MAA 5306 (531) Real Variables (Lebesgue integration) ~ BELLENOT

Office Hours MWF 2:00 – 3:30 pm Tu Th 5:30 – 6:45 pm

<u>Test Problems</u>: Given each Monday due the next Friday.
(1) To be your own work, using only the Notes
(2) How well you do on these will determine how good your grade will be.
(3) In Ink & one side of paper — No rough drafts

<u>Tests</u>: Two in class tests, closed book
(1) How bad you do on these will determine how bad your grade will be
(2) at least 50% definitions and statements of theorems
(3) Usually there would be a final — but not at 7:30 am.

<u>Notes</u>: Note taking and writing them up will rotate between 2 and 3 member groups.
(1) An example is the first Lecture notes
(2) Use the form given particularly with respect to name, page numbering, and consecutive numbering of DN, LM, THM, and EX.
(3) Each group will have the responsibility of running off enuff copies for the class plus 5 extra's for the Bellenot.
(4) Grades will suffer if note taking quality is not good
(5) Notes taken one week are due the next Monday

<u>Problem Sessions</u>: are availible (Tu 3:35pm) ~ just ask on Monday: (takes 4 students)

TEST PROBLEM 1 due 9/29/78

1. (3pts) Prove fact 14 [Hint use THM 4]

(increasing only)

2. (3pts) Prove fact 8 [Hint use THM4]

3 (4pts) Prove that a bounded function $f(x)$ which is continuous ~~&~~ except at a finite number of points in $[a,b]$ is R-$\int$'able on $[a,b]$.

Hints 1. Use induction.

~~2. $f(x)$ is bounded why?~~

or ~~3. Show a bounded function~~ on $[a,b]$ which is R-integrable on $[a, c-\varepsilon]$ and $[c+\varepsilon, b]$ for each $\varepsilon > 0$ is R-$\int$able on $[a,b]$ by THM4.

4 use Fact 9.

Lecture 1: Review of Riemann Integration 9/25/78

An general review of the properties and limitations of Riemann integration is given.

<u>DN 1</u>: A <u>division</u> π of the interval $[a,b]$ is a finite sequence of reals $x_0, x_1, \dots x_n$ with $a = x_0 < x_1 < x_2 < \dots < x_n = b$. The norm of π, $\|\pi\| = \max\{x_i - x_{i-1} : i = 1, 2, \dots n\}$.

<u>DN 2</u>: The function $f(x)$ on $[a,b]$ is said to be <u>Riemann Integratable</u> (written R-$\int$able) if there is a real number L, so that for each $\varepsilon > 0$, there is $\delta > 0$, such that for each division π with $\|\pi\| < \delta$ and for each choice of $\{\xi_i : i = 1, 2, \dots n\}$ with $\xi_i \in [x_{i-1}, x_i]$ we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - L \right| < \varepsilon.$$

In which case the number L is said to be the integral of $f(x)$ from a to b (written $\int_a^b f(x)\,dx = L$ or $\int_a^b f = L$.)

This definition is impractical to use, and there is some evidence that it is the wrong statement (of many equivalent ones) to use as the definition. For our purposes, we will use the statement of Thm 4 (below) as our definition. It is certainly the most useful from our viewpoint. Before stating THM4, we need a definition.

DN 3: The function $f(x)$ is said to be a step function on $[a,b]$ if there is a division $x_0, x_1, \dots x_n$ and numbers $\alpha_1, \alpha_2, \dots \alpha_n$ so that for $\xi_i \in (x_{i-1}, x_i)$ we have $f(\xi_i) = \alpha_i$.

Note that no restriction is placed on the values of $f(x_i)$ $i = 0, 1, \dots n$. In pacticular, if $f(x)$ is given by $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$, then $f(x)$ is a step function on each finite closed interval of the real line.

THM4: The function $f(x)$ is R-$\int$able on $[a,b]$, if and only if, for each $\varepsilon > 0$ there are step functions $\varphi(x)$ and $\psi(x)$ on $[a,b]$ so that $\varphi(x) \leq f(x) \leq \psi(x)$ and

$$\int_a^b \psi(x) - \varphi(x)\, dx < \varepsilon.$$

Reminder $\varphi(x) \leq f(x)$ (on $[a,b]$) means that for each $\xi \in [a,b]$, $\varphi(\xi) \leq f(\xi)$; or in words at each point of $[a,b]$ f is greater than or equal to φ; or $f \geq \varphi$ pointwise.

Let us make an important sidetrack before reviewing the properties of the R-$\int$.

LM 5: (The Fundemental Principle of Analysis)
If a and b are real numbers so that

$$\forall \varepsilon > 0 \qquad a \geq b - \varepsilon$$

Then $a \geq b$.

Proof: Suppose it is not true that $a \geq b$. Then $a < b$. Let $\varepsilon = (b-a)/2$. Then it is false that $a \geq b - \varepsilon$.

Warning, note the difference between LM5 & THM4. In LM5 a & b are given before ε is chosen while in THM4 φ & ψ are chosen after ε is chosen. This means that the step-functions φ and ψ can be considered functions of ε.

In fact a common error of analysis students is to combine LM5 & THM4 to ERRONEOUSLY

conclude that for any Riemann integrable $f(x)$ there exist step functions φ and ψ so that $\varphi \leq f(x) \leq \phi$ and $\int \psi - \varphi = 0$. Note that this statement is false, since $f(x) = x$ is R-$\int$'able but not a step function.

We now pass to some properties of the R-$\int$, but for completeness sake we list some definitions.

DN6: $f(x)$ is said to be bounded (on $[a,b]$) if there is a real number B so that for all x $(x \in [a,b])$ $|f(x)| \le B$.

DN7: $f(x)$ is said to be monotonely increasing (decreasing) on $[a,b]$ if $x, y \in [a,b]$ and $x < y$ implies $f(x) \le f(y)$ $(f(x) \ge f(y))$.

FACTS ABOUT R-$\int$'able functions

F8: Montone increasing or decreasing functions on $[a,b]$ are R-$\int$'able functions
pf: TEST PROBLEM

F8½ R-$\int$'able fcns are bounded

F9: Continuous functions are R-$\int$'able

F10: $\int f + \int g = \int (f+g)$

F11: $\int cf = c\int f$ where c is real number

F12: if $f \ge 0$, then $\int f \ge 0$ $\longrightarrow |\int f| \le \int |f|$

F12½

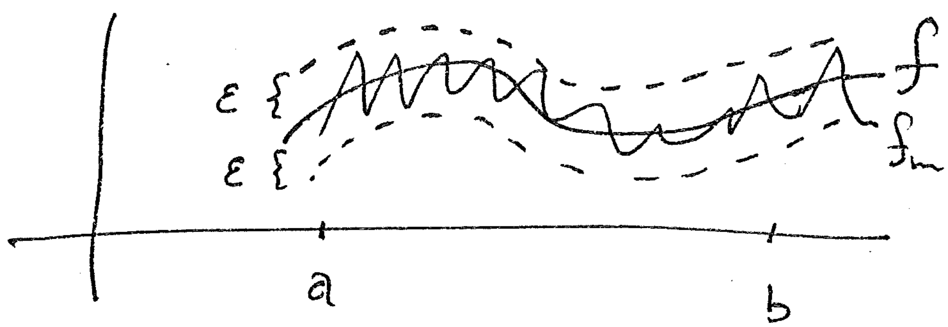
F13: $\int_a^b f + \int_b^c f = \int_a^c f$.

F14: If $f_n(x)$ are R-$\int$'able on $[a,b]$ and if $f_n(x) \to f(x)$ uniformly on $[a,b]$ then $f(x)$ is R$\int$'able on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

pf: TEST PROBLEM.

DN 15 $f_n(x) \to f(x)$ uniformly on $[a,b]$ if for each $\varepsilon > 0$ $\exists N$ so that $m \ge N$ and $\xi \in [a,b]$ imply $|f_m(\xi) - f(\xi)| < \varepsilon$.



Remarks. To find the R-$\int$ of a function, we will resort to Freshman Calculus. In fact even when given a Lebesque integral of a function that is familar we will use Freshman Riemann techniques to integrate it. There is an exception, Riemann improper integrals do not necessary give the same answer as Lebesque integrals.

We observe that if $f(x)$ is the step function given in DN 3 then

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \alpha_i (x_i - x_{i-1}).$$

WHAT IS WRONG WITH THE R-$\int$?

This is a hard question to answer at this point. Perhaps the best answer is by analogy, namely the Lebesque integral is to the Riemann integral as the reals are to the rationals. That is, there are limit processes which we would like to consider which throws use outside of the collection of R-$\int$ functions; Just like consideration of $\sqrt{2}$ throws us outside of the rationals. Unfortunately, I can't pick out a function which is not R-$\int$ which should be and is clearly important, nor can I pick out the limit process which yields the Lebesque integral functions from the R-$\int$'able ones at this point. So what follows is meerly an indication of the problem.

EX 16. Let $\mathbb{Q}$ = the set of rationals and let $\chi_{\mathbb{Q}}$ be the function which is 1 at each rational and zero at each irrational. It is well-known that $\chi_{\mathbb{Q}}$ is not R-$\int$ (say on $[0,1]$ yet $\chi_{\mathbb{Q}}$ is the point-wise limit of R-$\int$'able functions with zero integral.

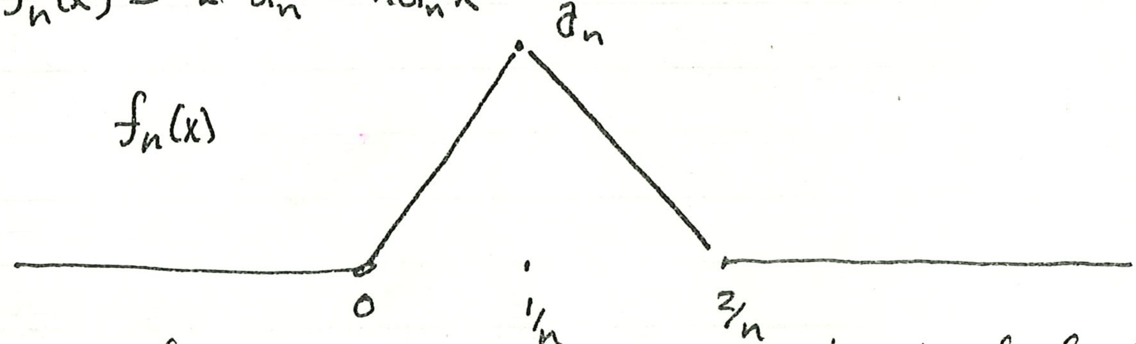
DN 17: $f_n(x) \to f(x)$ pointwise (on $[a,b]$) means that for each ξ ($\xi \in [a,b]$) $\lim_{n\to\infty} f_n(\xi) = f(\xi)$.

EX 18: The fact that R-$\int$ are closed with respect to uniform limits (F14) is not good enuff, because uniform limits are too strong. Consider $f_n(x) = x^n$ on $[0,1]$ then $f_n(x) \to f(x)$ pointwise where $f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$. Furthermore $\lim_{n\to\infty} \int_0^1 f_n(x) = \int_0^1 f(x) = 0$.

but the convergence of $f_n(x) \to f(x)$ is not uniform on $[0,1]$.

EX19: However pointwise convergence is not quite the correct limit process to consider (thought it is with some limitations) consider the following functions. Let $\{a_n\}$ be a sequence of positive reals and let $f_n(x)$ be the function which is zero except on $[0, \frac{2}{n}]$. For $0 \le x \le \frac{1}{n}$ let $f_n(x) = n a_n x$ and for $\frac{1}{n} \le x \le \frac{2}{n}$ let

$$f_n(x) = 2a_n - na_n x$$



The zero function is the pointwise limit of $f_n(x)$ (no matter what the a_n are) and $\int f_n(x) = \frac{a_n}{n}$.

Thus if $\lim \frac{a_n}{n} = 0$ this is like EX 18, but if $a_n = n^2$ then $\lim_{n\to\infty} \int f_n(x) = +\infty \neq \int 0 = 0$.

R-∫ vs Lebesque integration

At this point the best way to describe the difference between Riemann & Lebesque Integration is the following two ways to count your money. Both Riemann & Lebesque have horded alot of money. Each keeps his money in stacks of coins all identical. Riemann counts his coins by totalling the number of coins in each stack and then adding them to get a grand total. Lebesque counts the number of stacks with the same height and then sums up their totals to get a grand total.

Both ways give the same answers for coins, but the Lebesque method is more general for functions. Unfortunately, the Lebesque method requires a good bit of preperation as the next few lectures will indicate.

Appendix I: Quantifers

The use of quantifers is important in Real Analysis. Quantifers come in two flavors: the universal "$\forall$" which is read "for all", or "for every", or "for each"; and the existential "$\exists$" which is read "there exists", or "there is", or "for some".

This appendix is concerned with two things. The first is how to correctly negate a statement that contains quantifers. The second is an example showing the important of having the correct quantifer and having the quantifers in the correct order. Students in the past have made many a serious blunder underestimating the importance of this lesson.

For concreteness, we will ~~apply~~ use the statement that the function $f(x)$ {which is a real-valued function of a real variable} is continuous at the point ξ. Let $P(f, \varepsilon, \delta)$ be the statement

$$(1) \quad \forall x \; (|x-\xi|<\delta \Rightarrow |f(x)-f(\xi)|<\varepsilon)$$

Then the definition of continuity of f at ξ is

$$(*) \quad \forall \varepsilon>0 \; \exists \delta>0 \; P(f, \varepsilon, \delta).$$

Negation:

To negate a statement of the form $\forall x\ S(x)$ or $\exists x\ T(x)$, we use the following rules

$$(A) \quad \text{not}\ (\forall x\ S(x)) \equiv \exists x\ (\text{not}\ S(x))$$

$$(B) \quad \text{not}\ (\exists x\ T(x)) \equiv \forall x\ (\text{not}\ T(x))$$

where the symbol "$\equiv$" means "is identical to" or "is equivalent to".

Thus the statement that $f(x)$ is not continuous at ξ is equivalent (by (A) & then (B)) to (2) or (3)

$$(2) \quad \exists \varepsilon > 0\ \forall \delta > 0\ (\text{not}\ P(f, \varepsilon, \delta))$$

$$(3) \quad \exists \varepsilon > 0\ \forall \delta > 0\ \exists x\ (|x - \xi| < \delta \text{ and } |f(x) - f(\xi)| \geq \varepsilon).$$

Order and Choice:

Our purpose here is to show that each of other seven possible order & choices of the quantifers for (*) defind properties of $f(x)$ different from continuity. The eight choices are $\forall \varepsilon > 0\ \forall \delta > 0$, $\forall \varepsilon > 0\ \exists \delta > 0$, $\exists \varepsilon > 0\ \forall \delta > 0$, $\exists \varepsilon > 0\ \exists \delta > 0$, $\forall \delta > 0\ \forall \varepsilon > 0$, $\forall \delta > 0\ \exists \varepsilon > 0$, $\exists \delta > 0\ \forall \varepsilon > 0$, $\exists \delta > 0\ \exists \varepsilon > 0$.

First we construct some functions that will show the difference. Let $\xi = 0$ and let

$$g(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$$h(x) = x \qquad\qquad k(x) = \begin{cases} 0 & |x| < 1 \\ x^2 - 1 & \text{otherwise} \end{cases}$$

$$j(x) = \frac{1}{x-1}$$

I. The two statements $\forall_{\varepsilon>0}\,\forall_{\delta>0}$ and $\forall_{\delta>0}\,\forall_{\varepsilon>0}$ are logically the same and the statement $\forall_{\varepsilon>0}\,\forall_{\delta>0}\, P(f, \varepsilon, \delta)$ singles out exactly the constant functions.

Proof. It is easy to see that constant functions satisfy the statement. To see the converse, suppose $f(x)$ is not constant and let $f(\eta) \neq f(\xi)$. Then f does not satisfy $P(f, \varepsilon, \delta)$ when $\varepsilon = (|f(\eta) - f(\xi)|)/2$ and $\delta = 2|\eta - \xi|$.

example. The function $h(x)$ is a function which is continuous at $\xi = 0$ but which fails I.

II The two statements $\exists_{\varepsilon>0}\,\exists_{\delta>0}$ and $\exists_{\delta>0}\,\exists_{\varepsilon>0}$ are logically the same and the statement $\exists_{\varepsilon>0}\,\exists_{\delta>0}\, P(f, \varepsilon, \delta)$ singles out exactly the functions which are bounded on some neighborhood of ξ.

proof: If ε, δ work for the statement then f is bounded on $\{x : |x-\xi| < \delta\}$ by the bound $B = \varepsilon + |f(\xi)|$. Conversely, if B is a bound for f on $\{x : |x-\xi| < \delta\}$, then $\varepsilon = B + |f(a)|$ works for the statement.

example: The function $g(x)$ is not continuous at $\xi = 0$, but which satisfies the statement of II.

III $\exists_{\varepsilon>0} \forall_{\delta>0} \; P(f,\varepsilon,\delta)$ singles out the bounded functions

proof: Is like case III

example: again $g(x)$ is discontinuous at 0 but bounded. Note that h & j satisfy II but not III

IV $\exists_{\delta>0} \forall_{\varepsilon>0} \; P(f,\varepsilon,\delta)$ singles out the functions which are constant on some neighborbood of ξ (namely $\{x : |x-\xi| < \delta\}$). The proof is similar to I.

example: $h(x)$ is continuous at 0, but does not satisfy IV. Note that $k(x)$ satisfies IV but not I.

V $\forall_{\delta>0} \exists_{\varepsilon>0} P(f,\varepsilon,\delta)$ singles out the functions which are bounded on each neighborhood of ξ of the form $\{x : |x-\xi| < \delta\}$. The proof is much like that of II.

<u>examples</u>: $g(x)$ is discontinuous but satisfies V. Note that $j(x)$ satisfies II but not V since it is bounded on $\{x : |x| < \frac{1}{2}\}$ but not on $\{x : |x| < 1\}$.

VI $\forall_{\varepsilon>0} \exists_{\delta>0} P(f,\varepsilon,\delta)$ is the definition of continuity.

A Logical chart of the properties I–VI is given below. The implications given are the only ones which are true.

$$\begin{array}{ccccc} \mathrm{I} & \Rightarrow & \mathrm{IV} & \Rightarrow & \mathrm{VI} \\ \Downarrow & & & & \Downarrow \\ \mathrm{III} & \Rightarrow & \mathrm{V} & \Rightarrow & \mathrm{II} \end{array}$$

Appendix I½ negation of mathematical statements

Table I summarizes the results

TABLE I

	Statement	negation
	A	not A
	not A	A
	A and B	(not A) or (not B)
	A or B	(not A) and (not B)
(*)	$A \Rightarrow B$	(not B) and A
	$\forall x\ P(x)$	$\exists x$ not $P(x)$
	$\exists x\ P(x)$	$\forall x$ not $P(x)$

Only the row marked (*) needs explaining. Consider how you would show that A or B is true. There are two ways to procede, (1) you could assume not A is true and use it to show B is true or (2) vice versa, i.e. show that not B is true $\Rightarrow$ A is true. Now the row mark (*) becomes clear if we note $[A \Rightarrow B]$ is logically the same as $[(\text{not } A) \text{ or } B]$. We could prove the second statement by showing not not $A \equiv A$ is true implies B is true or by showing not B implies not A; clearly each of these are the same as proving $A \Rightarrow B$.

Finally, we note that [not (A and B)] can be written as either $[A \Rightarrow (\text{not } B)]$ or $[B \Rightarrow (\text{not } A)]$. Furthermore it is the only negation (of the statements in Table I) that can be written using $\Rightarrow$ with any ease.

Lecture 2: Sets and Algebras 9/27/78

DN 1: If A and B are sets, then $A \cup B$, read A union B, is the set

$$\{x : x \in A \text{ or } x \in B\}.$$

DN 2: If A and B are sets, then $A \cap B$, read A intersect B, is the set

$$\{x : x \in A \text{ and } x \in B\}.$$

DN 3: The complement of B relative to A is the set

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

DN 4: The symmetric difference of A and B is the set

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

DN 5: If $\forall x \in A$, $x \in B$, then $A \subset B$, read A is a subset of B. $A \subset B$ if $x \in A$ implies $x \in B$, or equivalently, $x \notin B$ implies $x \notin A$.

DN 6: If A is understood to be a subset of some well-defined universal set X, we can speak of A complement, or the complement of A in X, written

$$A^c = X \setminus A.$$

DN 7: If X is a set, the power set of X is the set

$$P(X) = \{A : A \subset X\}.$$

DN 8: The set containing no elements is called the empty set, denoted ϕ.

DN 9: The Cartesian Product of A and B, denoted $A \times B$, is the set of ordered pairs

$$\{(a,b) : a \in A,\ b \in B\},$$

where $(a,b) = (c,d)$ iff $a = c$ and $b = d$.

We can generalize our ideas of unions and intersections of sets as follows:

Let I be an indexing set, and let

$$\mathcal{A} = \{A_\lambda : A_\lambda \text{ is a set } \forall \lambda \in I\}.$$

Then we define

$$\bigcup_{\lambda \in I} A_\lambda = \bigcup_{A \in \mathcal{a}} A = \bigcup \mathcal{a} = \{x : x \in A_\lambda \text{ for some } \lambda \in I\}.$$

If $\mathcal{a} \neq \emptyset$, we define

$$\bigcap_{\lambda \in I} A_\lambda = \bigcap_{A \in \mathcal{a}} A = \bigcap \mathcal{a} = \{x : x \in A_\lambda \ \forall \lambda \in I\}.$$

<u>Thm 10</u>: (De Morgan's Laws). Let $A_\lambda \subseteq X$, $\forall \lambda \in I$.

(1) $\left(\bigcap_{\lambda \in I} A_\lambda\right)^c = \bigcup_{\lambda \in I} A_\lambda^c$.

(2) $\left(\bigcup_{\lambda \in I} A_\lambda\right)^c = \bigcap_{\lambda \in I} A_\lambda^c$.

<u>Proof of (1)</u>: $x \in \left(\bigcap_{\lambda \in I} A_\lambda\right)^c$ iff $x \notin \bigcap_{\lambda \in I} A_\lambda$.

But this is true iff for some $\lambda \in I$, $x \notin A_\lambda$. This last statement is equivalent to the statement that for some $\lambda \in I$, $x \in A_\lambda^c$, and this is equivalent to $x \in \bigcup_{\lambda \in I} A_\lambda^c$.

<u>DN 11</u>: Two sets A and B are said to have the same <u>cardinality</u> if $\exists$ a 1-1 and onto function $f : A \longrightarrow B$. A set is <u>finite</u> if it has the same cardinality as

one of the following sets:

$$\phi, \{1\}, \{1,2\}, \dots, \{1,2,\dots,n\}.$$

<u>DN 12</u>: If A and B are non-empty sets, we say that <u>card $A \leq$ card B</u> iff $\exists$ function f mapping ~~B~~ B onto A.

If $A \subset B$, card $A \leq$ card B, for fix $a \in A$, and define $f: B \longrightarrow A$ by

$$f(x) = \begin{cases} x, & x \in A \\ a, & x \notin A. \end{cases}$$

Then f is certainly onto.

<u>Thm 13</u>: If A and B are sets and card A $\leq$ card B and card $B \leq$ card A, then card $A =$ card B.

<u>DN 14</u>: A nonempty set $\mathcal{A} \subset P(X)$ (i.e., the elements of $\mathcal{A}$ are subsets of X) is ~~set~~ said to be an <u>Algebra</u> (<u>of sets</u>) if

(1) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$,
(2) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$,
(3) $A \in \mathcal{A} \Longrightarrow X \setminus A \in \mathcal{A}$.

That is, $\mathcal{A}$ is closed with respect to unions, intersections, and complements.

