MAA 5617 "M & ∫ 2" part 2 of "M & ∫"

Instructor: Bellenot Office 218 Love Off HRS { MW 12:30 – 2:15, Th 12:30 – 1:15

Texts: Rudin Real & Complex & Taylor Intro to M & ∫

Prereq: MAA 5616 "M & ∫ 1"

GRADES: 20% : in class Test Tentatively Set for 10 Apr 85
80% : graded homework

A ≥ 80% B ≥ 62.5 % C ≥ 50%

Homework Rules:

(1) Each Problem on a separate page.
(2) Do not use both sides of a page.
(3) Additional pages clipped or stapled together.
(4) Written in ink.
(5) MUST BE YOUR OWN WORK

Each problem is worth 10 pts. Each rule (1), (2), (3) or (4) will cost 1 pt each if disobeyed. Rule (5) will cost everything. Homework is assigned at least 2 weeks before they are due. 3 problems will be due each Monday. Late homework will not be accepted. The lowest 10% of ~~the~~ your grades on homework will be ~~tho~~ tossed out before computing your homework average.

First Homework Assignment Due 21 JAN 85

*1. Show that if $f \in L_\infty[0,1]$ then T_f is given by $T_f(g) = \int_0^1 fg\,dm$. Show $\tilde{T} : L_\infty[0,1] \to (L_1[0,1])^*$ is 1-1, onto, and norm preserving where $\tilde{T}(f) = T_f$

*2. Find $\Theta \in (L_\infty[0,1])^*$ s.t. $\Theta \neq T_f$ for any $f \in L_1[0,1]$ where $T_f(g) = \int_0^1 fg\,dm$.

*3. Rudin Ch 4 Ex 1

Second Homework Assignment Due 28 JAN 85

*4. Rudin Ch 4 Ex 6

*5. Rudin Ch 4 Ex 16

*6. Rudin Ch 5 Ex 11.

M&S II due Mon 18 Feb. 85

Consider two positive meas μ and ν on (X, Σ) and suppose for some measurable f

$$(*) \quad \forall E \in \Sigma \quad \mu(E) = \int_E f \, d\nu$$

13 A. Show $f \geq 0$ a.e. $[\nu]$ and $\mu \ll \nu$

B. Show $f \in L_1(\nu) \iff \mu$ is a finite measure

C. Show $f \leq 1$ a.e. $[\nu] \iff \mu \leq \nu$

D. Show $f \neq 0$ a.e. $[\nu] \iff \nu \ll \mu$.

14 A. Show for $g \geq 0$, $\int g \, d\mu = \int g f \, d\nu$

B. Suppose g meas s.t. $\mu(E) = \int_E g \, d\nu$ then $f = g$ a.e. $[\nu]$

C. If $\nu \ll \mu$ then $\nu(E) = \int_E f^{-1} \, d\mu$ $\quad$ ($f^{-1} = 1/f$ not inverse image)

15 ~~If~~ Let λ, μ are finite measures on (X, Σ) with $\lambda \ll \mu$

A. Let $\nu = \lambda + \mu$ then $\lambda, \mu \ll \nu$ and $\nu \ll \mu$

B. By RN special case proved in class $\exists\, g, f \in L_1(\nu)$ s.t. $\forall E \in \Sigma$ $\quad \lambda(E) = \int_E g \, d\nu$; $\mu(E) = \int_E f \, d\nu$

show $\lambda(E) = \int_E g f^{-1} \, d\mu$.

C. Prove: RN Thm If λ, μ are finite measures on (X, Σ) with $\lambda \ll \mu$, then there is a unique element $f \in L_1(\mu)$ so that $\forall E \in \Sigma$ $\lambda(E) = \int_E f \, d\mu$.

D. Suppose λ, μ are σ-finite positive measures on (X, Σ) with $\lambda \ll \mu$ show there is a ~~positi~~ non-negative measurable f s.t. $\forall E \in \Sigma$ $\lambda(E) = \int_E f \, d\mu$.

All functions are real-valued and measurable on the real line. dm = Lebseque measure.

