<u>Due</u> 14 Dec 81 1:00pm in 211 Love (Monday of Final's week)

TP25: If $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and (X, Σ, μ) is σ- ~~fintet~~ finite
Then the dual of $L_p(\mu)$ is $L_q(\mu)$

TP26: If X is uncountable, $\Sigma = \{A \subset X : A \text{ or } A^c \text{ is countable}\}$, μ
is ~~the~~ counting measure,
$\ell_1^* = \ell_\infty$ Then the dual of $L_1(X, \Sigma, \mu)$ is $L_\infty(X, \mathcal{P}(X), \mu)$ and
this last space is different from $L_\infty(X, \Sigma, \mu)$.

TP27: If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then the dual of $L_p(\mu)$ is $L_q(\mu)$

[Hint: If $T \in L_p(\mu)^*$ show there is a μ-σ-finite set A so
that if $f \in L_p(\mu)$ then $T(f) = T(f \chi_A)$.]

TP28: Let $L_\infty =: L_\infty([0,1], \mathcal{L}, m;$ real valued fcns$)$ consider
for $g \in L_\infty$

$$T(g) = \lim_{n \to \infty} \int \left(g \, \frac{n}{2} \chi_{[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]} \right) dm$$

show T is a continuous linear functional $L_\infty \to \mathbb{R}$.
and if g is continuous on $[0, 1]$ then $T(g) = g(\frac{1}{2})$.

[Hint: Do it first for g simple. Then for general g show ~~there~~
for each $\varepsilon > 0$ there is simple s s.t. $\|g - s\|_\infty < \varepsilon$.]

M & S 2 due Mon 4 Feb 85

7. Show the closed graph theorem implies the open mapping theorem

8. M, N are closed subspaces of the Banach space X

consider $T: M \oplus N \to X$ given by $T((m,n)) = m+n$

show T is linear and continuous

T is 1-1 $\iff$ $M \cap N = \{0\}$

T is onto $\iff$ $M + N = X$

and if both $M \cap N = 0$ and $M+N = X$ then T is a isomorphism.

9. A B-space X is a BK-space if the elements of X are sequences of scalars and the functionals $f_m((x_n)) = x_m$ are continuous. Show if X and Y are BK-spaces with $X \subset Y$ as sets of sequences then the inclusion map $T: X \to Y$ is continuous

M&S 2 due Mon 4 Feb 85

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Use a separate sheet of paper for each problem.

① (a.) Define : f is integrable over E

(b.) Show : f integrable over $E \Rightarrow |f|$ integrable over E

(c.) Show $\left|\int_E f\right| \le \int_E |f|$

② Give an example of an open dense subset O of $\mathbb{R}$ with $m(O) < \epsilon$, where ϵ is a preassigned positive number. Hint : $\mathbb{Q}$

③ Suppose f is continuous. Show $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = f(x)$. Do this "from scratch".

④ The function f satisfies a Lipschitz condition on $[a,b]$ if there is a finite number M such that $|f(x) - f(y)| \le M|x-y|$ for $a \le x, y \le b$. Show that then f is absolutely continuous.

⑤ Suppose f, f_n are measurable functions on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} |f - f_n|^p \to 0$ for some fixed $p \in (0,\infty)$. Show that then $m\{|f - f_n| > \epsilon\} \longrightarrow 0$ for each $\epsilon > 0$.

⑥ Suppose $0 < p < q < \infty$ and $\int_0^1 |f(x)|^q dx < \infty$. Show $\int_0^1 |f(x)|^p dx < \infty$.

⑦ Let E be a measurable set. Then $x \in E$ is said to be a <u>point of density</u> of E if $\lim_{h \to 0} \frac{m\{E \cap (x-h, x+h)\}}{2h} = 1$. Suppose $m(E) < \infty$ and show that almost every point of E is a point of density.

Hint: differentiate something.