Observe that inductively we have

(1) is equivalent to $\mathcal{A}$ is closed with respect to finite unions, i.e., $A_1, A_2, \ldots, A_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$;

(2) is equivalent to $\mathcal{A}$ is closed with respect to finite intersections.

Also,

(*) (1) and (3) $\Rightarrow$ (2),
(2) and (3) $\Rightarrow$ (1).

Proof of (*): $A, B \in \mathcal{A}$, then by (3), $A^c, B^c \in \mathcal{A}$. By (1), we have $A^c \cup B^c \in \mathcal{A}$, and by (3) and De Morgan's laws, $(A^c \cup B^c)^c = A^{cc} \cap B^{cc} = A \cap B \in \mathcal{A}$.

Note that for any algebra $\mathcal{A}$, $\emptyset$ & $X \in \mathcal{A}$.

Examples of Algebras

1. If X is a set, then $P(X)$ is an Algebra.
2. $\{\emptyset, X\}$ is an algebra for any set X.
3. $\mathcal{A} = \{A : A \in P(X)$ and either A is a finite set or $X \setminus A$ is finite$\}$.

(If $X \setminus A$ is finite, A is called cofinite.)

Check: <u>item (3)</u>: If A is finite, then $X \setminus A$ is cofinite. Hence $X \setminus A \in \mathcal{a}$.
If A is cofinite, then $X \setminus A$ is finite, and $X \setminus A \in \mathcal{a}$.
<u>item (1)</u>: If A and B are both finite, $A \cup B$ is finite, and therefore in $\mathcal{a}$.
If at least one of A and B is cofinite — say $A = X \setminus F$, where F is finite —, then $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \subset X \setminus A = F$, a finite set. Thus $A \cup B$ is cofinite and therefore in $\mathcal{a}$.

4. Let $X = \mathbb{R}^1$, the real numbers. Let

$\mathcal{a}_1$ = { all subsets of $\mathbb{R}^1$ that can be written as a finite disjoint union of sets of the type P and Q }

$P = [a, b)$, where $-\infty < a < b$, and $b \in \mathbb{R}^1$ or $b = +\infty$

$Q = (-\infty, c)$, where $c \in \mathbb{R}^1$ or $c = +\infty$.

<u>Note</u>: The collection of sets $\{A_\lambda : \lambda \in I\}$ is <u>disjoint</u> if $A_\alpha \cap A_\beta \neq \emptyset \Rightarrow \alpha = \beta$.

5. Let $X = \mathbb{R}^2$, the cartesian plane, and let

$\mathcal{A}_2 = \{$ all subsets of $\mathbb{R}^2$ that can be written as a finite disjoint union of sets $A \times B$, where A and B are sets of the type P or Q in example 4 above. $\}$

Addendum to Lecture 1:

Ex 19, line 7 reads $f_n(x) = na_n$, should read $f_n(x) = na_n x$.

DN 1 : $C_1 = \{\phi\} \cup \{(-\infty, b) : -\infty < b \leq +\infty\}$

$\cup \{[a, b) : -\infty < a < b \leq +\infty\}$

DN 2 : $C_2 = \{A \times B : A \,\&\, B \text{ both in } C_1\}$

ℓ_i = all finite disjoint unions of sets in C_i, $i = 1, 2$

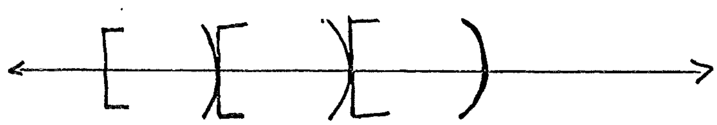
[i.e. all finite unions of sets in C_1]

Claim ℓ_i is an algebra on $\mathbb{R}^i$, $i = 1, 2$.

1^{st} Claim if $A \in \ell_1$, $A \neq \phi$ then A has a "best" form

i.e. $A = [a_1, b_1) \cup [a_2, b_2) \cup \ldots [a_n, b_n)$ where

$a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n$



"best" form

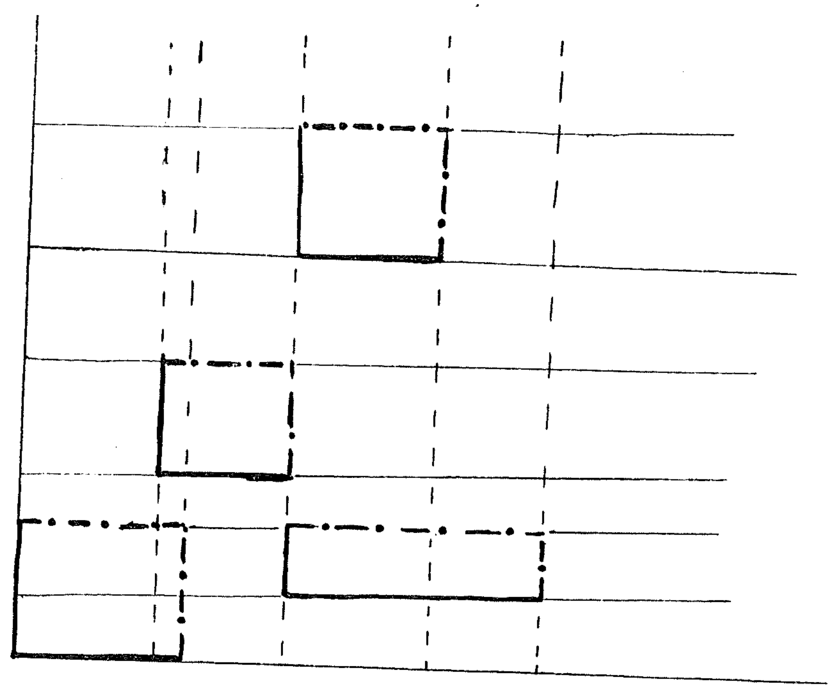
If $A \in \ell_1$ is in best form then $A \cup [a, b) \in \ell_1$

To verify ℓ_1 is an algebra one must show $A \cup B$ is in ℓ_1 if $A \,\&\, B$ are in ℓ_1; show $A \cap B$ and $R^1 \setminus A$ are both in ℓ_1 where $A, B \in \ell_1$.

Likewise for ℓ_2, though we delete the proof due to the

cases involved of each

An idea of the proof is to extend all horizontal and vertical boundaries, thus dividing the plane into rectangles either completely contained in or disjoint from the given sets.



Thm 3: Let $C \subset P(\bar{X})$, $C \neq \emptyset$, then there is a smallest algebra $\mathcal{A}(C)$ w/resp to $\mathcal{A}(C) \supset C$. [If $\mathcal{B}$ is any other algebra with $\mathcal{B} \supset C$ then $\mathcal{B} \supset \mathcal{A}(C)$]

Pf: Let $\{\mathcal{B}$ be algebra on $\bar{X}$ with $\mathcal{B} \supset C\} = \Lambda$

$\Lambda \neq \emptyset$ because $P(\bar{X})$ is algebra $\supset C$

$$\mathcal{A}(C) = \bigcap_{\mathcal{B} \in \Lambda} \mathcal{B}$$

Claim this $\mathcal{A}(C)$ works.

$\forall \mathcal{B} \in \Lambda$, $\mathcal{A}(\mathcal{C}) \subset \mathcal{B}$ by definition

$\mathcal{A}(\mathcal{C}) \supset \mathcal{C}$, take $C_1 \in \mathcal{C}$ then $C_1 \in \mathcal{B}$ for each $\mathcal{B} \in \Lambda$, thus $C_1 \in \bigcap_{\mathcal{B} \in \Lambda} \mathcal{B} = \mathcal{A}(\mathcal{C})$ $\therefore$ $\mathcal{C} \subset \mathcal{A}(\mathcal{C})$

Show $\mathcal{A}(\mathcal{C})$ is algebra

$A \in \mathcal{A}(\mathcal{C})$, show $\bar{X} \setminus A \in \mathcal{A}(\mathcal{C})$, But $A \in \mathcal{A}(\mathcal{C}) \Rightarrow$ $A \in \mathcal{B}$ for each $\mathcal{B} \in \Lambda$ since $\mathcal{B}$ is an algebra, $\bar{X} \setminus A \in \mathcal{B}$ for each $\mathcal{B} \in \Lambda$, thus $\bar{X} \setminus A \in \bigcap_{\mathcal{B} \in \Lambda} \mathcal{B} =$ $\mathcal{A}(\mathcal{C})$

$A, B \in \mathcal{A}(\mathcal{C}) \Rightarrow \forall \mathcal{B} \in \Lambda$, $A, B \in \mathcal{B}$ since $\mathcal{B}$ is algebra $A \cap B \in \mathcal{B}$. Thus $A \cap B \in \mathcal{A}(\mathcal{C}) = \bigcap_{\mathcal{B} \in \Lambda} \mathcal{B}$

<u>EX 4</u>

1. $\{\bar{X}, \phi\} = \mathcal{A}(\{\phi\})$.
2. $\{$finite and Co-finite sets$\} = \mathcal{A}(\{\{x\} : x \in \bar{X}\})$
3. $\mathcal{C}_i = \mathcal{A}(\mathcal{C}_i)$, $i = 1, 2$.

<u>DN 5</u> $\quad$ Card $\phi = 0$

DN 6 $\quad$ If A is finite, card $A = n$ where n is the number of elements in A.

F7: Card$(\mathbb{N}) = \aleph_0$ (aleph-null) ; $\mathbb{N}$ is the set of integers ≥ 0.

DN 8: A set A is said to be infinite if it is not finite.

DN 9: A set A is countable if card $A \leq$ Card$(\mathbb{N})$

[i.e. There is an onto fcn, $f : \mathbb{N} \longrightarrow A$]

DN 10: A set is said to be countably infinite if it is both countable and infinite.

DN 11: A set is said to be uncountable if it is not countable.

EX 12 $[0, 1]$ is uncountable.

Card $[0, 1] = c$, c is the cardinality of the continuum

F13: If $A \subset B$ and B is countable, so is A.

F14: If $\{A_n : n \in \mathbb{N}\}$ is a sequence of countable sets then $\bigcup_{n \in \mathbb{N}} A_n$ is countable. [Countable union of countable sets is countable]

Pf:

$$\mathbb{N} \xrightarrow{f} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \ \cdots \\ (1)\ x_1 & (3)\ y_1 & (6)\ z_1 & \\ (2)\ x_2 & (5)\ y_2 & z_2 & \\ (4)\ x_3 & \vdots & z_3 & \\ \vdots & & \vdots & \end{array}$$

Want to find f

$x_i = i$ element of the j^{th} set. Find n so that

$f(n) = x_i$

$$n = \frac{(i+j-2)(i+j-1)}{2} + j$$

Rationals are countable :

$(0,0) \quad (1,0) \quad (2,0) \quad \cdots$

$(0,1) \quad (1,1) \quad (2,1) \quad \cdots$

$\vdots \quad (1,2) \quad \vdots$

$(1,3)$

$\vdots$

$$(p,q) \longrightarrow \frac{p}{q}$$

This shows the positive rationals are countable. The same argument works for the negative rationals, and by F14; the rationals are countable [Being the union of two countable sets].

<u>DN 15:</u> $\mathcal{S} \subset \mathcal{P}(\mathbb{X})$; $\mathcal{S} \neq \emptyset$, is said to be a σ- algebra on $\mathbb{X}$ if

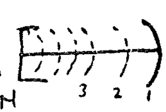
1. $A \in \mathcal{S} \Rightarrow \mathbb{X} \setminus A \in \mathcal{S}$
2. $\{A_n\} \subset \mathcal{S} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{S}$
3. $\{A_n\} \subset \mathcal{S} \Rightarrow \bigcap A_n \in \mathcal{S}$

(1) & (2) $\Longrightarrow$ (3)
(1) & (3) $\Longrightarrow$ (2)   by De Morgan's Laws.

<u>Pf</u>: Same as for algebras.

<u>Thm 16</u>: If $C \neq \phi$, $C \subset \mathcal{P}(\bar{X})$, then there is a smallest σ-algebra $S(C)$ w/resp to $S(C) \supset C$. [i.e. If $\mathcal{B}$ is any other σ-algebra with $\mathcal{B} \supset C$, then $\mathcal{B} \supset S(C)$]

<u>Ex 17</u>:

1. $\mathcal{P}(\bar{X})$ is a σ-algebra
2. $\{\underline{X}, \phi\}$ is the smallest σ-algebra
3. $\{$countable or co-countable sets$\}$ where A is co-countable if $\bar{X} \setminus A$ is countable
4. $\mathcal{B}_i$, called Borel sets in $\mathbb{R}^i$, $i = 1, 2$. is the σ-algebra generated by $\mathcal{C}_i$, $i = 1, 2$. $\mathcal{B}_i$ is much larger than $\mathcal{C}_i$, for example, one point sets $\bigcap_{i\in\mathbb{N}}$  belong to $\mathcal{B}_i$

Test Problem #2 Due 6 Oct 1978

1. ~~Show~~ If f is a function $f: X \to Y$ and $B \subset Y$ define $f^{-1}(B) = \{x \in X : f(x) \in B\}$, if $\mathcal{B} \subset P(Y)$

1.5pts define $f^{-1}(\mathcal{B}) = \{f^{-1}(B) : B \in \mathcal{B}\}$. Show

A. $f^{-1}(Y \setminus B) = X \setminus (f^{-1}(B))$.

B. $f^{-1}(\cup \mathcal{B}) = \cup f^{-1}(\mathcal{B})$

C. If $\mathcal{B}$ is a σ-algebra on Y then $f^{-1}(\mathcal{B})$ is a σ-algebra on X

2. If $f: X \to Y$ is a function and $\mathcal{A}$ & $\mathcal{B}$ are

2.0pts σ-algebra's on X & Y respectively, we will say that f is $\mathcal{A}$-$\mathcal{B}$ measurable if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. Suppose $\mathcal{C}$ is a collection of subsets of Y which generate $\mathcal{B}$ [i.e. $\mathcal{B}$ is $S(\mathcal{C})$, the smallest σ-algebra which contains $\mathcal{C}$]. Show f is $\mathcal{A}$-$\mathcal{B}$ measurable, if and only if for each $C \in \mathcal{C}$, $f^{-1}(C) \in \mathcal{A}$.

(HINT: Use the principle of good sets, that is look at $\mathcal{D} = \{D \subset Y : f^{-1}(D) \in \mathcal{A}\}$.)

3. A non-empty $\mathcal{M} \subset P(X)$ is said to be a <u>monotone class</u> if (1) & (2) hold

(1) $\{A_n\} \subset \mathcal{M}$ and $A_1 \subset A_2 \subset \dots \subset A_n \subset A_{n+1} \subset \dots$ † imply that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$

(2) $\{B_n\} \subset \mathcal{M}$ and $B_1 \supset B_2 \supset \dots \supset B_n \supset B_{n+1} \supset \dots$ * imply that $\bigcap_{n=1}^{\infty} B_n \in \mathcal{M}$.

† this is called $\{A_n\}$ is a nested increasing sequence of sets

* this is called $\{B_n\}$ is a nested decreasing sequence of sets

0.5 pts (A) Show that if $\mathcal{C}$ is a non-empty $\subset \mathcal{P}(X)$ then there is a smallest monotone class $\mathcal{M}(\mathcal{C})$ containing $\mathcal{C}$

2.0 pts (B) Show that if $\mathcal{C}$ is closed with respect to complements [i.e. $C \in \mathcal{C} \Rightarrow X \setminus C \in \mathcal{C}$], then $\mathcal{M}(\mathcal{C})$ is closed with respect to complements

(Hint: Look at the good sets of $\mathcal{M}(\mathcal{C})$)

1.5 pts 4. If $\{A_n\} \subset \mathcal{P}(X)$ define

$$\limsup A_n = \bigcap_{i=1}^{\infty} \left[\bigcup_{j=i}^{\infty} A_j \right] \quad \text{and}$$

$$\liminf A_n = \bigcup_{i=1}^{\infty} \left[\bigcap_{j=i}^{\infty} A_j \right].$$ If $\limsup A_n = \liminf A_n = A$ we will say that $\lim A_n = A$. Show

(A) $\limsup A_n = \{x \in X : x \in A_n \text{ for infinitely many } n\}^{\dagger}$

(B) $\liminf A_n = \{x \in X : \text{there is an } N \text{ so that } n \geq N \Rightarrow x \in A_n\}$

(C) $\liminf A_n \subset \limsup A_n$

2.5 pts 5. Show that if $\{A_n\} \subset \mathcal{P}(X)$ then $\lim A_n = A$, if and only if $\lim (A_n \triangle A) = \emptyset$.

$\dagger$ Note that a subset A of $\mathbb{N}$ is infinite, if and only if for each $n \in \mathbb{N}$ there is an $m \geq n$ with $m \in A$.

$\boxed{10/2}$

(Advance form of Fund. Principle of Analysis)

LM 1: If a & b are no.

If $f(\varepsilon)$ is a function of ε alone

$f(\varepsilon) > 0$ for $\varepsilon > 0$

If $\lim_{\varepsilon \to 0^+} f(\varepsilon) = 0$

If for all $\varepsilon > 0$ $a \geq b - f(\varepsilon)$

Then $a \geq b$

proof: $\forall \delta > 0$, $\exists \varepsilon > 0$ s.t $f(\varepsilon) < \delta$

Want to show $\forall \delta > 0$ $a \geq b - \delta$

$$b - \delta \leq b - f(\varepsilon) \leq a$$

Notation: $\mathbb{R}^1$ reals $\mathbb{R}^2$ real plane

extended reals $\mathbb{R}^*$ — all real no.'s and ∞ & $-\infty$

$r \in \mathbb{R}$

$r + \infty = \infty$

$r - \infty = -\infty$

$r\infty = \infty$ if $r > 0$, $r\infty = -\infty$ if $r < 0$

$\infty + \infty = \infty$

$-\infty - \infty = -\infty$

$\infty - \infty$ undefined

$0 \cdot \infty = 0$ by convention in all measure theory sup's & inf's

THM 2: If S is a subset of $\mathbb{R}$ or $\mathbb{R}^*$, then
sup S always exists as an extended reals
Same for inf's ;

DN 3: $\sup S = +\infty$
if $\forall n \in \mathbb{N} \quad \exists s \in S \quad s \geq n$
(unbounded on the right)

DN 4: $\sup S = A \in \mathbb{R}$
if (1) for each $s \in S$, $s \leq A$
(A is an upper bound for the set S)

and (2) $\forall \epsilon > 0$, there is $s \in S$ with $s > A - \varepsilon$
[(1) and (2) A is the smallest upper bound of S]

** $\sup S = -\infty \iff S = \phi$ or $S = \{-\infty\}$
$\inf S := -\sup(\{-s : s \in S\})$

DN 5: I be the collection of open intervals of $\mathbb{R} \sqcup \{\phi\}$
~~A~~ $A \in I \iff A = (a, b)$ w/ $a < b$ or $A = \phi$

DN 6: J be the collection of subsets of the plane of the form
$A \times B$: $A, B \in I$
J - set of open boxes of the plane $\sqcup \{\phi\}$

DN 7: If $A = (a,b) \in I$

$\ell(A) = b-a$ length of A

$\ell(\phi) = 0$

If $B = (a,b) \times (c,d) \in J$

area of B : $a(B) = (b-a)(d-c)$

$a(\phi) = 0$

---

o, Lebesgue ~~measure~~ outer measure

$m_1^* : P(\mathbb{R}^1) \longrightarrow \mathbb{R}^*$

$m_2^* : P(\mathbb{R}^2) \longrightarrow \mathbb{R}^*$

DN 8: If $S \subseteq \mathbb{R}$,

$m_1^*(S) = \inf \{ \sum_{n=1}^{\infty} \ell(A_n) : \text{where } A_n \in I \text{ \& } S \subset \bigsqcup_{n=1}^{\infty} A_n \}$

$T \subseteq \mathbb{R}^2$

$m_2^*(T) = \inf \{ \sum_{n=1}^{\infty} a(B_n) : \text{where } B_n \in J \text{ and } T \subset \bigsqcup_{n=1}^{\infty} B_n \}$

DN 9: $\mathcal{N}_i = \{ A \subseteq \mathbb{R}^i : m_i^*(A) = 0 \}$

$\mathcal{N}_i$ are called the sets of measure zero (null sets)

Prop. 10: $A \in \mathcal{N}_1$ iff for each $\varepsilon > 0$, there are $A_n \in I$ with

$\sum_{n=1}^{\infty} \ell(A_n) < \varepsilon$ & $A \subset \bigsqcup_{n=1}^{\infty} A_n$

Corollary: if $A \subseteq B$ & B is null

then A is a null set

Examples 11: 1. ϕ is null

2. singletons are null

$\{r\} \subset (r-\frac{\epsilon}{2}, r+\frac{\epsilon}{2}) = A_1$

let $A_n = \phi$. $n \geq 2$

Lemma 12: If $\{B_n\}_{n=1}^{\infty}$ is a sequence of null sets, then $\bigcup_{n=1}^{\infty} B_n$ is null

(Count union of sets of measure zero has measure zero)

proof: Let $\varepsilon > 0$ be given.

Since B_n has meas zero, there are $\{U_i^n\}_{i=1}^{\infty} \subset I$

so that $B_n \subset \bigcup_{i=1}^{\infty} U_i^n$

and $\sum_{i=1}^{\infty} \ell(U_i^n) < \epsilon/2^n$

$\{U_i^n ;\ n=1,2,\cdots ;\ j=1,2,\cdots\}$

a countable set

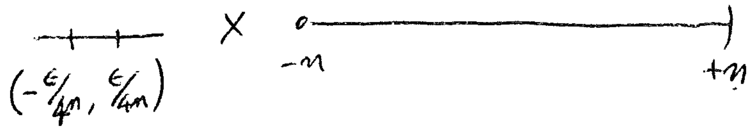
$$\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \ell(U_i^n) < \sum_{n=1}^{\infty}(\varepsilon/2^n) = \varepsilon \cdot \sum_{n=1}^{\infty}\frac{1}{2^n} = \varepsilon.$$

examples 13. Countable sets have measure zero

4. In the plane, the set $\{(r,0) : r \in \mathbb{R}^1\}$ has measure zero

Pf: It suffices to show that for each pos. integer n, the set $\{(r,0) \in \mathbb{R}^2 : |r| < n\}$ has measure zero.

Let $\varepsilon > 0$ be given



$m^*((0,1)) = 1$? hard to prove

DN 13: $\mathcal{L}_i = \{ A \subset \mathbb{R}^i : \exists B \in \mathcal{B}_i$ with $m_i^*(A \triangle B) = 0 \}$, $i = 1, 2, \dots$

$m_i^* |_{\mathcal{L}_i}$ $i = 1, 2, \dots$ is Lebesgue measure

5^{th} meeting.

Oct. 4 '78. Wed.
MAT 531 Fall.
Jill Perzley & Ding Huang

In $\mathbb{R}^1$ $I = \{(a,b) : a \le b\}$ (includs void set) is the basic open set in $\mathbb{R}^1$.

In $\mathbb{R}^2$ $\mathcal{J} = \{(a,b)\times(c,d) : a \le b,\ c \le d\}$ basic open sets in $\mathbb{R}^2$.

<u>Def</u> <u>Open set</u>.

(i) If for each $x \in U$ there is a basic open set I with $x \in I \subset U$.

or (ii) (a). $\mathbb{R}^1$: If for each $x \in U$, there is $\epsilon > 0$ s.t. $|x-y| < \epsilon \Longrightarrow y \in U$

(b) $\mathbb{R}^2$: If for each $(x,y) \in U$, there is $\epsilon > 0$, s.t. $|w-x| < \epsilon$ & $|z-y| < \epsilon$ $\Longrightarrow (w,z) \in U$

(i) & (ii) are equivalent.

<u>Prop</u>

U is open iff it is a union of basic open sets.

<u>Pf</u>:

Suppose U is open, then for each $x \in U$ Choose I_x basic open set. $x \in I_x \subset U$.