1. Prove both the following statements (They are simple results from one or another convergence theorem) and for one of them, find a counterexample to show that the underlined hypothesis is needed.

A. If f is a measurable function and $\underline{\underline{f \geq 0}}$, then
$$\int_0^\infty f\,dm = \lim_{n\to\infty} \int_0^n f\,dm.$$

B. If $\{f_n\}$ and f are measurable functions, $f_n \to f$ point-wise, $f_1 \geq f_2 \geq f_3 \geq \dots \geq 0$ and $\underline{\underline{f_1 \in L_1}}$, then
$$\lim_{n\to\infty} \int f_n\,dm = \int f\,dm.$$

2. Let S, T be non empty sets of positive reals and let $S+T = \{s+t \mid s \in S, t \in T\}$

A. Show $\sup(S+T) = \sup S + \sup T$.

In constructing Lebseque intergration, we defined, for $f \geq 0$, f measurable
$$\int f\,dm = \sup\left\{\int \varphi\,dm \,/\, 0 \leq \varphi \leq f,\ \varphi \text{ simple}\right\}$$

B. Using the definition above and part A it is possible to show one of the following inequilities, determine which one and prove it.

If $f, g \geq 0$
$$\int (f+g)\,dm \ \begin{matrix}\leq \\ \geq\end{matrix}\ ? \ \int f\,dm + \int g\,dm.$$

Do any ___ out of the below problems

1. Suppose $\bar{\mu}$ is a positive measure obtained by extending μ, a measure on an algebra $\mathcal{A}$, via outer measure; show:

   A) $\bar{\mu}$ is complete

   B) $\bar{\mu}(E) = 0$ iff $\forall \epsilon > 0$ $\exists$ sequence $(A_n) \subset \mathcal{A}$ such that $\bigcup_n A_n \supset E$ and $\sum_n \mu(A_n) < \epsilon$

2. Let μ be a σ-finite positive measure show for measurable f: $f = 0$ a.e. $[\mu]$ iff $\int_E f \, d\mu = 0$ for all measurable E.

3. Show that a positive σ-finite measure is semi-finite and give an example to show that the converse is false.

4. Suppose $f \geq 0$, f Σ measurable, μ a positive measure on Σ. Define $\nu(E) = \int_E f \, d\mu$ for $E \in \Sigma$. Show ν is a measure which is absolutely continuous with respect to μ. Show ν is finite iff $f \in L_1(\mu)$.

5. Suppose $f \in L_1(0,1)$, $(f_n) \subset L_1(0,1)$ and $f_n \to f$ pointwise. Show that $\|f_n\| \to \|f\|$ implies $\|f_n - f\| \to 0$.

Define or State:

1. The Lebesque Dominated Convergence Theorem
2. An outer measure on a set $\mathbb{X}$
3. a countable set
4. $\inf A$, when $\emptyset \neq A \subset \mathbb{R}$
5. when a measure $(\mathbb{X}, \Sigma, \mu)$ is σ-finite
6. A borel set in $\mathbb{R}^n$
7. $f_n \rightarrow f$ in measure
8. when a measure is complete
9. The function f^- when $f: \mathbb{X} \rightarrow \mathbb{R}$
10. The "ε-is-arbitrary" agruement.

Show

A. For $f: \mathbb{X} \rightarrow \mathbb{R}$ measurable

show $\left|\int f\,d\mu\right| \leq \int |f|\,d\mu$

B. $\sup(A+B) \geq \sup A + \sup B$ when $\emptyset \neq A, B \subset \mathbb{R}$

C. $f: [0,1] \rightarrow \mathbb{R}$ meas, $0 < p \leq q \leq \infty$, show $\|f\|_p \leq \|f\|_q$

D. Find a sequence of functions f_n in $L_1([0,1])$

s.t. $f_n \rightarrow$ pointwise but $\|f_n\|_1 \longrightarrow \infty$

E. Given a measurable $f: [0,1] \longrightarrow [-\infty, \infty]$

s.t. $m(\{x : |f(x)| = \infty\}) = 0$ Show

$\forall \varepsilon > 0\ \exists M$ s.t. $m(\{x : |f(x)| \geq M\}) < \varepsilon$

