges is a measurable set.

11. Assume the measure space (X, S, μ) is σ-finite. Prove that there can not be uncountably many disjoint measurable sets $\{E_\alpha : \alpha \in A\}$ for which $\mu(E_\alpha) > 0$ for all $\alpha \in A$.