$\bigcup_{x \in U} I_x$ this set is U. Since $I_x \subset U \Rightarrow \bigcup_{x \in U} I_x \subset U$

Conversely, since $x \in U \Rightarrow x \in I_x$ & hence $x \in \bigcup_{x \in U} I_x$. We have $U \subset \bigcup_{x \in U} I_x$, i.e. we proved these two sets are equal.

<u>Prop:</u>

The union of a collection of open sets is open.

<u>Pf.</u>

$\mathcal{O} = \{U :$ some open sets$\}$

$V = \bigcup_{U \in \mathcal{O}} U$, if $V = \phi$, it is open.

Otherwise, let $x \in V$, then $x \in U$ for some $U \in \mathcal{O}$. Since U is an open set, there exist basic open set I with $x \in I \subset U$.

But, $U \in V$, so $x \in I \subset V$. And V is open.

<u>Prop.</u>

If U & V are open, then $U \cap V$ is open.

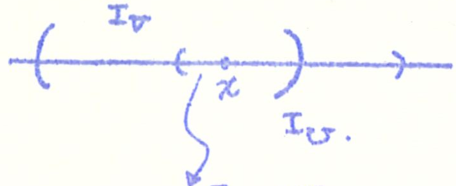
<u>Pf.</u>

If $U \cap V = \phi$, it is open.

Otherwise, let $x \in U \cap V$.

Since U is open (V is open), there exist basic open set I_U (I_V), s.t. $x \in I_U \subset U$ ($x \in I_V \subset V$).

Then, look at $I_U \cap I_V$.



$I_U \cap I_V$ is a basic open set.

And since $I_U \subset U$,

$$(I_U \cap I_V) \subset I_U \subset U$$
$$(I_U \cap I_V) \subset I_V \subset V.$$

thus, $x \in (I_U \cap I_V) \subset (U \cap V)$.

& $U \cap V$ is open.

<u>Note</u>:

1. ϕ is open, the whole space $\mathbb{R}^1$ or $\mathbb{R}^2$ is open.
2. Finite intersections of open sets are open.
3. The union of any number of open sets is an open set.

<u>Def</u>: <u>G_δ-set</u>

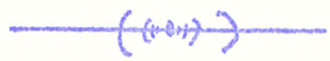
A set in $\mathbb{R}^1$ (or $\mathbb{R}^2$) is a G_δ (gee-delta) set if it is the intersection of countably many open sets in $\mathbb{R}^1$.

Examples of G_δ-sets.

1. Each open set is G_δ

2. Singleton are G_δ.

Pf.
$$\{x\} = \bigcap_{n \in \mathbb{N}} \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$$



3. The rationals are not G_δ-set (rationals are F_σ)

4. $$[a, b) = \bigcap_{n \in \mathbb{N}} \left(a - \frac{1}{n}, b\right)$$

— Show open intervals are Borel sets

$$B_i = S(\mathcal{J}_i)$$
$$(a, b) = \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right)$$

— $\mathcal{J}_1 = \{(a, b)$, a & b are rationals$\}$

$\mathcal{J}_2 = \{(a, b) \times (c, d)$ a, b, c, d are rationals$\}$

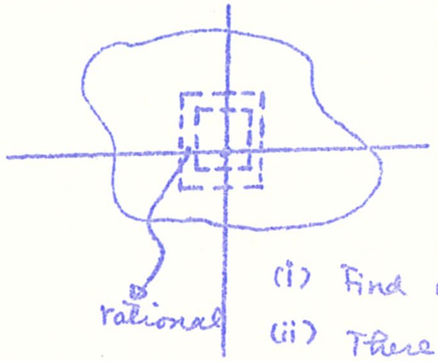
<u>Note</u>.

Each element of $\mathcal{J}_i$ is open. & $\mathcal{J}_1, \mathcal{J}_2$ are countable sets.

$$\mathcal{J}_1 = \bigcup_{a \in Q} \{(a, b) : b \in Q\}.$$ Q: rationals.

Prop.

Each open set in R^i is a union of sets in $\mathcal{J}_i$, $i=1,2$.



(i) Find a box around x.

(ii) There exist a rational number between any two reals.

Corr:

Every open set is the ~~random~~ union of a countable collection of open intervals.

Corr:

Open sets are Borel sets.

Corr:

The Borel sets is the smallest σ-algebra containing the open sets.

Pf.

Let $\mathcal{B}$ be the Borel-sets

$S(\mathcal{O})$ - smallest σ-algebra contains open sets.

Since open sets are Borel sets.

$\{\text{Open sets}\} \subset \mathcal{B}$, thus $S(\mathcal{O}) \subset \mathcal{B}$.

$S(\mathcal{O}) \supset \{(a,b) : a \le b\}$

$S(\mathcal{O}) \supset S(\mathcal{J}) = \mathcal{B}$. (Borel set).

Corr

G_δ-sets are Borel sets

Def. closed

A set F is closed iff its complement is open.

Note: $\phi = ($ ~~open~~ whole space$)^c$

whole space $= \phi^c$ are closed & open.

Arbitrary intersection of closed sets is closed & finite unions of closed sets are closed.

Pf.

Let A be a losed set. By definition,

$A = B^c$ B is some open set.

$\cap A = \cap B^c = (\cup B)^c$, by De Morgan's law. where $\cup B$ is still in open. Hence $(\cup B)^c$ is closed.

Def. F_σ-set.

A set which is the union of countably many closed sets.

Example of F_σ-sets.

1. Each closed set is F_σ
2. $\{x\}$ singleton is F_σ
3. The rationals is F_σ

Note F_σ sets are Borel sets.

Def.

In $\mathbb{R}'$, If $A \subset \mathbb{R}'$ & $x \in \mathbb{R}'$

$d(x, A) = \inf \{ |x-a| : a \in A \}$.

Lemma.

A is closed iff $\forall x \ni d(x, A) = 0 \Longrightarrow x \in A$.

(a, b) is not closed, because

$d(a, (a, b)) = 0$, but $a \notin (a, b)$

Prop.

Each closed set is a G_δ

Pf.

$$A = \bigcap_{n \in \mathbb{N}} \underbrace{\{ x : d(x, A) < \tfrac{1}{n} \}}_{= O_n},$$

$$O_n = \bigcup_{x \in A} (x - \tfrac{1}{n}, x + \tfrac{1}{n})$$

— Open sets are F_σ-sets.

— By DeMorgans Laws, it is obvious that a set is G_δ-set $\Longleftrightarrow$ its complement is an F_σ-set.

Lecture 6 $\qquad$ 10-6-78
P.1.

In $\mathbb{R}^2$ define the dist.

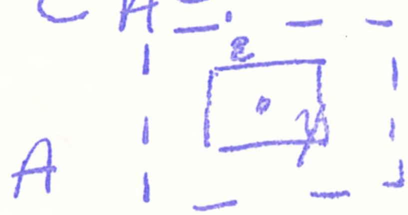
$$d((x,y),(w,z)) = \max\{|x-w|, |y-z|\}$$

PROP. A is closed $\iff$ (if $d(x,A)=0$, then $x\in A$)

FACT. $a\in A$ then $d(a,A)=0$.

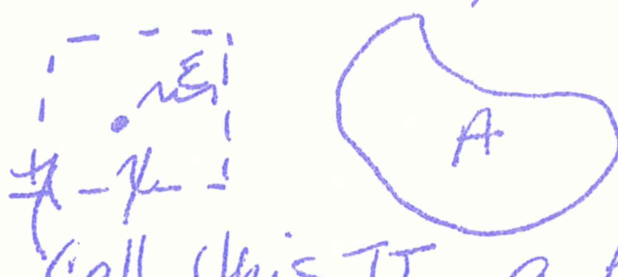
($\Longrightarrow$) $x\notin A$ $\quad x\in A^c$ $\quad A^c$ open by hypothesis

Thus, there is a basic open set U with $x\in U\subset A^c$.



So $d(x,A)>0$.

($\Longleftarrow$) $x\notin A$, thus $d(x,A)=\varepsilon>0$.



Call this U, a basic open set.

$x\in U\subset A^c$. True for arbitrary $x\in A^c$.

$\therefore$ A^c is open and A is closed.

LEMMA: $\{x : d(x,A) < \frac{1}{n}\} = \bigcup_{a\in A} (a-\frac{1}{n}, a+\frac{1}{n})$

Pf. $\supset$ easy.

$\subset$ suppose x is s.t. $d(x, A) < \frac{1}{n}$. Let r s.t. $d(x, A) < r < \frac{1}{n}$. By def. $\exists a \in A$ with $d(x, a) \leq r$. $x \in (a-\varepsilon, a+\varepsilon)$,

PROP. A is closed $\iff \{x_n\} \subset A$ & $x_n \to x$; $x \in A$.

Pf. $d(x_n, x) \to 0 \iff x_n \to x$.

$\vdash$ A set is said to be bounded in $\mathbb{R}^1$ (in $\mathbb{R}^2$) if for some integer n $A \subset [-n, n]$ or for $\mathbb{R}^2$, $(A \subset [-n, n] \times [-n, n])$

DN: A collection of subsets $\mathcal{O}$ of X is said to cover (or to be a cover of the set A) if $A \subset \cup\mathcal{O} = \bigcup_{G \in \mathcal{O}} G$.

THM: If $\mathcal{O}$ is a collection of open sets in $\mathbb{R}^i$ then there is a countable subcollection of $\mathcal{O}$ which covers $\cup\mathcal{O}$.

Pf. for $\mathbb{R}^1$, $G = \bigcup_{B \in \mathcal{O}} B$ for each $x \in G$ Choose $B_x \in \mathcal{O}$ with $x \in B_x$. Since B_x is open there is an open interval (a_x, b_x)

with a_x, b_x rational and $x \in I_x \subset B_x$.
The collection of all such I_x's
$\{I_x : x \in G\} = D$. For each $I \in D$ pick
a countable set
some $B_x \in Q$, call it B_I with $I \subset B_I$.
Claim: $G = \bigcup_{x \in G} I_x = \bigcup_{I \in D} I \subset \bigcup_{I \in D} B_I$.

But $\bigcup_{I \in D} B_I \subset \bigcup_{B \in Q} B = G$.

So $\{B_I : I \in D\} \subset \mathcal{O}$ and covers G
(in $\mathbb{R}^1$, $\mathbb{R}^2$)

<u>DN.</u> A set A is said to be compact iff
it satisfies one of the following equivalent statements:

1.) A is closed and bounded.

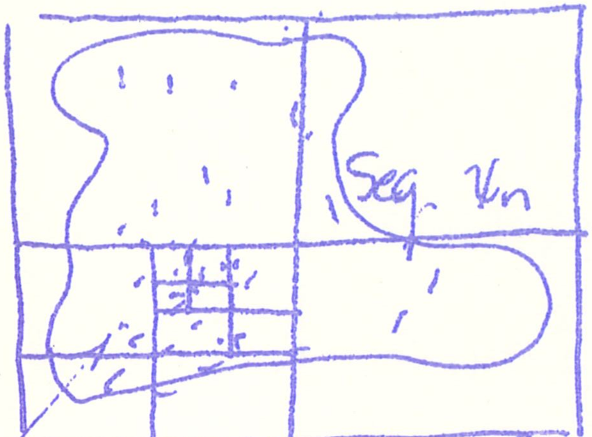
2.) For each seq $\{x_n\} \subset A$ there is a subsequence $\{x_{n_i}\}$ so that $x_{n_i} \to x$ as $i \to \infty$ and $x \in A$.

3.) Every open cover of A has a finite sub-cover [if $\mathcal{O}$ covers A then $\exists\, U_1, \ldots, U_n \in \mathcal{O}$ and $\{U_i, 1 \le i \le n\}$ covers A.

Show: (1.) $\Leftrightarrow$ (2.)

(3.) $\Leftrightarrow$ (1.)

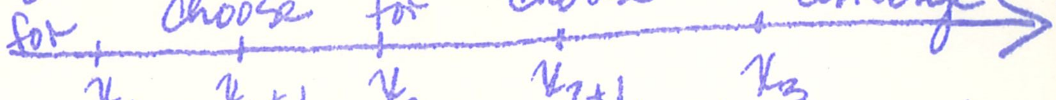
(1.) $\Rightarrow$ (2.) Lion catching.



We created a subsequence $\to x$, $x \in A$. (By subdividing the square and picking an x_{n_i} in each subdivision) Since A is closed.

(square containing infinitely many x_i)

(2.) $\Rightarrow$ (1.)



If A is not bounded $\exists$ a seq. that doesn't converge.

converging to $x \notin A$.

If A is unbd. then $\{(x-1, x+1) : x \in A\}$ is a cover with no finite subcover.

Pf.

(a.) $[a, b]$ is compact by (3.)

(b.) $[a, b] \times [c, d]$ is compact (3.)

(c.) Closed subsets of sets compact by (3.) are compact by (3.)

(i.) A is compact $B \subset A$, is closed. Let $\mathcal{O}$ be an open cover of A.

B^c is open so $\mathcal{O}' = \{B^c\} \cup \mathcal{O}$ is a cover of A so there is a finite subcover.

$\mathcal{O}_1, \dots, \mathcal{O}_n \subset \mathcal{O}'$; $B \subset A \subset \bigcup_{i=1}^{n} \mathcal{O}_i$

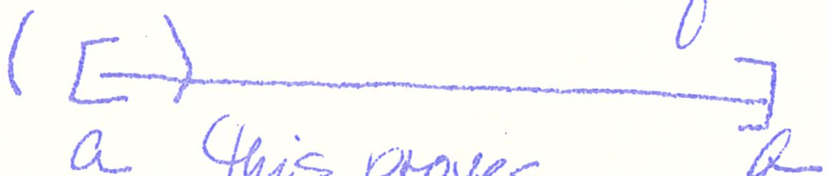
the worst that can happen is B^c is one of the $\mathcal{O}_i$ throw it out if it is. $B \subset \bigcup_{i=1}^{n} \mathcal{O}_i$

this is a finite subcover $B^c \notin \mathcal{O}_i$

(ii) Let $\mathcal{O}$ be an open cover of $[a,b]$.

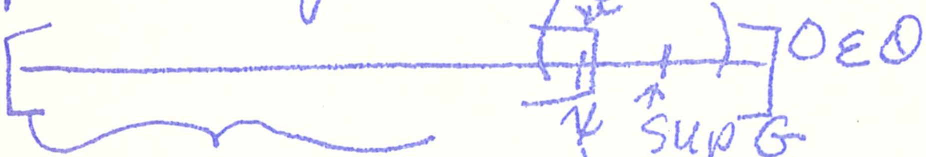
$G = \{x \in [a,b]$ the set $[a,x]$ can be covered with a finite subcover of $\mathcal{O}\}$.

$a \le \sup G$.



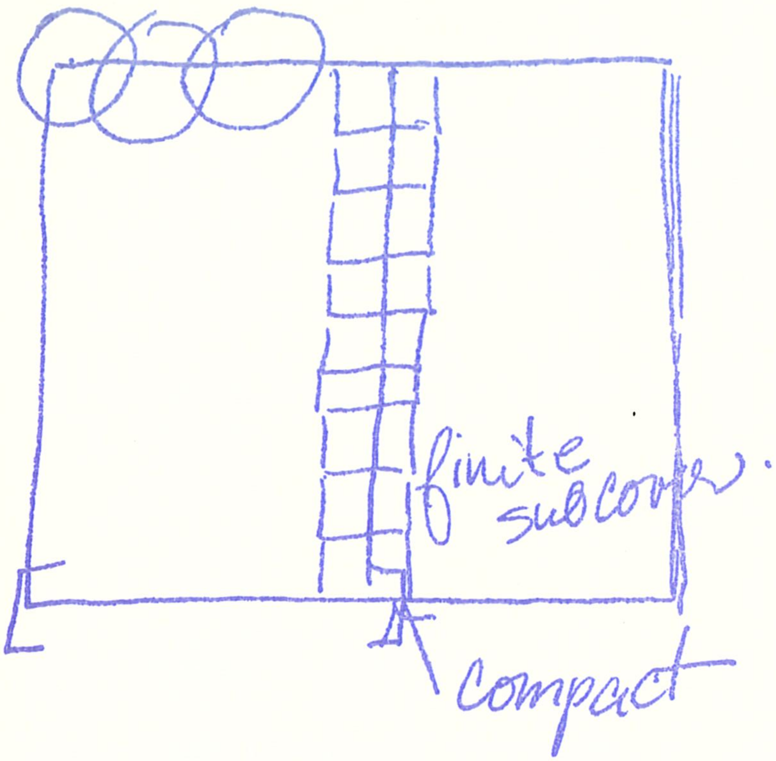
This proves $a \in G$.

Suppose $\sup G < b$. Claim: $\sup G \in G$.



$\exists\, O_1, O_2, \dots, O_n$ $x \in G$; There is an ε s.t. $\sup G + \varepsilon \in G$ True if $\sup G = b$.

(b.)



Same argument as in $\mathbb{R}^1$ for $\mathbb{R}^2$.

Test Problem * 3 Due Friday the 13th of Oct 1978

1. If $f: X \to Y$ is a function and $A \subset X$ define $f(A) = \{f(x) : x \in A\} \subset Y$.

0.8 pts Find examples of X, Y, f $A, B \subset X$, $C, D \subset Y$ with $f(A) = C$ but $f^{-1}(C) \neq A$ and $f^{-1}(D) = B$ but $f(B) \neq D$.

2. Suppose $\mathcal{D}$ is a σ-algebra on $\mathbb{R}^1$ and let

1.2 pts $\mathcal{S} = \{A \subset \mathbb{R}^1 : \exists B \in \mathcal{D}$ with $m_1^*(A \triangle B) = 0\}$. Show that $\mathcal{S}$ is a σ-algebra on $\mathbb{R}^1$

1 pt 3. If $\{A_n\}$ is a sequence of subsets of X and $B \subset X$ define functions $f_n, g : X \to \mathbb{R}^1$ by

$$f_n(x) = \begin{cases} 1 & x \in A_n \\ 0 & x \notin A_n \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

Show $\lim A_n = B$ if and only if for each $x \in X$ $\lim f_n(x) = g(x)$. [i.e. $f_n \to g$ pointwise]

4. If $A \subset \mathbb{R}^1$, define A' (the derived set of A) to be the set of $x \in \mathbb{R}^1$ for which there is a sequence $\{a_n\} \subset A \setminus \{x\}$ with $a_n \to x$. Let $A'' = (A')'$, $A''' = (A'')'$.

1 pt (a) Show $x \in A'$ if and only if $\forall \varepsilon > 0$ $A \cap (x - \varepsilon, x + \varepsilon)$ is an infinite set.

1 pt (b) For each $A \subset \mathbb{R}^1$, show that A' is closed.

1.5 pts (c) Let $A = \{0\} \cup \{1/n : n \in \mathbb{N}\} \cup \{1/n + 1/2^j : j \geq n;\ j, n \in \mathbb{N}\}$

Find and prove what A', A'' and A''' are.

3.0 pts 5. Let $\mathcal{A}$ be an algebra on X, Show that $\mathcal{S}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$

[the smallest σ-algebra $\supset \mathcal{A}$ is the smallest monotone class $\supset \mathcal{A}$]

Hint: Show $A, B \in \mathcal{M}(\mathcal{A}) \Rightarrow A \cup B \in \mathcal{M}(\mathcal{A})$.

Appendix II On limits of sets

Problem 3 of TP*3 can be used to prove many identities about limits of sets. A bit of notation (which is standard) will simplify the presentation. The characteristic function of a set A written χ_A is the function $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$, Note that

$$\chi_{A \cap B} = \chi_A \cdot \chi_B \,; \text{ If } A \supset B \text{ then } \chi_{A \setminus B} = \chi_A - \chi_B \,;$$

If $A \cap B = \emptyset$ then $\chi_{A \cup B} = \chi_A + \chi_B$. Thus by Problem 3 and the usual limit theorems about sums differences and products of real numbers:

1. $A_n \to A$; $B_n \to B$ then $A_n \cap B_n \to A \cap B$.
   pf $\chi_{A_n} \chi_{B_n} \to \chi_A \chi_B$ pointwise
2. $A_n \to A$; $B_n \to B$ and $A_n \supset B_n$, then $A_n \setminus B_n \to A \setminus B$
3. $A_n \to A$; $B_n \to B$ and $A_n \cap B_n = \emptyset$, then $A_n \cup B_n \to A \cup B$
4. $A_n \to A$; $B_n \to B$ then $A_n \setminus B_n = A_n \setminus (A_n \cap B_n) \to A \setminus B$ & $B_n \setminus A_n \to B \setminus A$ thus $A_n \triangle B_n \to A \triangle B$ and $A_n \cup B_n \to A \cup B$

L7 Pg 1

Lecture 7
Oct 9, 78

Def $m_i^*(E) = \inf\{\sum_{n=1}^\infty \ell(A_n) : A_n$ are basic open sets with $\bigcup_{n=1}^\infty A_n \supset E\}$

Thm 2 If (a,b) is bounded, then $m_i^*[(a,b)] = b-a$

Pf: Let $A_1 = (a,b)$, $A_n = \Phi$ for $n \geq 2$

$$m_i^*[(a,b)] \leq b-a$$

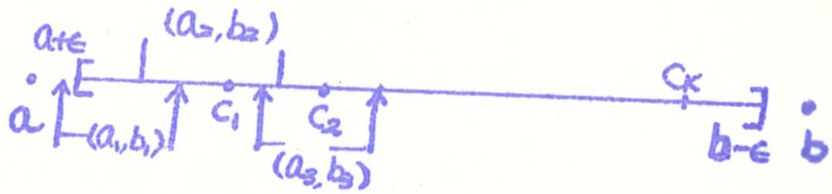
Let $\epsilon > 0$, A_n be basic open sets with $\bigcup_{n=1}^\infty A_n \supset (a,b)$

Want to show $m_i^*[(a,b)] \geq b-a-2\epsilon$

or equivalently $\sum_{n=1}^\infty \ell(A_n) \geq b-a-2\epsilon$

The set $[a+\epsilon, b-\epsilon]$ is compact and $\{A_n\}_{n=1}^\infty$ is an open cover of the set. Then, there exist $\{A_{n_1}, \cdots, A_{n_K}\}$, a finit open cover for $[a+\epsilon, b-\epsilon]$.

