INTEGRATION ON LOCALLY COMPACT SPACE

I. Preliminaries

1) X will always be a locally compact Hausdorff space. C will be the space of continuous real valued functions with compact support. S will be the set of extended real valued lower semi-continuous functions. In this and the next section I is a positive linear functional on C.

2) Urysohn's Lemma: If K is a compact subset of an open set V, then there exists an f in C with: $f(x) = 1$ on K; support $f \subseteq V$; and $0 \leq f \leq 1$ for all x. [6, p. 39], [5, problems 8-19, 9-16, 9-17].

3) Elementary properties of S [3, pp. 22-27], [4, p. 117f], [2, pp. 89f].

4) Lemma: For each compact K, there exists a positive number M_K for which $|I(f)| \leq M_K \|f\|$, for all f in C whose support is contained in K. [2, p. 115], [2, p. 21], [4, p. 116].

5) Extension of I to S^+ and definition of $\bar{I}$. [2, pp. 116f], [3, pp. 18f], [4, pp. 118f]. Definition: $\mu^*(A) = I(\chi_A)$.

6) Definition of almost regular measure space. [2, p. 177, def. (12.39)] uses this as his definition for regular measure space. Nearly the same definition is given in [6, p. 41, Th. (2.14) (b)(c)(d)].

II. Functionals and Measures

The following three theorems, whose proof we sketch in this section, are the main results of the subject.

1) Theorem: μ^* is an outer measure which induces an almost regular measure space $(X, \mathcal{M}, \mu)$.

2) Theorem: If f is a non-negative μ-measureable function then

$$\bar{I}(f) = \int f \, d\mu.$$

3) Theorem: If $(X, \mathcal{N}, \nu)$ is another almost regular measure space for which $I(f) = \int f \, d\nu$ for all f in C, then $\mathcal{N} \subseteq \mathcal{M}$ and ν is a restriction of μ.

Outline of Proofs:

4) Lemma: If $\mathcal{F} \subseteq S^+$ and $f = \sup \{g: g \in \mathcal{F}\}$, then $\bar{I}(f) = \sup \{\bar{I}(g): g \in \mathcal{F}\}$.

Proof: First consider f and $\mathcal{F}$ in C^+. [3, pp. 25f, prop. 25], [2, p. 117], [4, p. 118].

5) A) If f and g belong to S^+, then $\overline{I}(f+g) = \overline{I}(f) + \overline{I}(g)$. [2, p. 118, cor. (9.13], [4, p. 119], [3, p. 29].

B) If $\{f_\alpha\}_{\alpha\in A} \subseteq S^+$, then $\overline{I}(\sum_\alpha f_\alpha) = \sum_\alpha \overline{I}(f_\alpha)$. [2, p. 118], [4, p. 119], [3, p. 29]. We are mainly interested in ordinary countable sums.

6) A) If $\{f_n\}$ is a sequence of non-negative functions, then $\overline{I}(\Sigma f_n) \leq \Sigma\, \overline{I}(f_n)$. To prove this assume $\overline{I}(f_n)$ finite and choose g_n in S^+ with $f_n \leq g_n$ and $\overline{I}(f_n) > \overline{I}(g_n) - \varepsilon/2^n$.

B) μ^* is an outer measure. To prove this apply A) to characteristic functions. [2, p. 120, thm. (9.21)], [4, p. 121].

7) Regularity

A) If K is compact, then $\mu^*(K)$ is finite. Just choose f in C^+ with $f \geq \chi_K$.

B) $\mu^*(E) = \text{Inf}\ \{\mu^*(V) : V \supseteq E,\ V \text{ open}\}$. [2, p. 121, Th. (9.24)], [4, p. 121].

C) If V is open, $\mu^*(V) = \sup\ \{\mu^*(K) : K \subseteq V,\ K \text{ compact}\}$. To prove this apply the lemma of 4) to $\mathcal{F} = \{g \in C^+ : g \leq 1,\ \text{support } g \subseteq V\}$ then use [6, p. 43, step III].

8) All Borel sets are μ^* measureable [2, pp. 123f, th. (9.32)] [1, pp. 208f]. This will prove Theorem 1).

A) μ^* is additive on disjoint opens, by 5) A).

B) μ^* is additive on disjoint compacts.

C) If μ and V are open $\mu^*(U) \geq \mu^*(U \cap V) + \mu^*(U - V)$.

D) All open sets are measureable.

9) Proof of Theorem 2)

A) Theorem 2) is true for constant multiples of characteristic functions.

B) If f is a simple function, $\overline{I}(f) \leq \int f\, d\mu$.

C) If f is a simple function, $\overline{I}(f) = \int f\, d\mu$. Easy to prove if $\int f$ is infinite. If finite choose g, a multiple of a characteristic function, with $g \geq f$ and $\int g$ finite. Then apply B) to $g - f$.

D) $\int f \leq \overline{I}(f)$. Proof: Choose simple $f_n \uparrow f$.