F. Let $f \in L_1(\mathbb{R})$; $\mathbb{R} \rightarrow \mathbb{R}$, $t \in \mathbb{R}$ show

$$\int_{\mathbb{R}} f(x)\,dm(x) = \int_{\mathbb{R}} f(x+t)\,dm(x)$$

6 ~~8~~ ~~1~~ Either produce a measurable function $f : [a,b] \to \mathbb{R}$ with the properties

(a) f is strictly increasing

? (b) $\exists\, M \subset [a,b]$, $m(M) > 0$ and $x \in M \Rightarrow f'(x) = 0$

(c) $\forall\, C$ measurable $\subset [a,b]$, $m(f(C)) \geq m(C)$

Or show that none exists.

simple fcns ~~2~~ 5 Either produce a sequence of measurable functions (f_n) with the properties

step fcns

χ intervals

(a) $f_n : [0,1] \to \mathbb{R}$

(b) $\forall x \in [0,1]$, $\lim_n f_n(x)$ converges

(c) $\forall\, S \subset [0,1]$ with $m(S) = 1$, (f_n) does not converge uniformly on S

or show none exists.

3. Either produce a measurable function $f : [0,1] \to \mathbb{R}$ with the properties ?

(a) $f'(x)$ exists on a subset of $[0,1]$ with measure one

f'? $\longrightarrow$ (b) f' is discontinuous on a dense subset of $[0,1]$

or show none exists.

4. Is the product of two absolutely continuous functions also absolutely continuous? Verify your assertion

1 ~~2~~ ~~5~~ If $\mu(X) < \infty$ and $f_n \to f$ uniformly on X

(A) Prove $\int f_n \, d\mu \to \int f \, d\mu$

(B) Is $\mu(X) < \infty$ necessary in part A? Verify your assertion.

2 ~~1~~ ~~6~~ Show $\lim_{n \to \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx = 1$

Hint: show $(1+y)^{1/y} \leq e$ for $y > 0$

Assume there is $\phi \in C[0,1]$ such that

$$(*) \quad \int_0^1 \int_0^1 f(rs)\, dr ds = \int_0^1 f(t)\phi(t)\, dt$$

for all simple functions $f(t)$ on $[0,1]$.

A. Show $(*)$ is true for any $f(t) \in C[0,1]$.

B. Show $\phi(t) = \ln\left(\frac{1}{t}\right)$.

$\{2\pi k + r\}$ comes arb close to an integer

if not $\varepsilon > 0$ s.t. $|2\pi k + r - n| > \delta$
for all n, k

or $\left|2\pi + \frac{r}{k} - \frac{n}{k}\right| > \frac{\delta}{k}$

Approx $2\pi \approx \frac{h}{k}$ s.t.

$$\left|2\pi - \frac{h}{k}\right| \le \frac{1}{2k} \qquad \left|\frac{r}{k} - \frac{j}{k}\right| \le \frac{1}{2k}$$

$$\left|2\pi + \frac{r}{k} - \frac{h+j}{k}\right| \le \frac{1}{k}$$

thus $|2\pi k + r - (h+j)| \le 1$

The sequence $\{2\pi k \bmod 1 : k = 0, 1, \dots\}$
is dense in $[0,1]$

$$2\pi k = n(k) + r(k) \qquad \cancel{2}\pi = \frac{n(k)}{2k} + \frac{r(k)}{2k}$$

$$\pi - \frac{n(k)}{2k} = \frac{r(k)}{2k}$$

$$\left|\pi - \frac{p}{q}\right| < \varepsilon$$

6. Either produce a measurable function $f : [a,b] \to \mathbb{R}$ with the properties:

(a) f is strictly increasing;

(b) there is $M \subset [a,b]$ with $m(M) > 0$, so that $x \in M \Rightarrow f'(x) = 0$; and

(c) for each measurable $C \subset [a,b]$, $m(f(C)) \geq m(C)$.

or show that none exists.