We may assume that $\forall\ 1 \leq i \leq K$, there is $c_i \in [a+\epsilon, a-\epsilon]$ such that $c_i \in A_{n_i}$ but $c_i \notin A_{n_j}$ $i \neq j$. We may also assume that $c_1 < c_2 < \cdots < c_K$



Let $A_{n_i} = (a_i, b_i)$, then

$$\sum_{n=1}^\infty \ell(A_n) \geq \sum_{i=1}^K \ell(A_{n_i}) = \sum_{i=1}^k (b_i - a_i)$$
$$\geq (b-\epsilon) - (a+\epsilon) = b-a-2\epsilon$$

THM 3 If (a,b) and (c,d) are bounded, then

$$m_2^*[(a,b)\times(c,d)] = (d-c)(b-a)$$

Note: Properties of $m_i^* : P(R^i) \to$ positive extended Reals

(1) $m_i^*(\Phi) = 0$

(2) If $A \subset B$, $m^*(A) \leq m^*(B)$ "monotone"

Pf: $\bigcup A_n \supset B \Rightarrow \bigcup A_n \supset A$

remark: $m_i^*(a,+\infty) = m_i^*(-\infty, b) = \infty$ $[(a, a+n) \subset (a,+\infty)\ \forall n]$

$m_2^*[(a,+\infty)\times(c,d)] = \infty$

(3) Countable Subadditivity: If $\{A_n\}$ is any sequence of sets $\subset R^i$, then $m_i^*[\bigcup_{n=1}^{\infty} A_n] \leq \sum_{n=1}^{\infty} m_i^*(A_n)$

pf: Let $\epsilon > 0$ be given

For each n, let $\{I_i^n\}_{i=1}^{\infty}$ be basic open sets with $\bigcup_{i=1}^{\infty} I_i^n \supset A_n$ and

$$m_i^*(A_n) \geq \sum_{i=1}^{\infty} l(I_i^n) - \frac{\epsilon}{2^n}$$

the collection $\{I_i^n ; n = 1, 2, \cdots, i = 1, 2, \cdots\}$ is countable

$\bigcup_{n=1}^{\infty} [\bigcup_{i=1}^{\infty} I_i^n] \supset \bigcup_{n=1}^{\infty} A_n$ so that

$$\begin{aligned} m_i^*(\bigcup_{n=1}^{\infty} A_n) &\leq \sum_{n=1}^{\infty} [\sum_{i=1}^{\infty} l(I_i^n)] \\ &\leq \sum_{n=1}^{\infty} [m_i^*(A_n) + \tfrac{\epsilon}{2^n}] \\ &= \sum_{n=1}^{\infty} m_i^*(A_n) + \epsilon \sum_{n=1}^{\infty} \tfrac{1}{2^n} \\ &= \sum_{n=1}^{\infty} m_i^*(A_n) + \epsilon \end{aligned}$$

<u>THM 4</u> $[0,1]$ has uncountably many points

pf: $m_i^*([0,1]) \neq 0$

but countable sets have measure zero

<u>DN 5</u> Let $\mathcal{M}_i$ be the collection of sets $A \subseteq R$ with the property:

$$\forall E \subset R^i : m_i^*(E) = m_i^*(E \cap A) + m_i^*(E \setminus A)$$

These are called the measurable sets.

Let us forget i in m_i^*.

A set A is measurable if for all sets E

$$m^*(E) = m^*(E \cap A) + m^*(E \setminus A).$$

Let M be the collection of all measurable sets.

<u>Note</u>

(1) $A \in M \Rightarrow A^c \in M$.

Pf:- $E \setminus A^c = E \cap A$ & $E \cap A^c = E \setminus A$.

(2) To prove measurability of A it suffices to show that for all sets E

$$m^*(E) \geq m^*(E \cap A) + m^*(E \setminus A).$$

Pf:- By countable subadditivity

$B_1 = E \cap A \quad B_2 = E \setminus A, \quad B_n = \phi \quad n \geq 3.$

$E = \bigcup B_i$

$$m^*(E) \leq \sum_i m^*(B_i) = m^*(E \cap A) + m^*(E \setminus A) + 0 \dots$$

(3) It suffices to check only for sets E with $m^*(E) < \infty$.

4) sets of measure zero are measurable.

<u>Pf</u>:- Let A s.t $m^*(A) = 0$ then for any E $A \supset A \cap E$,

$E \supset E \setminus A$.

So $m^*(A \cap E) = 0 \quad m^*(E) \geq m^*(E \setminus A)$ by monotonicity.

$$m^*(E) \geq m^*(E \setminus A) + m^*(E \cap A)$$

<u>Corollary</u>:- $M \neq \phi$ since $\{\phi\} \in M$.

5) $A, B \in M \Rightarrow A \cup B \in M$.

pf:- let E be given. use the fact $A \in M$ and get

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c).$$

use the fact $B \in M$ w/ "$E = E \cap A$".

$$m^*(E \cap A) = m^*(E \cap A \cap B) + m^*(E \cap A \cap B^c)$$

use $B \in M$ w/ "$E = E \cap A^c$".

$$m^*(E \cap A^c) = m^*(E \cap A^c \cap B) + m^*(E \cap A^c \cap B^c)$$

Combining

$$m^*(E) = m^*(E \cap A \cap B^c) + m^*(E \cap A \cap B) + m^*(E \cap A^c \cap B) + m^*(E \cap A^c \cap B^c)$$

$$\geq m^*(E \cap (A \cup B)) + m^*(E \setminus (A \cup B)).$$

$\therefore$ $A \cup B$ is measurable

Corollarary:- If $A \cap B = \phi$ & $A, B \in M$ then for any E

$$m^*(E) = m^*(E \cap A) + m^*(E \cap B) + m^*(E \setminus (A \cup B)).$$

By induction if $\{B\}_{i=1}^{k}$ is a finite collection $\in M$ and if $\{B_i\}$ are pairwise disjoint then for any E, $m^*(E) = \sum_{i=1}^{k} m^*(E \cap B_i) + m^*(E \setminus (\bigcup_{i=1}^{k} B_i))$

on M m^* is "finitely additive"

(ie) If $\{B_i\}_{i=1}^{K}$ are pairwise disjoint $\subset M$ then

$$m^*(\bigcup_{i=1}^{K} B_i) = \sum_{i=1}^{K} m^*(B_i).$$

6) ~~B~~ ~~B~~ M is an algebra.

7) $\mathcal{M}$ is a σ-algebra.

we need to show $\{A_i\}_{i=1}^{\infty} \subset M$ then $\bigcup_{i=1}^{\infty} A_i \in M$.

Lemma: If $\{A_i\}_{i=1}^{\infty} \subset$ algebra $\mathcal{M}$ then there exist pairwise disjoint $\{B_i\}_{i=1}^{\infty} \subset M$ with the property

$$\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i \qquad N = \infty, 1, 2, 3 \ldots$$

Pf: Define B_i inductively

Let $B_1 = A_1 \quad \ldots\ldots \quad B_{n+1} = A_{n+1} \setminus \bigcup_{i=1}^{n} A_i$.

Suppose $B_1 \ldots B_n$ pairwise disjoint

and $\bigcup_{i=1}^{j} B_i = \bigcup_{i=1}^{j} A_i \qquad j \leq n$.

Have to show $B_1 \ldots B_{n+1}$ is Pairwise disjoint

& $\bigcup_{i=1}^{j} B_i = \bigcup_{i=1}^{j} A_i \qquad j \leq n+1$

All we have to show $\bigcup_{i=1}^{n+1} B_i = \bigcup_{i=1}^{n+1} A_i$.

Since $B_i \subset A_i$

$$\Rightarrow \bigcup_{i=1}^{n+1} B_i \subset \bigcup_{i=1}^{n+1} A_i.$$

$x \in \bigcup_{i=1}^{n+1} A_i \Rightarrow x \in A_{n+1}$ or

$x \in \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$ by induction.

Assume $x \notin \bigcup_{i=1}^{n} A_i$ must be in A_{n+1} & thus B_{n+1}.

$x \in \bigcup_{i=1}^{n+1} B_i$ and the two sets are equal.

Let E be any set

Since $\{B_i\}_{i\geq 1}^{n}$ is pairwise disjoint

$$m^*(E) = \sum_{i\geq 1}^{n} m^*(E \cap B_i) + m^*(E \setminus \bigcup_{i\geq 1}^{n} B_i).$$

note $E \setminus (\bigcup_{i\geq 1}^{n} B_i) = E \setminus (\bigcup_{i\geq 1}^{n} A_i) \supset E \setminus (\bigcup_{i\geq 1}^{\infty} A_i)$

So $m^*(E) \geq \sum_{i\geq 1}^{n} m^*(E \cap B_i) + m^*(E \setminus (\bigcup_{i\geq 1}^{\infty} A_i))$.

Since this is true for all n

$$m^*(E) \geq \sum_{i\geq 1}^{\infty} m^*(E \cap B_i) + m^*(E \setminus (\bigcup_{i\geq 1}^{\infty} A_i))$$

Define $C_i = B_i \cap E$ $\quad \bigcup_{i\geq 1}^{\infty} C_i = E \cap (\bigcup_{i\geq 1}^{\infty} B_i) = E \cap (\bigcup_{i\geq 1}^{\infty} A_i)$.

By countable subadditivity

$$m^*(\bigcup_{i\geq 1}^{\infty} C_i) \leq \sum_{i\geq 1}^{\infty} m^*(C_i)$$

$$m^*(E \cap (\bigcup_{i\geq 1}^{\infty} A_i)) \leq \sum_{i\geq 1}^{\infty} m^*(E \cap B_i).$$

$$\therefore m^*(E) \geq m^*(E \cap (\bigcup_{i\geq 1}^{\infty} A_i)) + m^*(E \setminus (\bigcup_{i\geq 1}^{\infty} A_i))$$

$\therefore$ M is σ-algebra.

8) If $\{B_i\}$ are pairwise disjoint $\subset M$ then

$$m^*(\bigcup_{i\geq 1}^{\infty} B_i) = \sum_{i\geq 1}^{\infty} m^*(B_i) \quad \text{[Countable additivity]}.$$

$$E = \bigcup_{i=1}^{\infty} B_i$$

$$m^*\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{n} m^*\left(\left(\bigcup_{j=1}^{\infty} B_j\right) \cap B_i\right) + m^*\left(\bigcup_{j=1}^{\infty} B_j \setminus \left(\bigcup_{i=1}^{n} B_i\right)\right)$$

$$m^*\left(\bigcup_{i=1}^{\infty} B_i\right) \geq \sum_{i=1}^{n} m^*(B_i) \qquad \text{true for all } n.$$

$$m^*\left(\bigcup_{i=1}^{\infty} B_i\right) \geq \sum_{i=1}^{\infty} m^*(B_i). \qquad \text{Count. Subadditivity.}$$

<u>Summary</u> $\mathcal{M}$ is a σ-algebra containing the sets of measure zero.

Let's write m for $m^*|_{\mathcal{M}}$.

$m : \mathcal{M} \longrightarrow$ non-neg extended reals.

(1) $m(\phi) = 0$

(2) If $\{B_i\}$ are pairwise disjoint then

$$m\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} m(B_i).$$

Real Analysis.
Friday the 13th. of Oct.
Part one
Lecture 9
Notes By Benedikt

THM. 1:

Prop. For each real a

$$(a,+\infty) \in \mathcal{M},$$

$$(a,+\infty) \times (-\infty,+\infty) \in \mathcal{M}_2$$

$$(-\infty,+\infty) \times (a,+\infty) \in \mathcal{M}_2$$

Note: This will imply that $\mathcal{M}_i \supset \mathcal{B}_i \quad i=1,2$

$\{a\}$ has measure zero

$(-\infty, b)$ is measurable for all b.

$\therefore$ (a,b) is measurable

$\therefore$ All open sets are measurable.

Actually, $\mathcal{M}_i = \mathcal{L}_i$ (TP 4-3)

Proof: We need to show that $\forall\, E \subset \mathbb{R}$

$$m^* E \geq m^*(E \cap (a,+\infty)) + m^*(E \setminus (a,+\infty)).$$

Let I_n be basic open set with

$\bigcup I_n \supset E$ and $m^*(E) \geq \sum_{n=1}^{\infty} \ell(I_n) - \epsilon$.

Define $I_n^r = I_n \cap (a,+\infty)$. This is an open interval.

$$E \cap (a,+\infty) \subset \bigcup_n I_n^r.$$

Define $I_n^\ell = I_n \cap (-\infty, a+\frac{\epsilon}{2^n})$. This is also an open interval.

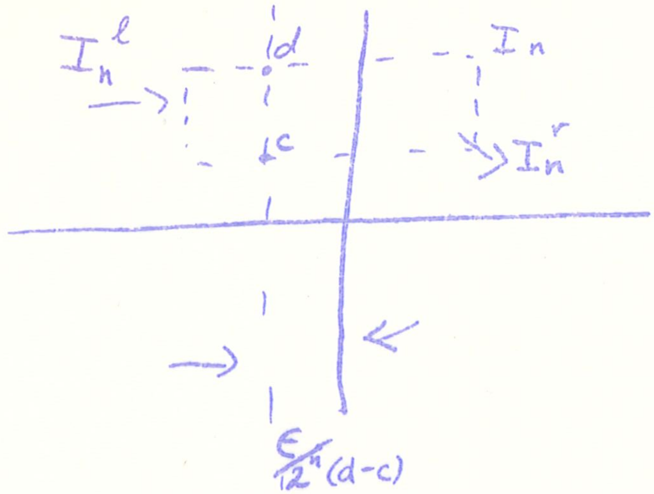
$$E \setminus (a,+\infty) \subset \bigcup_n I_n^\ell.$$

Then $\ell(I_n) \geq \ell(I_n^\ell) + \ell(I_n^r) - \frac{\epsilon}{2^n}$

and thus $\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{n=1}^{\infty} \ell(I_n^\ell) + \sum_{n=1}^{\infty} \ell(I_n^r) - \epsilon$

$$m^*(E) \geq m^*(E \cap (a,+\infty)) + m^*(E \setminus (a,\infty)) - 2\epsilon$$

Assume $m^*(E) < \infty$



DN2: <u>Definition</u> If X is a set ~~pair~~ & $\mathcal{S}$ is a σ-algebra on X, then a function μ

$\mu : \mathcal{S} \to$ non-negative extented reals s.t.

(1) $\mu(\phi) = 0$

(2) $\mu\left(\bigcup_{n=1}^{\infty} A_i\right) = \sum_{n=1}^{\infty} \mu(A_n)$ if $\{A_n\}$ is a pairwise disjoint sequence of sets.

Then μ is called a <u>measure</u>.

(1) & (2) imply:

(3) μ is <u>finitly additive</u>

(4) μ is <u>monotone</u>. If $A \subset B$

$\mu(B) = \mu(A) + \mu(B \setminus A)$

$\Rightarrow \mu(B) \geq \mu(A)$ (since $\mu(B\setminus A)$ is nonnegativ).

(5) μ is countably subadditive.

$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$ $\forall$ seq $\{A_n\}$.

Let $B_1 = A_1$ & $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$

$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$ by (2)

$\leq \sum_{n=1}^{\infty} \mu(A_n)$

THM 3: If μ is a measure on $\mathbb{B}$ & if $\mu((a,b)) = b-a$ $\forall$ $a+b$ then $\mu = m_1$.

(Similar statement for m_2 on $\mathbb{B}_2$ i μ gives the same as area on basic opens)

Note: $\mu(\{a\}) = 0$

$\{a\} \subset (a-\epsilon/2, a+\epsilon/2)$

$\mu(\{a\}) \leq \mu(a-\epsilon/2, a+\epsilon/2) = \epsilon$

Thus $\mu([a,b)) = b-a$

OCT. 13, 1978

define $\mathcal{G} = \{A \in \beta : \mu(A) = m_1(A)\}$.

claim (1) $\mathcal{G} \supset \mathcal{L}_1$

(2) $\mathcal{G}$ is monotone class

* A_n nest increasing sequence in $\mathcal{G}$

$$\mu\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

$$m_1\left(\bigcup_n A_n\right) = \lim_{n\to\infty} m(A_n).$$

By 1A Test Problem #4 & By 1B if $\{A_n\}$ nested increasing

** and $\mu(A_1)$ & $m(A_1)$ are finite

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

$$m_1\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$$

Suppose $m(A_1) = +\infty$

If $\{A_n\}$ are nested increasing

Letting $I_n = [-n, n]$

for each m $\{I_m \cap A_n\}_n$ is nested decreasing

so $\bigcap_{n=1}^{\infty} (A_n \cap I_m) \in \mathcal{G}$

$$= \left[\bigcap_{n=1}^{\infty} A_n\right] \cap I_m.$$

$\left\{\left[\bigcap_{n=1}^{\infty} A_n\right] \cap I_m\right\}_m$ is nested increasing

so by (*)

$$\bigcup_{m=1}^{\infty} \left(\left[\bigcap_{n=1}^{\infty} A_n\right] \cap I_m\right) \in \mathcal{G}$$

$\bigcap^{\infty} A_n$

$\therefore \mathcal{G}$ is a monotone class

so it $\supset m(\mathcal{A}_1) = \mathcal{S}(\mathcal{A}_1) = \mathcal{B}_1$.

<u>Thm</u>: m_1 is translatory invariant

If $E \subset \mathbb{R}$ is measure $r \in \mathbb{R}$

then $m(E) = m(E+r)$

where $E + r = \{ e + r : e \in E \}$.

Pf: Let r be given define

$\mu : \mathcal{B}_1 \to$ p.e.r.n.s.

$\mu(E) = m(E+r)$

μ is a measure

$\mu(\phi) = m(\phi + r) = 0$.

if $\{A_n\}$ are pairwise disjoint.

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = m\left(\left(\bigcup_{n=1}^{\infty} A_n\right) + r\right)$$

$$= m\left(\bigcup_{n=1}^{\infty}\left[A_n + r\right]\right)$$

$$= \sum_{n=1}^{\infty} m(A_n + r)$$

$$= \sum_{n=1}^{\infty} \mu(A_n).$$

$$\mu((a,b)) = m((a,b)+r)$$

$$= m((a+r, b+r))$$

$$= b + r - (a+r) = b - a$$

modulo: showing $E \in \mathcal{B} \Rightarrow E + r \in \mathcal{B}$

If $E \subset \mathbb{R}^2$ & $(r, s) \in \mathbb{R}^2$

$E + (r, s) = \{(e_1 + r, e_2 + s) : (e_1, e_2) \in E\}$

then $m_2(E) = m_2(E + (r, s))$

m_2 is also rotationally invariant

Thm: If μ is any measure on $\mathcal{B}_1$ that is translation invariant then for some $\lambda \in \mathbb{R}^+$

$$\mu(E) = \lambda\, m(E) \quad \forall E \in \mathcal{B}_1 .$$

Test Problem #4 Due 20^{th} Oct 1978

1. If μ is a measure defined on the σ-algebra $\mathcal{S}$ show

1pt A. If $\{A_n\} \subset \mathcal{S}$ is a nested increasing sequence of sets then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

1pt B. If $\{A_n\} \subset \mathcal{S}$ is a nested decreasing sequence of sets and $\mu(A_1) < \infty$, then $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

0.5pt C. Show by example, with $\mu = m_1$, $\mathcal{S} = \mathcal{B}_1$, that B can fail if $\mu(A_n) = \infty$ for all n, by showing that $\bigcap_{n=1}^{\infty} A_n$ could be $\emptyset$.

1.5pt 2. If S & T are non-empty sets of positive reals and let $S+T = \{s+t : s\in S, t\in T\}$, then show that

$$\sup S + \sup T = \sup(S+T)$$ [watch out for ∞ case].

3. Consider the following statements about subsets of $\mathbb{R}$

(a) E is measurable

(b) $\forall \varepsilon>0$ $\exists U$ open set w/ $U \supset E$ and $m^*(U\setminus E) < \varepsilon$

(c) $\exists G$ a G_δ-set w/ $G \supset E$ and $m^*(G\setminus E) = 0$

(d) $\forall \varepsilon>0$ $\exists C$ closed set w/ $C\subset E$ and $m^*(E\setminus C) < \varepsilon$

(e) $\exists F$ a F_σ-set w/ $F \subset E$ and $m^*(E\setminus F) = 0$

(f) $\forall \varepsilon>0$ $\exists$ finite collection of open intervals $I_1, \dots I_n$ so that $m^*\left(E \triangle \left(\bigcup_{i=1}^{n} I_i\right)\right) < \varepsilon$.

Show (a) – (e) are equivalent & if $m^*(E)<\infty$ then all are equivalent by

1pt A. Show $(b) \Rightarrow (c) \Rightarrow (a)$

1pt B. Show $(d) \Rightarrow (e) \Rightarrow (a)$

1pt C. Show if $m^*(E)<\infty$; $(a) \Rightarrow (b)$

1pt D. Show $(a) \Rightarrow (b)$

1pt E. Show: if $m^*(E)<+\infty$; $(b) \Leftrightarrow (f)$.

0.5pt F. use $(a)\Leftrightarrow(b)$ to show $(b) \Rightarrow (d)$

0.5pt G. Show $\mathcal{M}_1 = \mathcal{L}_1$

Appendix III The Borel sets have cardinality c.

This is a non constructive proof that there are a lot more Lebesgue measurable sets than Borel sets. Some preliminary counting results are needed.

If X is a set and $\text{Card}\, X = \alpha$ we will define $\text{Card}(\mathcal{P}(X)) = 2^\alpha$. Perhaps the best rational for this definition is the following proposition.

Proposition: If X is a finite set and $\text{Card}(X) = n$ then $\mathcal{P}(X)$ is a finite set and $\text{Card}(\mathcal{P}(X)) = 2^n$

proof: Suppose $n = 0$ so that $X = \emptyset$. The only subset of X is $\emptyset$ so that $\mathcal{P}(X) = \{\emptyset\}$ and $\text{Card}\, \mathcal{P}(X) = 1 = 2^0$.

Suppose the proposition is true for n. We complete the proof by induction. Let $\{x_1, x_2, \ldots x_{n+1}\}$ be a set with $\text{Card} = n+1$. We want to show this set has 2^{n+1} subsets. We know $\{x_1, x_2, \ldots x_n\}$ has 2^n subsets. We list them, and note that by adding x_{n+1} to each of these subsets we obtain them all.

$$\begin{array}{lll}
1 \to A & \qquad & 2^n+1 \to A \cup \{x_{n+1}\} \\
2 \to B & & 2^n+2 \to B \cup \{x_{n+1}\} \\
\vdots & & \vdots \\
2^n \to Z & & 2^{n+1} \to Z \cup \{x_{n+1}\}.
\end{array}$$

Proposition If X is a set then $\text{Card}\, X \lneq \text{Card}\, \mathcal{P}(X)$ (or $\alpha < 2^\alpha$ for all card's α.)

<u>proof:</u> Clearly, since for each $x \in X$, $\{x\} \in \mathcal{P}(X)$ we have $\text{Card}\, X \leq \text{Card}\,(\mathcal{P}(X))$.

Suppose $\text{Card}\, X = \text{Card}\,(\mathcal{P}(X))$, then there is a function $f: X \to \mathcal{P}(X)$ which is 1-1 and onto. Consider $T = \{x \in X : x \notin f(x)\}$. Since f is onto, there is a $t \in X$ so that $f(t) = T$. But this is impossible. Since either $t \in T$ or $t \notin T$ we have

(1) If $t \in T$ then by definition of T, $t \notin f(t) = T$ ※

(2) If $t \notin T$ then $t \notin f(t)$ and by definition of T, $t \in T$ ※.

<u>Remark</u> Since the Cantor set C has c elements and has measure zero each subset of C is measurable and there are 2^c of them. Thus there ~~et~~ are 2^c Lebesque measurable sets. (There are only 2^c subsets of $\mathbb{R}$).

The next step is to show that there is a large set with Card c. Let $\mathbb{R}^\infty = \{(x_n) : (x_n) \text{ is a sequence of reals}\}$. We will show $\text{Card}\, \mathbb{R}^\infty = c$. In particular this will imply that $\text{Card}\, \mathbb{R}^n = c$. for $n = 1, 2, \ldots$.

Consider the map $f: [0,1) \to [0,1)^\infty = \{\{x_n\} : \{x_n\}$ a sequence of reals in $[0,1)\}$ given by if $x = .d_1 d_2 d_3 \ldots$ in decimal expansion then

$x_1 = .d_1\, d_3\, d_6\, d_{10}$

$x_2 = .d_2\, d_5\, d_9$ is $f(x)$

$x_3 = .d_4\, d_8$

$x_4 = d_7$

$\vdots$

There are problems with this map. First we do not know if it is well-defined. (A number with two expansions may go to two different sequences). However, these are of little matter, since each sequence $\{x_n\}$ comes from some decimal expansion. And if we require each of $\{x_n\}$ to be in irrational form, the x it comes from must be in irrational form. In any case, f is onto & Card $[0,1)^\infty = c$.

Since the function $f: \mathbb{R} \to [0,1)$ given by $f(x) = (\arctan x + \pi/2)/2\pi$ is 1-1, the function $F: \mathbb{R}^\infty \to [0,1)^\infty$, $F(\{x_n\}) = \{f(x_n)\}$ is 1-1 and Card $\mathbb{R}^\infty$ is c.