E) $\int f \geq \overline{I}(f)$. Proof: Let $g_n = f_n - f_{n-1}$ and apply 5) B).

10) Proof of Theorem 3).

A) ν is a restriction of μ^*. [6, p. 41], [2, p. 178, Th. (12.41)].

B) $\mathcal{N} \subseteq \mathcal{M}$. This follows from the fact that $\mu^*(U) = \mu^*(U \cap E) + \mu^*(U - E)$, if U is open and E is in $\mathcal{N}$. [1, p. 208, Lemma].

11) Theorem: If $(X, \mathcal{N}, \nu)$ is an almost regular measure space and if we define $I(f) = \int f\, d\nu$ for f in C, then I is a positive linear functional and $\overline{I}(f) = \int f\, d\nu$ for all non-negative measureable f. In particular $\nu(E) = \overline{I}(\chi_E)$ for all E in $\mathcal{N}$. [2, pp. 178f, Th. (12.42)].

This sets up a one-to-one relation between positive linear functionals and almost regular Borel measures. In particular 4), 6B), and the definition of $\overline{I}$ in terms of sups and infs can be applied to $\int f\, d\nu$.

III. Approximations and Regularity

1) Theorem: Suppose that I is a positive linear functional on C, that $(X, \mathcal{M}, \mu)$ is the measure space constructed from $\overline{I}$, and that f is a real valued function, then the following are equivalent.

A) f is μ-integrable.

B) For all $\varepsilon > 0$, there exists a g in C with $I(|f-g|) < \varepsilon$.

C) For all $\varepsilon > 0$, there exists a g in $L^1(\mu)$ such that $\overline{I}(|f-g|) < \varepsilon$.

D) There exists a g in $L^1(\mu)$ with $\overline{I}(|f-g|) = 0$.

E) For all $\varepsilon > 0$, there exists $g, -h$ in S with $h \leq f \leq g$ and $\overline{I}(g-h) < \varepsilon$.

The proof requires knowing that $L^1(\mu)$ is complete and proving that $\overline{I}(|\phi|) = 0$ implies ϕ is zero almost everywhere.

2) Definition of inner regular, outer regular, and regular. [6, p. 47]

3) Theorem: If $(X, \mathcal{M}, \mu)$ is almost regular and if E is a sigma-finite measureable set, then E is inner regular [2, pp. 137f, Th. (10.30)].

4) Theorem: If $(X, \mathcal{M}, \mu)$ is outer regular and X is sigma-compact, then μ is regular. Moreover, if E belongs to $\mathcal{M}$ then:

A) For all $\varepsilon > 0$, there exists a closed set F and an open set V with $F \subseteq E \subseteq V$, and $\mu(V - F) < \varepsilon$.

B) There exists sets A and B such that A is a G_δ, B is an F_σ, $A \subseteq E \subseteq B$, and $\mu(B - A) = 0$.

A proof of roughly this result is in [6, pp. 47f, Th. (2.17)]. If μ is complete, any set E satisfying A) or B) is measureable.

5) Theorem: If X is sigma-compact and if every closed subset is a G_δ, then every Borel measure on X is regular. [5, pp. 305-307, prop. 6], [6, p. 48f, Th. (2.18)].

Note that every locally compact separable metric space satisfies the hypothesis.

IV. Signed Measures and Non-Positive Functionals

1) Definition of relatively bounded (i.e., satisfying I. 4) of $\|I\|$ and of regular signed measures. [7, pp. 372f] [5, pp. 308f]

2) Theorem: If I is a relatively bounded linear functional on C, then there exist positive linear functionals I^+ and I^- for which $I = I^+ - I^-$. If I is bounded, so are I^+ and I^-.

Proof: If f is in C^+, let $I^+(f) = \sup\{I(g) : 0 \leq g \leq f, g \in C\}$. In general $I(f) = I(f^+) - I(f^-)$, and $I^- = I^+ - I$. [7, pp. 371f, section 8-7], [5, pp. 309f, prop. 7]

3) Theorem: If I is a bounded linear functional on C, then there exists a unique finite regular signed Borel measure ν for which $I(f) = \int f\, d\nu$ for all f in C.

Proof: Apply Section II to I^+ and I^-. [5, pp. 310f, prop. 8]

4) Lemma: If I and ν are as above $I^+(f) = \int f\, d\nu^+$ and $I^-(f) = \int f\, d\nu^-$ for all f in C.

Proof: If f is in C^+, then the definition of I^+ implies $I^+(f) \leq \int f\, d\nu^+$. On the other hand, if I^+ defines the measure ν_1, $\nu_1 \geq \nu^+$.

5) Theorem: If I and ν are as above, then $\|I\| = |\nu|(X)$ [5, pp. 310 f].

References

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[2] is the closest to our approach. [6] is fairly close to our approach but never defines $\overline{I}$ or outer measure. [3] and [4] construct integrals without assuming any measure theory. [1] is more interested in measure than integration. [5] and [7] treat the integral on locally compact spaces as a special case of the abstract Daniell integral, but their approach in the non-positive case resembles ours.