<u>Corollary 1</u>: The union of c sets each with card $\le c$ has Card $\le c$

proof: Card $\mathbb{R}^2 = c$, and $\mathbb{R}^2 = \bigcup_{a \in \mathbb{R}} \{(a,b) : b \in \mathbb{R}\}$.

<u>Corollary 2</u>: The collection of a sequences of a set with card c has card c.

The final fact we need is the fact that every set can be well-ordered. That statement is equivalent to the Axiom of Choice, and is not obvious. A set X with an ordering $\le$ is said to be well ordered if each subset $A \subset X$ has an element $a \in A$ so that for each $b \in A$ $a \le b$. Well-ordering is stronger than linear ordering.

We apply this well-ordering principle to the set $\mathbb{R}$. So [illegible] of $\mathbb{R}$ this NOT the usual ordering

If a is in $\mathbb{R}$, the set of presessors of a is $P_a = \{b \in \mathbb{R} : b \leq a \text{ and } b \neq a\}$. Let $A = \{x \in \mathbb{R}, P_x \text{ is uncountable}\}$. By the well-ordering principle A has a smallest element α. let $\Omega = P_\alpha$ (if ~~no~~ $A = \phi$, let $\Omega = \mathbb{R}$). In any case Ω is an uncountable set and for each $a \in \Omega$, P_a is countable, Ω is sometimes called the first uncountable ordinal. We define two types of ~~all~~ elements of Ω. successor elements are elements $a \in \Omega$ so that there is $b \in \Omega$ and for each $c \in \Omega$ with $b \leq c \leq a$ implies $b = c$ or $c = a$. Otherwise a is a limit element.

We are now ready to "construct" the Borel sets. Let 0 be the first element of Ω and let $\mathcal{B}_0 = \{(a,b) : -\infty < a < b < +\infty ;\ a, b \in \mathbb{R}\}$ note that Card $\mathcal{B}_0 = c$ by Corollary 1; and that the smallest σ-algebra $\supset \mathcal{B}_0$ is the Borel sets.

For each $a \in \Omega$ we define $\mathcal{B}_a \subset \mathcal{P}(\mathbb{R})$ as follows.

(1) If a is a sucessor element to b we obtain $\mathcal{B}_a$ from $\mathcal{B}_b$ as follows.

(i) Let $\mathcal{B}_b^{(1)}$ be the unions of ~~all~~ sequences of elements of $\mathcal{B}_b$. By Corollary 2 if Card $\mathcal{B}_b = c$ then Card $\mathcal{B}_b^{(1)} = c$

(ii) let $\mathcal{B}_b^{(2)}$ be the sets of $\mathcal{B}_b^{(1)}$ or their complements. By Corollary 1 if Card $(\mathcal{B}_b^{(1)}) = c$ then Card$(\mathcal{B}_b^{(2)}) = c$.

(iii) let $\mathcal{B}_a$ be the intersections of sequences of elements of $\mathcal{B}_b^{(2)}$. By Corollary 2 if Card $\mathcal{B}_b^{(2)} = c$ then $\mathcal{B}_a = c$.

(2) If a is a limit element, then P_a is countable and define $\mathcal{B}_a = \bigcup_{b \in P_a} \mathcal{B}_b$. By Corollary 1

Card $\mathcal{B}_a = c$ if Card $\mathcal{B}_b = c$ for each $b \in P_a$.

Lemma: Card $\mathcal{B}_a = c$ for each $a \in \Omega$.

Proof: If not, there is a first such $a \in \Omega$, but by (1) & (2) it cannot be a limit element or a ~~susser~~ sucessor element a contradiction.

Thus let $\mathcal{B} = \bigcup_{a \in \Omega} \mathcal{B}_a$, Card $\mathcal{B} = c$ by the lemma and Corollary 1, We claim that $\mathcal{B}$ contains (it actually is) the Borel sets. It suffices to prove $\mathcal{B}$ is a σ-algebra. We need

Lemma: if $a_n \in \Omega$ then $\bigcup_n P_{a_n} \subsetneq \Omega$.

Proof: P_{a_n} is countable and so is the union, but Ω is uncountable.

Proposition: $\mathcal{B}$ is σ-algebra

Proof: If $A \in \mathcal{B}$ then $A \in \mathcal{B}_a$ some $a \in \Omega$ if b is the next element of Ω then $\mathbb{R} \setminus A \in \mathcal{B}_b$ by construction (1) If $A_n \in \mathcal{B}$, then $A_n \in \mathcal{B}_{a_n}$ some $a_n \in \Omega$, by the Lemma. $A_n \in \mathcal{B}_b$ some $b \geq$ all a_n; $b \in \Omega$. If a is the next element of Ω after b, then $\bigcup A_n$ is in $\mathcal{B}_a$ by construction (1).

Thus the Borel sets have Card $= c <$ Card Lebesgue sets $= 2^c$

Real Analysis
Grey + Maloney

10/16

THM 1. If μ is a translation invariant measure on $\mathcal{B}$, $\mu([0,1)) < \infty$, then $\mu(E) = \lambda m(E)$ for $E \in \mathcal{B}$, where $\lambda = \mu([0,1))$.

RMK 2. Take $\mu(E) = \begin{cases} 0 & \text{if } m^*(E) = 0 \\ \infty & \text{if } m^*(E) > 0 \end{cases}$

then this is a measure where $\mu([0,1))$ fails to be finite.

FACT 3. Every non-empty open subset of $\mathbb{R}$ contains a rational.

FACT 4. If $f : \mathbb{R} \to \mathbb{R}$ so that $\forall x, y \in \mathbb{R}$, $f(x+y) = f(x) + f(y)$, then $f(x) = x \cdot f(1)$ for all $x \in \mathbb{Q}$. (An idea of how this proof goes: $x = 0$ $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$.
By induction show $x \in \mathbb{N}$ $f(n) = f(\underbrace{1+1+\dots+1}_{n\text{-times}})$
$= \underbrace{f(1) + f(1) + \dots + f(1)}_{n\text{-times}} = n \cdot f(1)$

and for $x \in \mathbb{Q}$ $n \cdot f(1) = f(n) = f(\underbrace{\tfrac{n}{m} + \tfrac{n}{m} + \dots + \tfrac{n}{m}}_{m\text{-times}})$
$= f(\tfrac{n}{m}) + f(\tfrac{n}{m}) + \dots + f(\tfrac{n}{m}) = m f(\tfrac{n}{m})$

$\Rightarrow n \cdot f(1) = m \cdot f(\tfrac{n}{m})$ or $f(\tfrac{n}{m}) = \tfrac{n}{m} \cdot f(1)$.)

FACT 5. If $f(x)$ in FACT 4 is continuous, then $f(x) = x \cdot f(1)$, $\forall x \in \mathbb{R}$.

LEMMA 6. $f : \mathbb{R} \to \mathbb{R}$ is continuous at each pt. in $\mathbb{R}$ iff $f^{-1}(A)$ is open $\forall$ open sets $A \subseteq \mathbb{R}$.
(Note: this lemma along with verifying $\{x \mid f(x) - x \cdot f(1) \neq 0\} = g^{-1}\{\mathbb{R} \setminus \{0\}\}$ where $g(x) = f(x) - x \cdot f(1)$ is how one proves FACT 5).

Proof of LEMMA 6

($\Rightarrow$) Let $\mathcal{U}$ be an open subset of the reals and let x be $\ni$ $f(x) \in \mathcal{U}$. $\exists$ $\varepsilon > 0$ $\ni$ $(f(x)-\varepsilon, f(x)+\varepsilon) \subset \mathcal{U}$. $\exists$ $\delta > 0$ $\ni$ $y \in (x-\delta, x+\delta) = I$ implies $f(y) \in (f(x)-\varepsilon, f(x)+\varepsilon)$. We have $x \in I \subset f^{-1}(\mathcal{U})$. I is open $\therefore f^{-1}(\mathcal{U})$ is open.

($\Leftarrow$) Let x be given. let $\mathcal{U} = (f(x)-\varepsilon, f(x)+\varepsilon)$ $x \in f^{-1}(\mathcal{U})$ which is open. $\therefore \exists \delta > 0$ $\ni$ $(x-\delta, x+\delta) \subset f^{-1}(\mathcal{U})$.

Proof of THM 1

Define $f(x) = \mu([0,x))$, then

$$f(x)+f(y) = \mu([0,x)) + \mu([0,y))$$
$$= \mu([0,x)) + \mu([x, x+y)) \quad \text{(translation invariant)}$$
$$= \mu([0, x+y)) = f(x+y)$$

f is continuous!

If $x_n \searrow x$ (x_n is monotonically decreasing to x) $\mu([0,x)) = \lim_{n\to\infty} \mu([0,x_n))$. Same for $x_n \nearrow x$. (This is why $\mu([0,1)) < +\infty$ required).

Let $\gamma(E) = \dfrac{\mu(E)}{\mu([0,1))}$, then γ is a measure which agrees with m on $\mathcal{A}_i$, so by the uniqueness $\gamma \equiv m$. Also true if $\mu([0,1)) = 0$, additivity gives a zero measure on $\mathcal{B}$. Now $\mu = \lambda\gamma = \lambda m$, the proof is complete.

Construction of a Non-measurable set in $\mathbb{R}$!

Axiom of Choice: If $\{A_\alpha : \alpha \in \Omega\}$ is a non-empty family of non-empty sets then $\exists$ a choice function c

where $c : \Omega \to \bigcup_{\alpha \in \Omega} A_\alpha$ defined by $c(\alpha) \in A_\alpha$.

DFN We define an equivalance relation on $[0,1)$ by $x \sim y$ (means "x is equivalent to y") iff $x - y$ is a rational.

Note/ $x \sim x$ (since $0 \in \mathbb{Q}$)
{ if $x \sim y$ then $y \sim x$ $(x-y \in \mathbb{Q} \Rightarrow y-x \in \mathbb{Q})$
{ if $x \sim y$ and $y \sim z$ then $x \sim z$ $(x-z = (x-y)+(y-z) \in \mathbb{Q})$

DFN For $x \in [0,1)$ define $[\![x]\!] = \{y \mid y \sim x\} \cap [0,1)$ ($[\![x]\!]$ is the equivalance class of x).

FACT 7 $[\![x]\!] \cap [\![y]\!] = \phi$ or $[\![x]\!] = [\![y]\!]$.

Proof of Fact 7

Suppose $[\![x]\!] \cap [\![y]\!] \neq \phi$ then $\exists z \ni z \sim x$ and $z \sim y$

so $w \in [\![x]\!] \Leftrightarrow x \sim w$
$\Leftrightarrow z \sim w$
$\Leftrightarrow y \sim w$
$\Leftrightarrow w \in [\![y]\!]$. □

Thus $\Omega = \{[\![x]\!] : x \in [0,1)\}$ is a non-empty collection of non-empty sets. Define (*) $E = \bigcup_{\alpha \in \Omega} \{c(\alpha)\}$, where c is a choice funtion. Claim E is non-measurable in $\mathbb{R}$. (note that E is measurable in $\mathbb{R}^2$). If $x, y \in [0,1)$ Define $x \dot{+} y = \begin{cases} x+y, & \text{if } x+y < 1 \\ x+y-1, & \text{otherwise} \end{cases}$

If $E \subset [0,1)$ is measurable and $x \in [0,1)$ then $E \dot{+} x = \{e \dot{+} x : e \in E\}$

Take this set E (from *) & consider the collection $\{E \dot{+} r : r \in \mathbb{Q} \cap [0,1)\}$. It's a countable collection. Claim if $r \neq s$ then $(E \dot{+} r) \cap (E \dot{+} s) = \phi$. Spse not, let x be an element of the set. Then $\exists e_1, e_2 \in E \ni$ $e_1 \dot{+} r = x = e_2 \dot{+} s$ so $e_1 \sim e_2$. But $e_1 \neq e_2$ Contradiction. So $(E \dot{+} r) \cap (E \dot{+} s) = \phi$

<u>Claim</u> $[0,1) = \bigcup_{r \in \mathbb{Q} \cap [0,1]} (E \dot{+} r)$

So if E is measurable

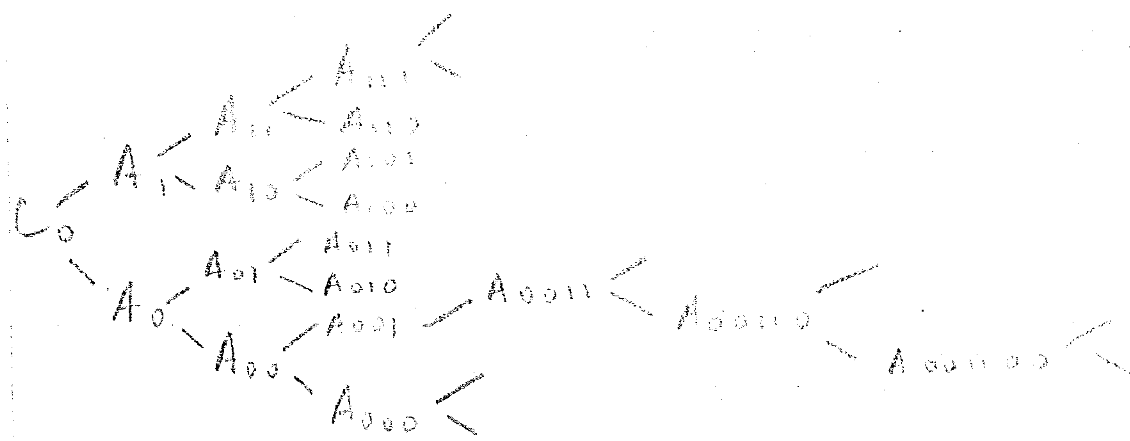
$$1 = m([0,1)) = \sum_{r \in \mathbb{Q} \cap [0,1)} m(E \dot{+} r)$$

$$= \sum_{n=1}^{\infty} m(E).$$

$m(E) \neq 0$, otherwise $m([0,1)) = 0$ ⊗.
$m(E) \not> 0$, otherwise $m([0,1)) = +\infty$ ⊗.
Contradiction. E is non measurable.

There are uncountable sets with measure zero in $\mathbb{R}^2$. E.g. $\mathbb{R}^1$.

Oct. 18 The Cantor Set / Function



C_0 $[0,1]$ 2^0 intervals of length $\left(\frac{1}{3}\right)^0$

O_1 $\left(\frac{1}{3}, \frac{2}{3}\right)$ 2^{1-1} intervals of length $\left(\frac{1}{3}\right)^1$

$C_1 = C_0 \setminus O_1$ $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

2^1 intervals of length $\frac{1}{3}$

O_2 2^{2-1} intervals of length $\left(\frac{1}{3}\right)^2$

$A_{\underbrace{00\ldots0}_{n}}, A_{\underbrace{00\ldots01}_{n}}, \ldots, A_{\underbrace{1\ldots1}_{n}}$

C_N 2^N closed intervals of length $\left(\frac{1}{3}\right)^N$

$C_N \subset C_{N-1}$

O_{N+1} 2^N open intervals of length $\left(\frac{1}{3}\right)^{N+1}$

C_{N+1} is 2^{N+1} closed intervals of length $\left(\frac{1}{3}\right)^{N+1}$

The Cantor Set $C = \bigcap_{n=0}^{\infty} C_n$

The Cantor set C (6):

(i) Closed, hence compact & measurable

(2) perfect (≅ C)

(3) uncountable

(4) nowhere dense (contains no non empty open interval) and hence totally disconnected

(5) has measure zero

(6) has dense open complement

$$m(O_N) = 2^{N-1}\left(\tfrac{1}{3}\right)^N$$

$$m\left(\bigcup_{N=1}^{\infty} O_N\right) = \sum_{N=1}^{\infty}\left(2^{N-1}/3^N\right) = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$$

For each $x \in [0,1]$, there is a binary representation of $x = .b_1b_2\ldots$ $b_i \in \{0,1\}$

So $x = \sum_{i=1}^{\infty}\left(b_i/2^i\right)$

Pf pick $b_1 \ldots b_N$ inductively so that

$$\left| x - \sum_{n=1}^{N}\left(b_n/2^n\right)\right| \le \frac{1}{2^N}$$

$b_{n+1} = 0$ if $\left| x - \sum_{1}^{N} b_n/2^n \right| \le \frac{1}{2^{N+1}}$

otherwise $b_{n+1} = 1$ will make

$$\left| x - \sum^{N+1}\left(b_n/2^n\right)\right| \le \frac{1}{2^{N+1}}$$

This is called the irrational representation

i.e. there are no representations which end in a sequence of 0's.

$$\frac{1}{2^N} = .00\ldots\underset{\substack{\uparrow \\ N\text{th place}}}{1}000$$

$$= .00\ \underset{\substack{\uparrow \\ N\text{th place}}}{0}11 \quad (\text{irrational rep.})$$

The numbers $\frac{P}{2^N}$ P an integer, N an integer have 2 representations

<u>**Corollary**</u> $\mathcal{P}_f(\mathbb{N}) = \{A \subset \mathbb{N} : A \text{ is finite}\}$

$$\text{card}(\mathcal{P}_f(\mathbb{N})) = \aleph_0 \ (\text{aleph})$$

<u>Pf</u> $T : \mathcal{P}_f(\mathbb{N}) \longrightarrow [0,1]$

$$A \longmapsto .b_1 b_2 b_3 \ldots$$

where $b_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$

If A is finite, $T(A)$ ends in a string of zeros

$A \neq B$ both finite $\Longrightarrow T(A) \neq T(B)$

if $A \in \mathcal{P}_f(\mathbb{N})$ then $T(A) = \frac{P}{2^N}$ for some $P/2^N$

$$T(\mathcal{P}_f(\mathbb{N})) \subset \mathbb{Q}$$

<u>Corollary</u> $2^{\aleph_0} = \text{card}(\mathcal{P}(\mathbb{N})) = \mathfrak{c}$
$= \text{card}(\mathbb{R})$

$$[0,1] \xrightarrow[\text{into}]{1-1} \mathcal{P}(\mathbb{N}) \xrightarrow[\text{into}]{1-1} [0,2]$$

$x \longmapsto A$ infinite
irrational binary representation $T(A)$
$.b_1 b_2$

A finite $\longrightarrow T(A)+1$

Similarly, each $x \in [0,1]$ has a ternary representation $x = .t_1 t_2 \ldots = \sum_{n=1}^{\infty} (t_i/3^i)$
where each $t_i \in \{0,1,2\}$ $\exists$ unique irrational representation $.t_1 t_2 \ldots t_N 00\ldots$

$.t_1 t_2 \ldots (t_N - 1) 222\ldots$

numbers of the form
$x = P/3^N$: $P, N \in \mathbb{Z}^+$ are the only $x \in [0,1]$ with 2 different representations.

<u>Claim</u> $C = \{x \in [0,1] : x$ has a representation of the form $.t_1 t_2 \ldots$ which does not use $1\}$

$1/3 \in C$ $\quad 1/3 = .100\ldots$
$= .022\ldots$

define $\varphi : [0,1] \longrightarrow \mathbb{N} \cup \{\infty\}$

by $\varphi(.t_1 t_2 \ldots) = \begin{cases} N & \text{if } N \text{ is the 1st } t_i = 1 \\ \infty & \text{otherwise} \end{cases}$

$$f(x) = \sum_{n=1}^{\varphi(.t_1 t_2 \ldots)} \left(b_i / 2^i \right) \qquad b_i = t_i / 2 \quad i \in \mathbb{N}$$

↑ the <u>Cantor function</u> (6)

(7) increasing

(8) continuous

(9) Constant on connected components of C

(10) maps C onto [0, 1]

The function <u>$g(x) = x + f(x)$</u>

(11) maps [0, 1] onto [0, 2]

(12) is strictly increasing, thus 1-1

(13) is continuous

(14) is a homeomorphism

(15) maps C onto a set with positive measure

notes by KERWIN / PARK

<u>LECTURE 12</u> : Measurable Functions 10/20/78

$$f : \mathbb{R}^1 \to \mathbb{R}^1$$

From today on through the end of the quarter, we say "GOOD-BYE" to the PLANE and concentrate on the REAL LINE.

<u>DN 1</u>: f is <u>Measurable</u> (<u>Borel Measurable</u>) if it is a $\mathcal{L}-\mathcal{B}$ measure ($\mathcal{B}-\mathcal{B}$ Measure).

that is, $f^{-1}(B) \in \mathcal{L}$ $(f^{-1}(B) \in \mathcal{B})$ for each $B \in \mathcal{B}$.

<u>LEMMA 2</u>: f is Measurable (Borel measurable) if and only if

$f^{-1}(c) \in \mathcal{L}$ $(f^{-1}(c) \in \mathcal{B})$ for each $c \in \mathcal{C}$,

where $\mathcal{C}$ is any of the following collections:

(a) $\{(x, +\infty) : x \in \mathbb{R}\}$ (e) $\{(r, +\infty) : r \in \mathbb{Q}\}$

(b) $\{[x, +\infty) : x \in \mathbb{R}\}$ (f) $\{[r, +\infty) : r \in \mathbb{Q}\}$

(c) $\{(-\infty, x) : x \in \mathbb{R}\}$ (g) $\{(-\infty, r) : r \in \mathbb{Q}\}$

(d) $\{(-\infty, x] : x \in \mathbb{R}\}$ (h) $\{(-\infty, r] : r \in \mathbb{Q}\}$.

<u>proof</u>: Each of the above collections generates the Borel sets. Then by problem #2, T.P. #2, the proof is complete. QED.

<u>FACTS</u> about Measurable Functions:

<u>F3</u>: Continuous functions are Borel Measurable.

<u>F4</u>: Borel Measurability $\Rightarrow$ Measurability

<u>F5</u>: There is a set F which is Measurable but not Borel Measurable.

<u>proof</u>:

consider the function from the last lecture:

$g =$ cantor function $+ x$ $\qquad g : [0,1] \to [0,2]$

we know that: g is a homeomorphism $\ni$ $m(g(C)) > 0$.

Then by problem #2, T.P. #5: $\exists$ $E \subset g(C)$ $\ni$ E is non-measurable.

Let $F = g^{-1}(E)$ $\ni$ let $h = g^{-1}$.

h is continuous $\Rightarrow$ h is Borel Measurable.

but $h^{-1}(F) = h^{-1}g^{-1}(E) = E$.

Thus $F \in \mathcal{L}$, but $F \notin \mathcal{B}$. QED.

<u>NOTE</u> that in the above proof h is an example of a continuous function that is <u>not</u> $\mathcal{L}$-$\mathcal{L}$ Measurable (which is why we don't consider $\mathcal{L}$-$\mathcal{L}$ Measurability).

The following <u>FACTS</u> deal with the <u>Question</u>:

What kind of operations on Measurable functions yield Measurable functions?

<u>F6</u>: f is measurable $\Rightarrow$ $-f$ is measurable.

<u>ie</u>: $\{x : f(x) > r\} = \{x : -f(x) < -r\}$.

F7: If f is measurable and $c>0$ is Real, Then cf is measurable.

ie: $\{x : f(x) > r\} = \{x : cf(x) > cr\}$.

F8: χ_ϕ is measurable (where χ_ϕ is the Characteristic fun. of ϕ).

note that F8 $\Rightarrow$ (if $c=0$ then cf is measurable).

Combining F6, F7, and F8 we obtain:

F9: If f is measurable and $c \in \mathbb{R}$, then cf is measurable.

F10: If f & g are measurable, Then so are

$h = \text{Max}(f, g)$ and $k = \text{Min}(f, g)$.

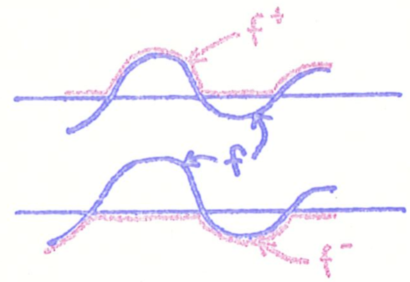
ie: $\{x : h(x) > r\} = \{x : f(x) > r\} \cup \{x : g(x) > r\}$

$\{x : k(x) > r\} = \{x : f(x) > r\} \cap \{x : g(x) > r\}$.

F11: If f is measurable, Then so are:

(a) $f^+ = \max(f, 0)$

(b) $f^- = \min(-f, 0)$.



F12: If $\{f_n\}$ are measurable, Then so are:

(a) $\sup_n \{f_n\} = g$

(b) $\inf_n \{f_n\} = h$.

ie: $\{x : g(x) > r\} = \bigcup_n \{x : f_n(x) > r\}$

$\{x : h(x) > r\} = \bigcap_n \{x : f_n(x) > r\}$.

F13: If $\{f_n\}$ are measurable, then so are:

(a) $\lim_{N\to\infty} \sup_{n \geq N} f_n$

(b) $\lim_{N\to\infty} \inf_{n \geq N} f_n$.

F14: by F13, If $f = \lim_{n\to\infty} f_n$ exists, Then f is measurable.

F15: Pointwise Limits of Measurable functions are Measurable.

LEMMA 16: If f & g are Measurable, Then $f+g$ is measurable.

proof: need to show: $\overset{A}{\{x: f(x)+g(x)>s\}} = \bigcup_{r\in\mathbb{Q}} \overset{B}{\left[\{x: f(x)>r\} \cap \{x: g(x)>s-r\}\right]}$.

($\supset$) Suppose $z \in B$. Then $f(z)>r$ & $g(z)>s-r \Rightarrow f(z)+g(z)>s$.

$\Rightarrow z \in A$. ✓

($\subset$) Suppose $z \in A$. Then $f(z)+g(z)>s$.

pick $\epsilon = f(z)+g(z)-s > 0$.

let $r \in \mathbb{Q}$ such that $0 < f(z)-r < \epsilon$,

$\Rightarrow 0 < g(z)+r-s$

$\Rightarrow s-r < g(z) \Rightarrow z \in B$. QED.

F16: If f is Measurable, Then so is $|f| = f^+ + f^-$.

F17: If f is Measurable, Then so is $f^2 = |f|^2$.

ie: $\{x: f^2(x) > r\} = \begin{cases} \mathbb{R} & \text{if } r<0 \\ \{x: |f(x)|>0\} & \text{if } r=0 \\ \{x: |f(x)|>\sqrt{r}\} & \text{if } r>0. \end{cases}$

F18: If f & g are Measurable, Then so is $f \cdot g$.

ie: $fg = \frac{1}{2}[(f+g)^2 + (-f^2) + (-g^2)]$, then combine previous facts.

DN 19: A property is said to hold Almost Everywhere (a.e.) if it is true everywhere except on a set of measure zero.

THM 20: If $f = g$ a.e., and f is measurable, Then g is also measurable.

proof: let $B = \{x : f(x) \neq g(x)\}$. Then $m(B) = 0$.

now $\{x : g(x) > r\} = [\{x : f(x) > r\} \cap B^c] \cup [\{x \in B : g(x) > r\}]$

this set is measurable, so is, and also.

$\Rightarrow \{x : g(x) > r\}$ is measurable $\Rightarrow$ g is measurable. QED.

COROLLARY 21: There is a Borel measure f such that $f = g$ a.e. by g IS NOT Borel Measurable.

proof: let $F \in \mathcal{L} \setminus \mathcal{B}$. Then $m(F) = 0$.

but $\chi_F = 0$ a.e.

$\Rightarrow F = \{x : \chi_F(x) > 0\}$.

QED.

COROLLARY 22: If f is measurable, then there is a Borel Measurable function g with $f = g$ a.e.

(proof next lecture)

F23: If $A \in \mathcal{L}$, Then χ_A is measurable.

DN 24: $s(x)$ is SIMPLE if there is a finite collection of $A_1, A_2, \ldots, A_n \in \mathcal{L}$ and numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that: $s = \sum_{i=1}^{n} a_i \chi_{A_i}$.

F25: Simple functions are Measurable and Measurable functions which take on a finite number of values are simple functions.

Test Problem #5 Due Fri 27 Oct 1978

1pt 1. If $r_\alpha \geq 0$ for each $\alpha \in I$, and $\sum_{\alpha \in I} r_\alpha = \sup\{r_{\alpha_1} + r_{\alpha_2} + \dots + r_{\alpha_n} ; \{\alpha_1, \alpha_2, \dots \alpha_n\}$ a finite subset of $I\}$, Show that $\sum_{\alpha \in I} r_\alpha < \infty$ implies that $\{\alpha \in I : r_\alpha \neq 0\}$ is countable. Hint use the principle of Bad sets, i.e show $\forall \varepsilon > 0$ the set $\{\alpha \in I : r_\alpha \geq \varepsilon\}$ is finite

1pt 2. Show that if $A \subseteq \mathbb{R}^1$, A is measurable and $m(A) > 0$, then there is a non-measurable $E \subset A$. Hint prove it for $A \subset [0,1)$.

2pt 3. Show that $\mathcal{B}_2$ and m_2 on $\mathbb{R}^2$ are invariant under rotations about $(0,0)$. Hint use the fact that m_2 is the unique measure agreeing with area on the basic open boxes. (i.e. those whose sides are parallel to x & y axis.

1.5pt 4. Let $0 < \alpha \leq 1$, Show that by mimicking the construction of the Cantor set only deleting "the middle" open intervals of length $\alpha(1/3), \alpha(1/3^2), \dots$ we obtain a set (sometimes called the generalized Cantor set) which is closed nowhere dense uncountable and has measure $1-\alpha$.

.25

5. Let f be a bounded function $\mathbb{R}^1 \to \mathbb{R}^1$, define $\omega(f,x,\delta)$ for $\delta > 0$ by $\omega(f,x,\delta) \doteq \sup_{|y-x|<\delta} f(y) - \inf_{|y-x|<\delta} f(y)$ Show that

0.5pt A. the $\lim_{\delta \to 0^+} \omega(f,x,\delta) = \omega(f,x)$ exists

1pt B. $\omega(f,x) = 0$ if and only if f is continuous at x.

2pt C. Let $[a,b] \subset \mathbb{R}^1$, and for $\eta > 0$ let $F_\eta = \{f(x)$ a bounded function on $[a,b]$ with the property that there are step functions φ and ψ with $\varphi \leq f \leq \psi$ and $\int_a^b \psi - \varphi < \eta\}$. Show that $\forall \delta > 0\ \forall \varepsilon > 0\ \exists_{\eta > 0}$ so that for $\forall f \in F_\eta$ $m^*(\{x \in [a,b] : \omega(f,x) \geq \delta\}) < \varepsilon$.

1pt. D. Show a R-$\int$'able function is continuous except on a set of

Notes by W. Chan
10/23/78

Lecture 13 : Measurable functions

Let $R^* = R \cup \{-\infty, \infty\}$. The extended reals

DN 1 : The basic open sets of ∞ are $\{(x, +\infty) : x \in R\}$
The basic open sets of $-\infty$ are $\{(-\infty, x) : x \in R\}$
The basic open sets of $x \in R$ are the same as in R

F2 : A set $\mathcal{U}$ is open $\Leftrightarrow \forall x \in \mathcal{U}, \exists$ a basic open set $B \subset \mathcal{U}$ s.t. $x \in B$

F3 : The Borel sets of R^*, $\mathcal{B}^*$, are exactly the smallest σ-algebra containing the open sets

$$\mathcal{B}^* = \{B \cup A_i : B \in \mathcal{B} \text{ and } i = 1, 2, 3, 4\}$$

$A_1 = \phi$ $\qquad A_3 = \{-\infty\}$
$A_2 = \{\infty\}$ $\qquad A_4 = A_2 \cup A_3 = \{\infty, -\infty\}$

F4 : $f : R \to R^*$ is measurable $\Leftrightarrow f^{-1}\{x \in R^* : x > r\}$ is measurable $\forall r \in R$

All properties of measurable FCNS. hold true for the extended m'ble fcns. Except for the following situation:

Consider $(f+g)(x)$ where $f(x) = \infty$
$g(x) = -\infty$
f, g meas. fcns.

EX 5. Examples of measurable functions.

(1) Characteristic Functions

$$\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & x \notin E \end{cases}$$

where E is measurable

(2) Simple functions

$s(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ where the E_i are measurable

(3) Elementary functions

$e(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{E_i}$ where the E_i are measurable and pairwise disjoint.

e measurable since $\sum_{i=1}^{n} \alpha_i \chi_{E_i} \to e$ pointwise

Note that each Simple Fcns has a Canonical Representation

Let $\beta_1, \cdots \beta_m$ be the list of values of $s(x) = \sum_1^n \alpha_i \chi_{E_i}$

Define $A_i = s^{-1}(\{\beta_i\})$, then A_i's are pairwise disjoint

and $s(x) = \sum_1^m \beta_i \chi_{A_i}$

THM 6: If $s_n(x)$ is a sequence of simple fcns, and

$s_n(x) \to f(x)$ pointwise a.e.

Then $f(x)$ is measurable

pf: Let B be a set s.t. $m(B) = 0$ and for $x \notin B$, $s_n(x) \to f(x)$

let $t_n(x) = \begin{cases} s_n(x) & x \notin B \\ 0 & x \in B \end{cases}$

Then $t_n(x) \to g(x) = \begin{cases} f(x) & x \notin B \\ 0 & x \in B \end{cases}$

Since $t_n \to g$ pointwise, g is measurable

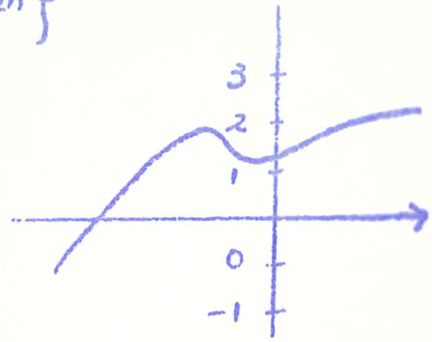
and since $g = f$ a.e. then f is measurable

THm 7 f measurable $\Rightarrow \exists \{s_n\}_{n=1}^{\infty}$ simple fcns s.t.

$s_n \to f$ pointwise

proof:

Let $A_n = \{\frac{i}{2^n} : i = 0, \pm 1, \pm 2, \pm 3, \dots, \pm 2^{2n}\}$



Partition the real line as follows:

for $[\frac{i}{2^n}, \frac{i+1}{2^n})$ where $0 \leq i \leq 2^{2n}$

let $E_i^n = f^{-1}([\frac{i}{2^n}, \frac{i+1}{2^n}))$ and

$\alpha_i^n = \frac{i}{2^n}$

for $(-\frac{i+1}{2^n}, -\frac{i}{2^n}]$ where $0 \leq i \leq 2^{2n}$

let $F_i^n = f^{-1}((-\frac{i+1}{2^n}, -\frac{i}{2^n}])$ and $\beta_i^n = -\frac{i}{2^n}$

Define $G_1 = f^{-1}([2^n, \infty))$ and $2^n = \gamma_1$

$G_2 = f^{-1}((-\infty, -2^n])$ and $-2^n = \gamma_2$

Let $S(x) = \sum_0^{2^{2n}} \alpha_i^n \chi_{E_i^n} + \sum_0^{2^{2n}} \beta_i^n \chi_{F_i^n} + \gamma_1 \chi_{G_1} + \gamma_2 \chi_{G_2}$

then $|S(x)| < |f(x)|$

Claim: $S_n \to f$ pointwise

let $x \in X$

Case I : $f(x) \in R$

Let $\epsilon > 0$ be given, choose n s.t. $|f(x)| < 2^n$ is s.t.

$\frac{1}{2^n} < \epsilon$

Note that this implies $m \geq n$ and that

$x \in E_i^m$ or F_i^m

$0 \leq f(x) - S^m(x) < \frac{1}{2^m} < \epsilon$

Case II $f(x) = +\infty$ (Similiar for $-\infty$)

then for all n, $x \in G_1$

$S_n(x) = 2^n \to +\infty$

Wednesday Oct-27, 1978 (Guess)

Theorem: (Egoroff) If E meas, $m(E) < \infty$, and $f_n \to f$ pt. wise a.e on E, then $\forall\, \delta > 0\ \exists\, B \ni m(B) < \delta$ and $f_n \to f$ uniformly on $E \setminus B$.

Reminder: $\forall\, \epsilon > 0\ \exists\, N(\epsilon) \ni \forall x \in E \setminus B$ and $\forall n \ge N$ $|f_n(x) - f(x)| < \epsilon$

Example: let $f_n(x) = (1 - \frac{1}{n})x$, $f(x) = x$; very measurable fcns

$f_n \to f$ pt. wise on $\mathbb{R}^1$. If $m(E) < \infty$, then $f_n \to f$ uniformly on $E \setminus B$

Proof fix $\epsilon > 0$, $B_N = \{x \mid \exists\, n \ge N, |f_n(x) - f(x)| \ge \epsilon\}$. Note: $B_n \supset B_{n+1}$ and $E \supset B_1$, so $m(B_1) < \infty$. Set $A = \bigcap_{n=1}^{\infty} B_n = \{x \mid \forall N \exists n \ge N \ni |f_n(x) - f(x)| \ge \epsilon\}$

$m(A) = 0$ since $f_n \to f$ pt. wise a.e.

(*) $\forall\, \epsilon > 0, \forall\, \delta > 0\ \exists\, B \subset E$ with $m(B) < \delta$ and

$\exists\, N \ni x \in E \setminus B \Rightarrow \forall n \ge N$ $|f_n(x) - f(x)| < \epsilon$

let $N \ni B_N : m(B_N) < \delta$

pick $\epsilon = \frac{1}{n}$, $\delta = \frac{\text{"}\delta\text{"}}{2^n}$ in (*) where "δ" is the 1st δ in the statement of the theorem.

$A_n, N_n \ni x \in E \setminus A_n$ $m > N_n$ $|f_m(x) - f(x)| < \frac{1}{n}$

$m(A_n) < \delta/2^n$

let $B = \bigcup_{n=1}^{\infty} A_n$, $m(B) \le \sum m(A_n) \le \sum \delta/2^n = \delta$

let $\epsilon > 0$ be given, then $\exists\, n \ni \frac{1}{n} < \epsilon$

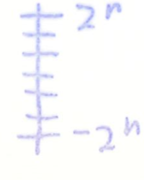
if $x \in E \setminus A_n$ then for $m > N_n$ $|f_m(x) - f(x)| < \frac{1}{n} < \epsilon$

and $x \in E \setminus B_n$ i.e. we have uniformly cgt. $\boxtimes$

Last time: If f is measurable, then $\exists$ simple fcns. $s_n \to f$ pt. wise

(1.) fixed n $\{x \mid |f_n(x) - s_n(x)| \ge \frac{1}{2^n}\} = \{x \mid |f(x)| > 2^n + \frac{1}{2^n}\}$

Picture:



(2.) If $f(x)$ is real valued and E is a set of finite measure, $\forall \epsilon > 0$, $\forall \delta > 0$, $\exists$ a simple fcn $s(x)$ with $m(\{x \in E \mid |f(x) - s(x)| \geq \epsilon\}) < \delta$

note: $\bigcap_{n=1}^{\infty} \{x \in E \mid |f(x)| > n\} = \emptyset$

__Definition:__ Support of a fcn $f: \mathbb{R} \to \mathbb{R}^*$, written $\operatorname{supp} f = \{x : f(x) \neq 0\}$

(3.) If f is measurable then $\exists$ simple fcns with support on a set of finite measure with $s_n \to f$ pt wise

__proof:__ If $t_n \to f$ pt.wise, t_n simple; let $s_n = t_n \cdot \chi_{[-n,n]}$. It is simple. $\operatorname{supp}(s_n) \subset [-n,n]$. $s_n \to f$ pt.wise i.e. for each x $s_n(x) \to f(x)$, but if $|x| < M$ $s_n(x) = t_n(x) \to f(x)$ for $n \geq m$

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remember: step fcn on $[a,b]$, there are pts $a = x_0 < x_1 < \cdots < x_n = b$, and numbers $\alpha_1, \alpha_2, \ldots, \alpha_n \ni$ for $x_{i-1} < t < x_i$ we have $\varphi(t) = \alpha_i$ $i = 1, 2, \ldots, n$

we want to do the show for step and continuous functions.

__Lemma:__ If $s(x)$ is a real valued simple fcn. with $\operatorname{supp}(s) \subset [a,b]$, then $\forall \epsilon > 0$ $\exists$ step fcn. $\varphi(x)$ on $[a,b]$ with $m(\{x \in [a,b] \mid s(x) \neq \varphi(x)\}) < \epsilon$ and $\inf_{x \in [a,b]} s(x) \leq \varphi \leq \sup_{x \in [a,b]} s(x)$

__proof:__ spse $E \subset [a,b]$ prove lemma for $\chi_E = s$, let $\epsilon > 0$ be given since E is measurable $\exists$ open intervals $I_1, I_2, \ldots, I_n \ni$ $m(E \,\Delta\, \bigcup_{i=1}^{n} I_i) < \epsilon$. we may assume $\bigcup_{i=1}^{n} I_i \subset (a,b)$ and that $I_1, I_2, \ldots I_n$ are pair wise disjoint

__Picture:__ a $\underset{I_1}{(\longrightarrow)}$ $\underset{I_2}{(\longrightarrow)}$ $\cdots$ $\underset{I_n}{(\longrightarrow)}$ b

__claim__ my step fcn is $\varphi(x) = \chi_{\bigcup_{i=1}^{n} I_i}$

If $x \ni s(x) \neq \varphi(x)$, then $x \in E \,\Delta\, \bigcup_{i=1}^{n} I_i$; hence done $s = \chi_E$

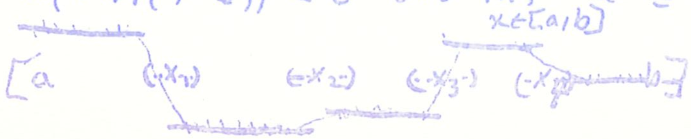
let $\epsilon > 0$ be given, use the lemma to define step fcns φ_i $i = 1, \ldots, m$ $\ni$ $m(\operatorname{supp}(\chi_{E_i} - \varphi_i)) < \epsilon/M$ let $\psi = \sum_{i=1}^{M} \alpha_i \varphi_i$ $\Leftarrow$ step fcn.

$\{x \mid \psi(x) \neq s(x)\} \subset \bigcup_{j=1}^{M} \operatorname{supp}(\chi_{E_j} - \varphi_j)$, $m(\operatorname{supp}(\psi - s)) < \epsilon$

$\psi' = \max\{\min\{\psi, \sup_{x \in [a,b]} s(x)\}; \inf_{x \in [a,b]} s(x)\}$ satisfies lemma $\boxtimes$

If φ is a real valued step fcn. on $[a,b]$ and $\epsilon > 0$, then $\exists$ a cont. fcn. f on $[a,b]$ with $m(\operatorname{supp}(f - \varphi)) < \epsilon$ and $\inf_{x \in [a,b]} \varphi \leq f \leq \sup_{x \in [a,b]} \varphi$

__picture:__



__Notation:__ $\ni$ = such that

$\forall$ = for every, for each for all, etc.

$\exists$ = there is (or are) there exist.

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Let $\epsilon > 0$ be given. Let $\epsilon_n = \frac{\epsilon}{2^{n+1}}$. If r_n is a listing of $\mathbb{Q}$, let $E = \bigcup_{r \in Q} (r_n - \epsilon_n, r_n + \epsilon_n)$

note $m(E) \le \epsilon$.

In fact by changing the order of r_n any pre-given sequence $\{x_n\} \subseteq R$ can be made to lie in E (i.e.) $x_n \in E \ \forall n \ge 1$

Let $s(x) = \sum_{j=1}^{m} \beta_j \chi_{F_j}$ be a simple function where $F_1 \dots F_m$ are measurable sets with $m(F_i) < \infty$ and $\beta_1 \dots \beta_m$ are reals.

Rewrite $s(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ in canonical form (i.e) E_i's are measurable and disjoint, assume no α

Then $\sum_{i=1}^{m} \beta_i m(F_j) = \sum_{i=1}^{n} \alpha_i m(E_i)$

Proof:- Let $\overline{X} = \{1, 2 \cdots m\}$ $\quad \mathcal{P} = \mathcal{P}(\overline{X})$

for $A \in \mathcal{P}$ define $F_A = \bigcap_{j \in A} F_j \setminus \bigcup_{j \notin A} F_j$

$r_A = \sum_{j \in A} \beta_j$

If $A = \phi$ $\quad F_\phi = R \setminus \bigcup_{j=1}^{m} F_j \quad r_\phi = 0$.

observe $A \ne B$ $\quad A, B \in \mathcal{P}$ then $F_A \cap F_B = \phi$

since $j \in A \setminus B$ $\quad F_A \subset F_j$ & $F_B \subset F_j^c$

$\therefore\ F_j = \bigcup_{\substack{A \in \beta \\ j \in A}} F_A$

It is easy to see that $F_j \supset \bigcup_{\substack{A \in \beta \\ j \in A}} F_A$

On the other hand if $x \in F_j$, let $A = \{ i \in \bar{X} : x \in F_i \}$

$j \in A$ & $x \in F_i \iff i \in A$

$x \in F_i^c$ if $i \notin A$

Thus $x \in F_A$ and hence we have the equality.

$$\sum_{A \in \beta} \gamma_A m(F_A) = \sum_{A \in \beta} \Big[\sum_{j \in A} \beta_j \Big] m(F_A)$$

$$= \sum_{A \in \beta} \sum_{j \in A} \beta_j \, m(F_A)$$

$$= \sum_{j=1}^{m} \sum_{\substack{A \in \beta \\ j \in A}} m(F_A) \cdot \beta_j$$

$$= \sum_{j=1}^{m} \beta_j \sum_{\substack{A \in \beta \\ j \in A}} m(F_A) = \sum_{j=1}^{m} \beta_j \, m(F_j)$$

For each i, $1 \le i \le n$ let $t_i = \{ A \in \beta : F_A \subset E_i \}$

<u>claim</u>. $E_i = \bigcup_{A \in t_i} F_A$ in which case $\gamma_A = \gamma_B = \alpha_i$ for any A, B in t_i.

$$\sum_{A \in \beta} \gamma_A m(F_A) = \sum_{i=1}^{n} \sum_{A \in t_i} \gamma_A \, m(F_A) = \sum_{i=1}^{n} \sum_{A \in t_i} \alpha_i \, m(F_A)$$

$$= \sum_{i=1}^{n} \alpha_i \, m(E_i)$$

Q.E.D.

Integration

If $s(x) = \sum_{i=1}^{n} \alpha_i \chi_{F_i}$ where for all i $m(F_i) < \infty$

define $\int s = \int s \, dm = \int s(x) \, dm(x) = \int s(x) \, dx = \int_{\mathbb{R}} s$

as $\int s = \sum_{i=1}^{m} \alpha_i m(F_i)$

This defination is unambigious by the earlier result.

Properties

(1) If c is a constant then $\int cs = c \int s$.

(2) If $s \geq 0$ then $\int s \geq 0$.

(3) If s & t are simple fns then $\int s+t = \int s + \int t$.

Pf Let $s = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, $t = \sum_{j=1}^{m} \beta_j \chi_{F_j}$

Then $s+t = \sum_{i=1}^{n} \alpha_i \chi_{E_i} + \sum_{j=1}^{m} \beta_j \chi_{F_j}$

$\int s+t = \sum_{i=1}^{n} \alpha_i m(E_i) + \sum_{j=1}^{m} m(F_j) \beta_j = \int s + \int t$.

Defination :- If f is a positive measurable function then

$\int f = \sup \{ \int s : s$ simple ≥ 0 with $m(\operatorname{Supp} s) < \infty$ and $s \leq f \}$.

This defination is consistent with our earlier defination clearly if f is s

then $\int_{\text{Def 2}} f \geq \int_{\text{Def 1}} s$.

<u>Conversely</u> If t is simple ≥ 0 & $t \leq s$

then $s - t \geq 0$

$$\int s - \int t = \int (s-t) \geq 0 \Rightarrow \int s \geq \int t$$

$$\Rightarrow \int_{\text{Def 1}} f \geq \int_{\text{Def 2}} s.$$

$$\Rightarrow \int_{\text{Def 1}} f = \int_{\text{Def 2}} f.$$

<u>Properties</u>

(1) If $c > 0$ then $\int cf = c \int f$.

If $c = 0$ $\int 0 = 0$.

(2) $f \geq 0 \Rightarrow \int f \geq 0$.

(3) f is positive & g is positive

$$\int f+g = \int f + \int g.$$

<u>Pf:-</u>

$S = \{ \int s : 0 \leq s \leq f, \ s \text{ is simple} \}$

$T = \{ \int t : 0 \leq t \leq g, \ t \text{ is simple} \}$

$S + T \subset \{ \int u : 0 \leq u \leq f+g, \ u \text{ simple} \}$

$\sup (S+T) \leq \sup \{ \int u : 0 \leq u \leq f+g, \ u \text{ simple} \}$

$= \int f+g$

$\therefore \sup S + \sup T = \int f + \int g$

$\Rightarrow \int f + \int g \leq \int (f+g)$.

Test Problem #6 Due Fri 3 Nov 1978

0.5 1 A. If $\{A_n\}$ is a sequence of measurable sets $\subset \mathbb{R}$, then

$$(*) \quad m(\liminf A_n) \le \liminf m(A_n). \text{ [Do not use Fatou's Lemma]}$$

1.0 B. For each λ, $0<\lambda<1$, Construct $A_n \subset [0,1]$ with $m(A_n) \equiv \lambda$ for all n and $\limsup A_n = [0,1]$ and $\liminf A_n = \emptyset$ & show both $m(\liminf A_n) < \liminf m(A_n)$ and $m(\limsup A_n) > \limsup m(A_n)$ are possible

0.5 C. Show that $m(\limsup A_n) < \limsup m(A_n)$ is possible.

2 pts

0.75 2. A. If for each $r \in \mathbb{Q}$, G_r is a Borel set $\subset \mathbb{R}$ and if $r<s$ then $G_r \supset G_s$, show that $g(x) = \sup\{r \in \mathbb{Q} : x \in G_r\}$ is Borel measurable fcn : $\mathbb{R} \to \mathbb{R}^*$

1.50 B. <u>USE PART A</u>, to "<u>construct</u>" for each measurable function $f : \mathbb{R} \to \mathbb{R}$ a Borel measurable $g : \mathbb{R} \to \mathbb{R}$ with $f = g$ a.e.

2.25 4.25

3. We will say that the meas fcns $f_n : E \to \mathbb{R}$ or $\mathbb{R}^*$ convergences to $f : E \to \mathbb{R}$ or $\mathbb{R}^*$ in <u>measure</u> (if $m(E)=1$ it's sometimes called convergence in probability) if $\forall \varepsilon>0\ \forall \delta>0$ (*) is true; where

$$(*) \quad \exists_N \text{ s.t. } \forall n \ge N \quad m\{x : |f(x) - f_n(x)| \ge \delta\} < \varepsilon.$$

0.25 A. Show that $f_n \to f$ in measure if and only if $\forall \varepsilon > 0$ (*) is true with $\delta = \varepsilon$.

The sequence of meas fcns $\{f_n\}$ are said to converge in measure if there is some meas f with $f_n \to f$ in meas.

1.50 B. Show that $\{f_n\}_n$ converges in measure if and only if every subsequence $\{f_{n_i}\}_i$ of $\{f_n\}_n$ has a subsequence $\{f_{n_{i_j}}\}_j$ (of $\{f_{n_i}\}_i$) which converges in measure. 80 70

[Watch out, the right hand side does not say that different subsequences must have same limit] [Hint: This is a quite general principle] 1.75

0.75 4 A. Show that if $F \subset \mathbb{R}$ is a closed set and $f: F \to \mathbb{R}$ is continuous, then there is a $g: \mathbb{R} \to \mathbb{R}$, g is continuous and $f(x) = g(x)$ if $x \in F$.

1.00 B. Show that if $f: \mathbb{R} \to \mathbb{R}$ is measurable then there are continuous functions $g_n: \mathbb{R} \to \mathbb{R}$ with $m(\text{supp } g_n) < \infty$ for each n and $g_n \to f$ pt-wise a.e.

Lusin's Thm: If $f: E \to \mathbb{R}$ is measurable and $\varepsilon > 0$ then there is a continuous function $g: E \to \mathbb{R}$ with $m(\text{supp}(f-g)) < \varepsilon$.

~~.50~~ 1.50 C. Prove Lusin's Thm for $E = [a,b]$

0.75 D. Prove Lusin's Thm for $E = \mathbb{R}$ and (meas) $E \subset \mathbb{R}$.

Correction for H.W. set 5

Oct. 30 Mon. '78.

$m(C_\alpha) = 1-\alpha$ uncountable

It is not true $C_\alpha \supset C_1$. e.g. $\alpha = \frac{1}{2}$, $\frac{2}{9} \in C_1 \setminus C_{\frac{1}{2}}$.

Notation:

$\mathbb{Z}$ = set of positive, negative + zero integers

$\forall n \in \mathbb{Z}$

$A_n = A \cap [n, n+1) \quad n \in \mathbb{Z}$

$A = \bigcup_{n \in \mathbb{Z}} A_n$, say $m(A_n) > 0$

$E \subseteq A_n - n \subset [0,1]$

$E + n \subset A_n$

[THM] Monotone Convergence thm.

f_n, f are meas. ≥ 0 & $0 \leq f_1 \leq f_2 \leq \cdots \leq f$ &

$f_n \longrightarrow f$ pointwise

$\lim_{n \to \infty} \int f_n = \int f$

Cor

$f \geq 0$. S_n are simple functions with $(\mathrm{supp}(S_n)) < \infty$

$f_n \longrightarrow f$ point-wise

Cor

[illegible] $\int f+g = \int f + \int g$ for $f, g \geq 0$

Pf.

Let s_n (simple) $s_1 \leq s_2 \leq \cdots \leq f$

$m(C_x)$ $s_n \longrightarrow f$ pointwise $m(\text{supp } s_n) < \infty$

$\exists$ $t_n \longrightarrow f$ pt-wise $t_1 \leq \cdots \leq g$ $m(\text{supp } t_n) < \infty$

$s_n + t_n \longrightarrow f+g$ pt-wise

$s_1 + t_1 \leq s_2 + t_2 \leq \cdots \leq f+g$

$$\int f + \int g = \lim_n \int s_n + \lim_n \int t_n$$

$$= \lim_n \left(\int s_n + \int t_n \right)$$

$$= \lim_n \int (s_n + t_n) = \int f+g.$$

Lemma

If $s(x) = \sum_1^n \alpha_i \chi_{E_i}$ $\alpha_i > 0$. simple $\{E_i\}$ p.w.d.

$m(E_i) < \infty$. if $0 \leq s \leq f$. then

$\forall \epsilon > 0$. $\exists N$ s.t. $n \geq N \Longrightarrow \int f_n \geq \int s - \epsilon$.

Pf. (given lemma.)

$\forall M$ $\exists$ simple $s(x)$ like hypothesis of lemma $s \leq f$

$0 \leq s \leq f$

$\int s \geq M+1$, use lemma $\epsilon = 1$ find N, s.t.

$n \geq N \Rightarrow \int f_n \geq s - 1 \geq M$

$\therefore \lim \int f_n = \infty$.

for the case where $\int f = M < \infty$

Let $\delta > 0$ be given. $\exists$ Simple function s

$0 \le s \le f$ with $m(\text{supp } s) < \infty$ s.t.

$$\int s + \frac{\delta}{2} \ge M.$$

Apply lemma $\epsilon = \frac{\delta}{2}$, $\exists N$ s.t. $n \ge N$

$\Rightarrow \int f_n + \frac{\delta}{2} \ge \int s$

$\Rightarrow \int f \ge \int f_n \ge \int s - \delta.$

$\therefore \lim_n \int f_n = \int f.$

Pf of lemma

$F = \text{supp } s$

$F_n = F \cap [-n, n]$

F_n nested increasing sequence $m(F) = \lim m(F_n)$

Let $\epsilon > 0$ be given $A = \max \alpha_i$

Choose n s.t. $m(F_n) > m(F) - \frac{\epsilon}{2A}$

$s' = s \chi_{F_n}$, $t = A \chi_{F \setminus F_n}$

simple $s' \le s \le s' + t$

$$\int s' \le \int s \le \int s' + t = \int s' + \int t \le \int s' + A\, m(F \setminus F_n)$$
$$\le \int s' + \frac{\epsilon}{2}.$$

Thus, it suffices to prove the lemma for s w/

Supp $s \subset [-n, n]$

Assume supp $s \subset [-M, M]$

Let $\epsilon > 0$ be given

By Egoroff's thm: $\exists B \subset [-m, m]$ with

$m(B) < \frac{\epsilon}{2A}$ $\quad A = \max \alpha_i$ s.t. on

$[-m, m] \setminus B$ $\quad f_n \longrightarrow f$ uniformly

Choose $\delta > 0$. s.t.

(1). $\delta \le \min \alpha_i$

(2) $2m\delta < \frac{\epsilon}{2}$

Pick N s.t. $n \geq N$.

$\Rightarrow \quad 0 \le f(x) - f_n(x) < \delta$ for $x \in [-m, m] \setminus B$

Define $\quad t(x) = \Sigma(\alpha_i - \delta)\chi_{E_i \setminus B}$

$0 \le t \le f_n$ for $n \geq N$.

$\int t \le \int f_n$

But $\int t = \sum_1^n (\alpha_i - \delta)\, m(E_i \setminus B)$

$\geq \sum_1^n (\alpha_i - \delta)\, m(E_i) - A\, m(B)$

$\geq \sum_1^n \alpha_i\, m(E_i) - \delta \Sigma\, m(E_i) - \frac{\epsilon}{2}$

$= \int s - \epsilon$

$\Leftarrow$ f is meas. $\quad f^+ = \max(f, 0)$

$f^- = f^+ - f = \max(0, -f)$

$|f| = f^+ + f^-$

For general meas. f. if not both

$\int f^+$ & $\int f^-$ are ∞, we define $\int f = \int f^+ - \int f^-$

if both $\int f^+, f^- < \infty$. we will say $f \in L_1(\mathbb{R})$

① If $f \geq 0$

$$\underset{\text{old}}{\int f} = \underset{\text{new}}{\int f}$$

$f = f^+ \quad f^- = 0.$

② If $s = \sum_1^n \alpha_i \chi_{E_i}$ is simple & $\{E_i\}$ p.w.d.

$m(E_i) < \infty \qquad \alpha_i > 0 \quad i = 1, \cdots m$

$\alpha_i < 0 \quad i = m+1 \cdots n$

$$s^+ = \sum_1^m \alpha_i \chi_{E_i}$$

$$s^- = -\sum_{m+1}^n \alpha_i \chi_{E_i}$$

$$\int s^+ = \sum_1^m \alpha_i m(E_i)$$

$$\int s^- = -\sum_{m+1}^n \alpha_i m(E_i)$$

$$\int s^+ - \int s^- = \sum_1^n \alpha_i m(E_i) = \int s$$

f measure one of $\underbrace{\int f^+ \;\&\; \int f^-}_{\text{if both finite } f \in L_1 \mathbb{R}} < +\infty$

(1) $\int cf = c\int f$ $\quad c$ is real

(a) $c > 0$ $\quad (cf)^+ = cf^+$

$(cf)^- = cf^-$

so $\int cf = \int (cf)^+ - \int (cf)^-$

$= \int cf^+ - \int cf^-$

$= c\left(\int f^+ - \int f^-\right) = c\int f$

(b) $c = 0$, then $(cf)^+ = 0 = (cf^-)$

$\int cf = 0 - 0 = 0\int f$.

(c) $c = -1$ (note finish all cases).

$(-f)^+ = f^-$

$(-f)^- = f^+$

$\int(-f) = \int f^- - \int f^+ = -\left(\int f^+ - \int f^-\right) = -\int f$.

(2) $f \geq 0 \Rightarrow \int f \geq 0$ done.

(3) If E is measure $\subseteq \mathbb{R}$.

define $\int_E f = \int f \chi_E$

$f^+ \geq f^+ \chi_E = (f\chi_E)^+$

$f^- \geq f^- \chi_E = (f\chi_E)^-$

If $E = A \cup B$ and $A \cap B = \phi$, then $\int_E f = \int_A f + \int_B f$.

$(f\chi_A)^+ = f^+ \chi_A$ similarly for B, E.

$(f\chi_A)^- = f^- \chi_A$ " —— " —— "

$\int_E f = \int f^+ \chi_E - \int f^- \chi_E$

$$\int_A f + \int_B f = \int f^+ \chi_A - \int f^- \chi_A + \int f^+ \chi_B - \int f^- \chi_B$$

$$= \int (f^+\chi_A + f^+\chi_B) - \int (f^-\chi_A + f^-\chi_B)$$

$$= \int f^+ \chi_E - \int f^- \chi_E$$

suppose $0 \le g \le f$ suppose $g \in L(\mathbb{R})$.

then $\int (f-g) = \int f - \int g$

$f - g + g = f$

positive function

$\int (f-g) + \int g = \int f$.

subst. $\int g$ from both sides.

(4) suppose $f, g \ge 0$. & one of them is in $L_1(\mathbb{R})$.

$\int (f-g) = \int f - \int g$.

$(f-g)^+ = (f-g)\chi_{\{x : f(x) \ge g(x)\}}$.

$(f-g)^- = (g-f)\chi_{\{x : f(x) < g(x)\}}$.

$\int (f-g) = \int (f-g)^+ - \int (f-g)^- = \int_A (f-g) - \int_B (g-f)$

on A $f \ge g$ $\int_A g < +\infty$.

suppose $\int_A g = \infty$, $\int g = \infty$

because $g \ge \chi_A g$

$f \ge g$ on A $\int f \ge \int_A f \ge \int_A g = \infty$

$\therefore \int_A g < \infty$.

$\int (f-g)\chi_A = \int f\chi_A - \int g\chi_A$

$\int (g-f)\chi_B = \int g\chi_B - \int f\chi_B$

$= \int_A f - \int_A g - \int_B g + \int_B f = \int f - \int g$.

⑤ If $\int f + \int g$ makes sense (i.e., we don't have $\infty - \infty$) then $\int (f+g) = \int f + \int g$.

<u>Note</u> The hypothesis $\Rightarrow$ either both $\int f^+$ and $\int g^+ < \infty$ $(*)$
or both $\int f^-$ and $\int g^- < \infty$.

$$(f+g)^+ \le f^+ + g^+$$
$$(f+g)^- \le f^- + g^-$$

Thus $\int (f+g)$ is defined, since either $\int (f+g)^- < \infty$ or $\int (f+g)^+ < \infty$ by $(*)$.

Since $f+g = (f^+ + g^+) - (f^- + g^-)$,

$$\int (f+g) \overset{\text{by (4)}}{=} \int (f^+ + g^+) - \int (f^- + g^-)$$
$$= \int f^+ - \int f^- + \int g^+ - \int g^-$$
$$= \int f + \int g.$$

<u>DN</u> If E (meas) $\subset \mathbb{R}$ and $\int_E f < \infty$ then we say $f \chi_E \in L_1(E)$.

⑥ Suppose f is R-Sable on $[a,b]$. Then $f \in L_1([a,b])$ and furthermore $\int_{[a,b]} f = R\int_a^b f(x)\,dx$.

Note that this says nothing about improper R-S's (interval is a closed & bdded one)

⑦ If $f \ge g$ and $\int f$, $\int g$ are defined, then $\int f \ge \int g$.

<u>Proof</u>
$f - g \ge 0$ Thus $\int g + \int (f-g) = \int f$
$\Rightarrow \int g \le \int f$.

⑧ If f, g are meas., $f = g$ a.e. and f, g are integrable, then $\int f = \int g$.

Proof Let B be a set such that $mB = 0$ and $f = g$ on $\mathbb{R} \setminus B$.

Then $\int_{\mathbb{R}\setminus B} f = \int_{\mathbb{R}\setminus B} g$. [If h is any integrable fn., $\int_B h = \int_B h^+ - \int_B h^- = 0 - 0 = 0$]

So $\int f = \int_B f + \int_{\mathbb{R}\setminus B} f = \int_{\mathbb{R}\setminus B} f = \int_{\mathbb{R}\setminus B} g = \int_B g + \int_{\mathbb{R}\setminus B} g = \int g$.

Example Since $\chi_{\mathbb{Q}} = 0$ a.e., $\int \chi_{\mathbb{Q}} = 0$.

Monotone Convergence Thm: (a restatement w/ weaker hypotheses)

Let f_n, f be meas, $0 \underset{\text{a.e.}}{\le} f_1 \underset{\text{a.e.}}{\le} f_2 \underset{\text{a.e.}}{\le} \cdots$, $f_n \to f$ ptwise a.e.

(note that the set where all these inequalities hold is a set of meas. zero, since it's a cntable collection of sets of meas. zero)

then $\lim_{n\to\infty} \int f_n = \int f$.

Thm: For every meas. set E, $\int_E f = 0$ iff $f = 0$ a.e.

Proof ($\Leftarrow$) Let B be such that $B = \{x \mid f(x) \neq 0\}$. Then $mB = 0$.

So $\int f = \int_B f + \int_{\mathbb{R}\setminus B} f = 0 + 0 = 0$.

($\Rightarrow$) Suppose $f \neq 0$ a.e. Either ⓐ $\{x : f(x) > 0\}$ or ⓑ $\{x : f(x) < 0\}$ has positive measure. Say ⓐ has +ve meas.

$\{x : f(x) > 0\} = \bigcup_{n\in\mathbb{N}} \{x : f(x) > \frac{1}{n}\}$ rhs is a nested incr. seq.

Thus for some n $m\{x : f(x) > \frac{1}{n}\} > 0$. Define $s(x) = \frac{1}{n} \chi_{\{x : f(x) > \frac{1}{n}\}}$.

Then $s(x) \le f$. But then $\int_{\{x : f(x) > \frac{1}{n}\}} f > 0$. Contradiction, so the proof is complete.

Notes by Pacheco, Jackson

11/3/78

Lecture 18

F1: Let f and g be measurable fcns, $f, g \in L_1(\mathbb{R})$ and spse $f \geq g$. Then

$$\int f \geq \int g$$

Pf: Since $f \geq g$, $f^+ \geq g^+$ and $g^- \geq f^-$. Since $f \in L_1(\mathbb{R})$, $\int g^+ < +\infty$. So

$$\int g = \int g^+ - \int g^-$$

exists. Since $g \in L_1(\mathbb{R})$, $\int f^- < \infty$. So $\int f$ exists and

$$\int f = \int f^+ - \int f^- \geq \int g^+ - \int g^- = \int g.$$

F2: If $\int f$ exists, then

$$\left| \int f \right| \leq \int |f|.$$

Pf:

$$\begin{aligned} \left|\int f\right| &= \left|\int f^+ - \int f^-\right| \\ &\leq \left|\int f^+\right| + \left|\int f^-\right| \\ &= \int f^+ + \int f^- \\ &= \int f^+ + f^- \\ &= \int |f|. \end{aligned}$$

F3: If $f \in L_1(\mathbb{R})$, $|f| \in L_1(\mathbb{R})$

Cor: If $f, g \in L_1(\mathbb{R})$

$$\int |f+g| \leq \int |f| + \int |g|$$

Pf: Since $|f+g| \leq |f| + |g|$

$$\int |f+g| \leq \int (|f| + |g|)$$
$$= \int |f| + \int |g|$$

Cor: If $f, g \in L_1(\mathbb{R})$

$$\left| \int |f| - \int |g| \right| \leq \int |f-g|.$$

Pf: Since $|f| = |f-g+g| \leq |f-g| + |g|$,

$$\int |f| \leq \int |f-g| + \int |g|.$$

So

$$\int |f| - \int |g| \leq \int |f-g|.$$

Since $|f-g| = |g-f|$, a similar argument shows that

$$\int |g| - \int |f| \leq \int |f-g|.$$

Hence

$$\left| \int |f| - \int |g| \right| \leq \int |f-g|.$$

Fatou's Lemma: If $\{f_n\}_{n=1}^{\infty}$, f are measurable fcns, $f, f_n \geq 0$, and $f_n \to f$ pt. wise a.e. Then

$$\int f \leq \liminf \int f_n.$$

Pf: Let $B \subseteq \mathbb{R}$, w/ $m(B) = 0$ and $f_n \to f$ pt. wise on $\mathbb{R} \setminus B$. Let

$$g_n = f_n \chi_{\mathbb{R}\setminus B}, \quad g = f \chi_{\mathbb{R}\setminus B}.$$

Note that $g_n \to g$ pt. wise, $\int g_n = \int f_n$, $\int g = \int f$, and $g_n, g \geq 0$. Let

$$h_n(x) = \inf_{i \geq n} g_i(x).$$

Then each h_n is measurable & ≥ 0. Note that $h_n \leq h_{n+1}$, and

$$\begin{aligned} g(x) &= \lim g_n(x) \\ &= \lim\inf g_n(x) \\ &= \lim_{n\to\infty} \inf_{i \geq n} g_i(x) \\ &= \lim_{n\to\infty} h_n(x). \end{aligned}$$

So $h_n \to g$ pt. wise and $h_n \leq g$. By the monotone convergence Thm.

$$\int g = \lim_{n\to\infty} \int h_n.$$

Since $h_n \leq g_n$, $\int h_n \leq \int g_n$. Hence

$$\inf_{i \geq n} \int h_i \leq \inf_{i \geq n} \int g_i,$$

and

$$\lim_{n\to\infty} \inf_{i \geq n} \int h_i \leq \lim_{n\to\infty} \inf_{i \geq n} \int g_i.$$

Now

$$\begin{aligned}\int f = \int g &= \lim_{n\to\infty} \int h_n \\ &= \lim_{n\to\infty} \inf_{i\ge n} \int h_i \\ &\le \lim_{n\to\infty} \inf_{i\ge n} \int g_i \\ &= \liminf \int g_n \\ &= \liminf \int f_n .\end{aligned}$$

<u>Lebesgue</u> <u>Dominated</u> <u>Convergence</u> <u>Thm</u>: Spse that $\{f_n\}_{n=1}^{\infty}$, f, $g \in L_1(\mathbb{R})$, and $g \ge 0$. Spse also that $|f|, |f_n| \le g$. If $f_n \to f$ pt. wise a.e., then $\int f = \lim \int f_n$.

<u>Pf</u>: $0 \le f_n + g$ and $f_n + g \longrightarrow f+g$ pt. wise a.e.
So, by Fatou's Lemma,

$$\begin{aligned}\int f + \int g = \int (f+g) &\le \liminf \int (f_n+g) \\ &= \liminf \left(\int f_n + \int g\right) \\ &= \liminf \int f_n + \int g .\end{aligned}$$

Hence

$$\int f \le \liminf \int f_n .$$

Also, $0 \le g - f_n$ and $g - f_n \to g - f$ pt. wise a.e.
Using Fatou's Lemma again, we have

$$\begin{aligned}\int g - \int f = \int (g-f) &\le \liminf \int (g-f_n) \\ &= \liminf \left(\int g - \int f_n\right)\end{aligned}$$

$$= \int g - \limsup \int f_n .^{*}$$

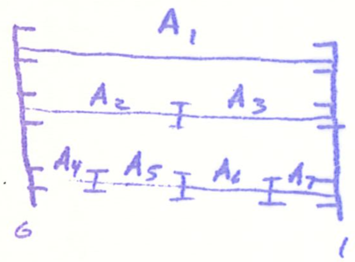
So

$$\limsup \int f_n \le \int f ,$$

and

$$\int f = \lim \int f_n .$$

<u>Example 4</u>: Let $A_m = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]$, where $m = 2^n + j$, when $0 \le j \le 2^n - 1$, n a nonnegative integer. Then $\lim_{n\to\infty} m(A_n) = 0$. So $\chi_{A_n} \longrightarrow 0$ in measure, but χ_{A_n} does not converge for each $x \in [0,1]$.



<u>F5</u>: Spse $f_n \longrightarrow f$ in measure. Then $\exists$ a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ $\ni$ $f_{n_i} \longrightarrow f$ pt. wise a.e.

$*$ $$\liminf (G - F_n) = \lim_{n\to\infty} \inf_{i \ge n} (G - F_i)$$
$$= \lim_{n\to\infty} \left(-\sup_{i\ge n} (F_i - G)\right)$$
$$= -\limsup (F_n - G)$$
$$= -(\limsup (F_n - G))$$
$$= G - \limsup F_n$$

Pf: $\forall\, i \in \mathbb{Z}^+$, choose N_i so that $\forall\, n \geq N_i$

$$m\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^i}\right\} < \frac{1}{2^i}.$$

Consider f_{N_i}. Let

$$B_i = \left\{x : |f_{N_i}(x) - f(x)| \geq \frac{1}{2^i}\right\}.$$

We want to show that $f_{N_i} \to f$ pt. wise a.e. Let

$$B = \bigcap_{N=1}^{\infty}\left[\bigcup_{i \geq N} B_i\right].$$

Note that $m\left(\bigcup_{i \geq N} B_i\right) \leq \sum_{i \geq N} \frac{1}{2^i} = \frac{1}{2^{N-1}}$. So $m(B) = 0$.

If $x \notin B$, and $\varepsilon > 0$, choose $n \ni$

(1) $\frac{1}{2^n} < \varepsilon$

(2) $x \notin \bigcup_{i \geq n} B_i$.

Then for $i \geq n$, $x \notin B_i$ and

$$|f(x) - f_{n_i}(x)| < \frac{1}{2^i} \leq \frac{1}{2^n} < \varepsilon.$$

TEST PROBLEM #7 Due FRI 10 NOV 1978 4:00pm in 211 LOVE T

1\. A Prove the generalized Lebesgue Dominated Convergence Theorem:

0.5pt If f_n, f are measurable; g_n, g positive functions in $L_1(\mathbb{R})$, $f_n \to f$ pointwise a.e; $g_n \to g$ pt-wise a.e.; $|f_n| \le g_n$ & $|f| \le g$ and if $\int g = \lim_n \int g_n$ then $\int f = \lim_n \int f_n$. lim inf ≤ lim inf + lim inf −.20

1pt B. Suppose $f_n, f \in L_1(\mathbb{R})$ and $f_n \to f$ pt-wise a.e. Show $\lim_n \int |f - f_n| = 0$ if and only if $\lim_n \int |f_n| = \int |f|$

2\. A Let $L \in \mathbb{R}$ and f a real-value function defined on $[1, +\infty)$

5.5 Show that the following are equivalent

(i) $\lim_{x \to \infty} f(x) = L$

1pt

.50 (ii) For every ~~sub~~sequence $\{x_n\}$ with $\lim_n x_n = \infty$ then $\lim_n f(x_n) = L$

(iii) For every sequence $\{x_n\}$ with $\lim_n x_n = \infty$ and satisfying $x_1 < x_2 < x_3 < \dots < x_n < x_{n+1} < \dots$ then $\lim_n f(x) = L$

HINT Show (i) ⇒ (ii) ⇒ (iii) & not (i) ⇒ not (iii)

1pt B. Let $f \ge 0$ and for $t \in [1, \infty]$ let $I(t) = [1, t]$ Show
$$\lim_{t \to \infty} \int_{I(t)} f = \int_{I(\infty)} f \qquad \text{HINT USE A(iii)}$$
one such −.40

0.5pt C. Show that if f is a bounded positive function and the improper

.20 Riemann $\int_1^\infty f(x)dx$ exists $< +\infty$ $\left[\text{i.e. } \lim_{t \to \infty} \int_1^t f(x)dx = \int_1^\infty f(x)dx\right]$

then $f \in L_1([1, \infty))$ and $\int_{[1,\infty)} f = \int_1^\infty f(x)dx$ −.10

1pt D. Show if $g(x) = \dfrac{\sin x}{x}$ then the improper R-$\int$ $\int_1^\infty g(x)dx$

.20 exists and is finite

1pt E. Show that $g(x)$ in D is not only not in $L_1([1, \infty))$ but $\int_{[1,\infty)} g(x)dx$ is not even defined!

0.5pt F. Use E to construct a sequence of functions $f_n \in L_1([1, \infty))$ such that $f_n \to g$ (in D) uniformly on $[1, \infty)$ and hence there is no uniform convergence theorem

0.5pt G. Show that if $m(E) < \infty$ and $f_n \to f$ uniformly on E then $\lim_n \int_E f_n = \int_E f$ [all functions f_n, f are measurable] −.15 not considering $\int f = +\infty$

TP 7. Pg 2

1pt 3. If f is measurable on $[0,1]$ define $\sigma(f) = \int_{[0,1]} \frac{|f|}{1+|f|}$
Show $f_n \to f$ in measure on $[0,1]$, if and only if $\lim_n \sigma(f_n) = 0$

2pts 4. Suppose X is a set and $\mathcal{A}$ is a σ-algebra on X and $\mathcal{A}$ has infinitely many elements show $\mathcal{A}$ has uncountably many elements

1,05
3,95
,98
,60
6,58

1,30
4,00
zip
1,40
6,70

2,1
1,45
3.80
,50
1.95
70

1,48
2,75
,35
2.00
6,58

.90
3,00
.40
.30
4,60

Notes: Monday
11/6/78
G. Gray

Note: $\chi_{\mathbb{Q}}$ is continuous on $\mathbb{Q}$
& continuous on $\mathbb{R}\setminus\mathbb{Q}$

also $f_n(x) = \frac{1}{n}x$ $\quad f_n \to f \equiv 0$ pt. wise
$\quad f_n \not\to f$ in measure.

Let $0<p<\infty$ and E be a measurable subset of $\mathbb{R}$

DEF 1: $L_p(E) = \{f: E \to \mathbb{R} \mid \int |f|^p < +\infty\}$. If $E=[0,1]$ we will write L_p for $L_p([0,1])$.

If $f: E \to \mathbb{R}$ then

DEF 2: $\|f\|_p = \left(\int_E |f|^p\right)^{1/p}$

DEF 3: $\|f\|_\infty = \text{ess sup } f = \inf\{M \geq 0 : m\{x : |f(x)| \geq M\} = 0\}$
(ess sup is for "essential sup").

DEF 4: $L_\infty(E) = \{f : \|f\|_\infty < +\infty\}$.

We have shown $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$. It is also true for $p \geq 1$ the triangle inequality holds. i.e. $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. For $p<1$ the triangle inequality is false in geural.

EX 5. $L_p(\mathbb{R})$, $p<1$, $f = \chi_{[1,2]}$, $g = \chi_{[3,4]}$

$\|f\|_p = 1 = \|g\|_p \quad \left(\int |f+g|^p\right)^{1/p} = \left(\int (\chi_{[1,2]\cup[3,4]})^p\right)^{1/p}$
$= (2)^{1/p} > 2.$

So $\|f+g\|_p > \|f\|_p + \|g\|_p$.

RMK 6. If $0 < p \leq q \leq \infty$ then $L_p \supseteq L_q \supset L_\infty$.

RMK 7. If $p \neq q$ then both $L_p(\mathbb{R}) \subset L_q(\mathbb{R})$ and $L_p(\mathbb{R}) \supset L_q(\mathbb{R})$ are false.

RMK 8. $\forall\, 0 < p < \infty$ $(a+b)^p \leq 2^p(a^p + b^p)$ for $a, b \geq 0$.

This remark implies

LEMMA 9. $\int |f+g|^p \leq 2^p\left(\int |f|^p + \int |g|^p\right)$

Proof

$|f+g|^p \leq 2^p\left(|f|^p + |g|^p\right)$

$\int |f+g|^p \leq \int 2^p\left(|f|^p + |g|^p\right)$ ← positive functs.

$\leq 2^p\left(\int |f|^p + \int |g|^p\right)$. □

So $\int |f+g|^p < +\infty$ and if $\alpha \in \mathbb{R}$ $\int |\alpha f|^p = |\alpha|^p \int |f|^p$

Hence $\|\alpha f\|_p = |\alpha| \cdot \|f\|_p$. This shows that $L_p(E)$ is a vector space over $\mathbb{R}$.

RMK 10. $\left(\int_E |f|^p\right)^{1/p} = 0 \iff \int_E |f|^p = 0$

$\overset{(*)}{\iff} |f|^p = 0$ a.e. on E

$\iff |f| = 0$ a.e. on E

$\iff f = 0$ " " ".

(*) in detail

$\Leftarrow$: If $|f|^p = 0$ a.e. on E

$\int_E |f|^p = \int_E 0 = 0$.

$\Rightarrow$: $\forall\, A \subset E$ $0 \leq \int_A |f|^p \leq \int_E |f|^p = 0$

because $|f|^p \chi_A \leq |f|^p$. □

Summarizing these results yeilds

LEMMA 11. If $0 < p < \infty$

(i.) $\|f\|_p \geq 0$

(ii.) $\|\alpha f\|_p = |\alpha| \cdot \|f\|_p \quad \forall \alpha \in \mathbb{R}$.

(iii.) $\|f\|_p = 0 \Leftrightarrow f = 0$ a.e.

and if $p > 1$ (we will show)

(iv.) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Def 12: any mapping $\| \ \| :$ vector space $\to \mathbb{R}^+ \cup \{0\}$ satisfing (i.), (ii.), (iv.), and (iii.)' [where (iii.)' $\|f\| = 0 \Leftrightarrow f = 0$] is called a norm.

To be precise with calling $\| \ \|_p$ a norm we could go through equivalance classes where $f \sim g$ iff $f = g$ a.e..

RMK 13. For $p \geq 1$ we use the natural metric
for $p < 1$ we can construct a "nice metric".

THM 14 (Hölders Inequality). If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $\int |fg| \leq (\int |f|^p)^{1/p} \cdot (\int |g|^q)^{1/q}$ is true for all functions f, g.

Proof [of LEMMA 11 (iv.) assuming THM 14]

$\int |f+g|^p \leq \int (|f| + |g|)^p$. And we can write for $p > 1$

$(|f| + |g|)^p = (|f| + |g|)^{p-1} |f| + (|f| + |g|)^{p-1} \cdot |g|$. So

$\int (|f| + |g|)^p = \int (|f| + |g|)^{p-1} |f| + \int (|f| + |g|)^{p-1} |g|$.

Note that $p > 1 \Rightarrow \frac{1}{p} < 1 \Rightarrow 1 - \frac{1}{p} > 0$.

$\exists\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ [called "conjugate exponents"]

using Hölders Inequality

$$\int (|f|+|g|)^p \le \left(\int (|f|+|g|)^{(p-1)q}\right)^{1/q}\left(\left[\int|f|^p\right]^{1/p}+\left[\int|g|^p\right]^{1/p}\right)$$

we observe that $(p-1)q=p$ and $\frac{1}{q}=1-\frac{1}{p}$ and

get $\int (|f|+|g|)^p \le \left(\int (|f|+|g|)^p\right)^{1-p}\left(\left[\int|f|^p\right]^{1/p}+\left[\int|g|^p\right]^{1/p}\right)$

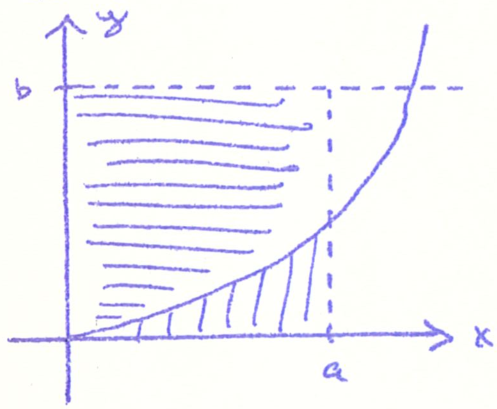
dividing thru by $\left(\int(|f|+|g|)^p\right)^{1-p}$ yeilds

$$\left(\int (|f|+|g|)^p\right)^{1/p} \le \left(\int|f|^p\right)^{1/p}+\left(\int|g|^p\right)^{1/p}$$

$\therefore \|f+g\|_p \le \|f\|_p + \|g\|_p$ for $p>1$. And we have already shown true for $p=1$. So we must now prove Hölders Inequality by means of

<u>LEMMA 15.</u> If $a,b\ge 0$ $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ then $ab \le \frac{a^p}{p}+\frac{b^q}{q}$.

A graphical interpetation of this lemma is a comparison of area



$y = x^{p-1}$

$x = y^{q-1}$

Note by Kerwin Park

11/8/98

Lemma : If $a, b \geq 0$ $\quad p, q > 1$ $\quad \frac{1}{p} + \frac{1}{q} = 1$

then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

Pf : $f(x) = \frac{a^p}{p} + \frac{x^q}{q} - ax \qquad a \geq 0$

If I can show $f(x) \geq 0$, for $x \geq 0$ it is sufficient.

$f'(x) = x^{q-1} - a$

only critical pt is $x = a^{\frac{1}{q-1}}$

$x > a^{\frac{1}{q-1}}$ $\quad f'(x) > 0$

$x < a^{\frac{1}{q-1}}$ $\quad f'(x) < 0$

that the min of $f(x)$ occurs at $x = a^{\frac{1}{q-1}}$

$$f(a^{\frac{1}{q-1}}) = \frac{a^p}{p} + \frac{a^{\frac{q}{q-1}}}{q} - a\,a^{\frac{1}{q-1}}$$

$$p = \frac{q}{q-1} \quad \Rightarrow \quad \frac{1}{q-1} = p-1$$

$$= \frac{a^p}{p} + \frac{a^p}{q} - a\,a^{p-1}$$

$$= a^p - a^p = 0$$

$\therefore f(x) \geq 0$ for $x \geq 0$

∘ Hölder's inequality if $p, q > 1$ & $\frac{1}{p} + \frac{1}{q} = 1$

$$\|fg\|_1 = \int |fg| \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q} = \|f\|_p \|g\|_q$$

Pf : If either $\|f\|_p$ or $\|g\|_q = 0$

then one of $f = 0$ a.e or $g = 0$ a.e

Hence $fg = 0$ a.e. in either case

thus $\|fg\|_1 = 0$

If $\|f\|_p$ or $\|g\|_q$ is ∞, then inequality is now trivially true.

We may assume $0 < \|f\|_p, \|g\|_q < \infty$

Claim: it is enough to show that the inequality with $\|f\|_p = \|g\|_q = 1$

$$\left\| \frac{f}{\|f\|_p} \right\|_p = \frac{1}{\|f\|_p} \|f\|_p = 1$$

$$\int \left| \frac{f}{\|f\|_p} \frac{g}{\|g\|_q} \right| \leq \left\| \frac{f}{\|f\|_p} \right\|_p \left\| \frac{g}{\|g\|_q} \right\|_q$$

$$\frac{1}{\|f\|_p \|g\|_q} \int |fg| \leq \frac{1}{\|f\|_p \|g\|_q} \|f\|_p \|g\|_q$$

Suffices to show $\|f\|_p = \|g\|_q = 1$

$$\Rightarrow \int |fg| \leq 1$$

$$\boxed{ab \leq \frac{a^p}{p} + \frac{b^q}{q}}$$

$$|f(x)||g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} \quad \text{for each } x.$$

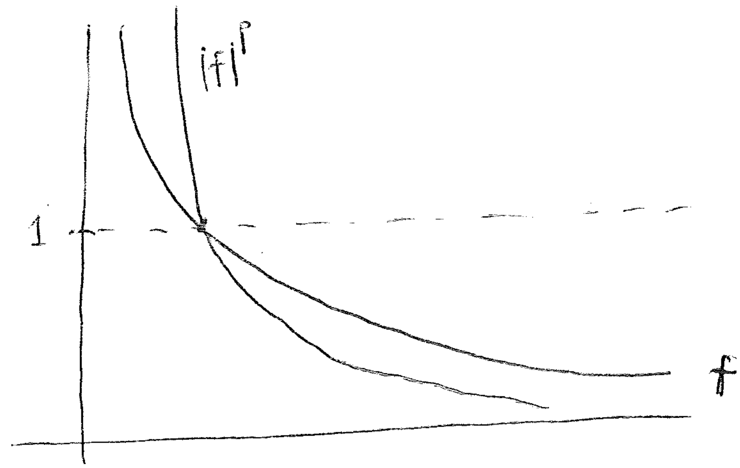
$$\int |fg| \leq \int \left(\frac{|f|^p}{p} + \frac{|g|^q}{q} \right) = \int \frac{|f|^p}{p} + \int \frac{|g|^q}{q}$$

$$= \frac{1}{p} \int |f|^p + \frac{1}{q} \int |g|^q = \frac{1}{p} + \frac{1}{q} = 1 .$$ done

$p = 1$ & $q = \infty$ conjugate exp

$f : [0,1] \to \mathbb{R}$ $\quad 1 \le p < \infty$

Claim: $\|f\|_1 \le \|f\|_p$



$g = \chi_{[0,1]}$ $\quad \frac{1}{p} + \frac{1}{q} = 1$

$\|f\|_1 = \|fg\|_1 \le \|f\|_p \|g\|_q = \|f\|_p$

$L_q(E)$ $\quad \|\chi_E\|_q = [m(E)]^{1/q}$

o. functions $f_n, f \in L_p$ $\quad 0 < p \le \infty$

we will say $f_n \to f$ in the p-norm. if

$\|f_n - f\|_p \to 0$ as $n \to \infty$.

If $p \ge 1$ and $f_n \to f$ in p-norm

$\Rightarrow \|f_n\|_p - \|f\|_p$ $\quad$ (problem 3C TD #8)

$p > 0 \Rightarrow f_n \to f$ in measure

<u>Pf</u>: Suppose $f_n \not\to f$ in meas.

then $\exists\ \varepsilon > 0 \quad \forall N \quad \exists\ n \ge N$

$m \underbrace{\{x : |f_n(x) - f(x)| \ge \varepsilon\}}_{A_n} \ge \varepsilon$ for any such n

$$|f_n - f| \geq \varepsilon \chi_{A_n}$$

$$|f_n - f|^p \geq \varepsilon^p \chi_{A_n}$$

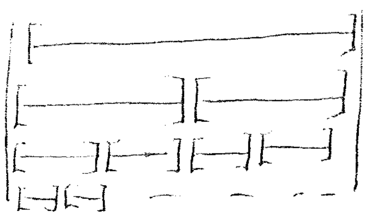
$$\|f_n - f\|_p^p \geq \varepsilon^p m(A_n)$$

$$\|f_n - f\| \geq \varepsilon\, \varepsilon^{1/p} = \varepsilon^{1-1/p}$$

thus $\|f_n - f\|_p \not\to 0$

example : $m = 2^n + j \qquad 0 \leq j \leq 2^n$

$$f_m = \chi_{A_m} \qquad A_m = \left[j/2^n , \frac{j+1}{2^n} \right]$$



we have shown $f_n \to 0$ in meas

$f_n(x)$ does not converge for any $x \in [0,1]$ & thus $f_n \not\to f$ pt-wise

$$\|f_m\|_p = [m(A_m)]^{1/p} = (1/2^n)^{1/p}$$

$f_m \to 0$ in p-norm $\quad p < \infty$

Norm space is vector space X & function $\|\cdot\| : X \to$ non-neg. reals

(1) $\|x\| = 0 \iff x = 0$

(2) $\|\alpha x\| = |\alpha| \|x\| \qquad \alpha \in \mathbb{R}$

(3) $\|x + y\| \leq \|x\| + \|y\|$

$x_n \to x$ in norm $\Longleftrightarrow \|x_n - x\| \to 0$ as $n \to \infty$

$\{x_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0$ $\exists M$ s.t $m, n \geq M$

$\Rightarrow \|x_n - x_m\| < \varepsilon$.

the norm space is a Banach space if it is complete

(i.e every c.s converges to a point in the space)

→ Show L_p $\infty > p > 1$ is complete.

∘ Examples of normed space

(1) $\mathbb{R} = X$, $| \, | = \| \cdot \|$.

(2) $\mathbb{R}^n = X$

$$\|(x_1, \dots, x_n)\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

this is a norm for $p \geq 1$

$$\|(x_1, \dots, x_n)\|_\infty = \sup_{1 \leq i \leq n} |x_i|$$

$$\mathbb{R}^n \longrightarrow L_p(\mathbb{R})$$

$$(x_1, x_2, \dots, x_n) \longrightarrow \sum_{i=1}^{n} x_i \chi_{[i, i+1)}$$

(3) l_p set of sequences $\{x_n\}$ s.t. $\|\{x_n\}\|_p < \infty$

$$(x_1, x_2, \dots\dots) \longleftrightarrow \sum_{i=1}^{\infty} x_i \chi_{[i, i+1)} \in L_p$$

$$\|(x_1, \dots)\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

$l_1 \subset l_p$ $p > 1$

a. $\xi = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

$\|\xi\|_1 = \infty$

$p > 1 \qquad \|\xi\|_p = \left(\sum \frac{1}{n^p}\right)^{1/p} < \infty$

Test Problem #8 Due 17 Nov 1978

1. f & g are measurable fcns: $E \to \mathbb{R}$

1 pt A Show $\|f\|_\infty = M \in \mathbb{R}^*$ if and only if there is a fcn g with $g = f$ a.e and $\sup_{x \in E} |g(x)| = \|g\|_\infty = M$

1/4 pt B. If $\alpha \in \mathbb{R}$, show $\|\alpha f\|_\infty = |\alpha| \, \|f\|_\infty$

1/4 pt C. Show $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

1/2 pt D. Show $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$

.166

∞. ~.75

.833 5/6 pt 2. A. If $f: [0,1] \to \mathbb{R}$ is meas and $0 < p \leq q \leq \infty$ then $\|f\|_p \leq \|f\|_q$.

5/6 pt B. Show if $0 < p \leq q \leq \infty$ $L_p \not\supseteq L_q$

5/6 pt C. Show if $0 < p \neq q \leq \infty$ $L_p(\mathbb{R}) \not\supset L_q(\mathbb{R})$

1 pt 3. A Show if $f \in L_\infty$ then $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$

1 pt B. Show if $0 < r, p, q < \infty$ and $1/r = 1/p + 1/q$ & $f: E \to \mathbb{R}$ is meas, the $\|fg\|_r \leq \|f\|_p \|g\|_q$

1 pt C. Suppose $\infty > p \geq 1$ and $f_n, f \in L_p$ and $f_n \to f$ point-wise a.e. Show $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|$

1 pt 4. A Monotone Conv. Thm: If f_n, f meas. $0 \leq f_1 \leq f_2 \leq \ldots \leq f$ and $f_n \to f$ in measure then $\lim_n \int f_n = \int f$

3/2 pts B. Fatou Lemma: If f_n, f meas ≥ 0 $f_n \to f$ in meas, then $\int f \leq \liminf_n \int f_n$.

MAA 5306

Lecture: 11/13/78

Note: Consider $\|\cdot\|_p$ on R^N, $0 < p \le \infty$

for $p \ge 1$, $d(x,y) = \|x-y\|_p$ is a metric

for $0 < p < 1$, we can define $\underline{x_n \to x}$ by

$$\|x_n - x\|_p \to 0$$

In any case, they all give the same convergent sequences.

LEMMA: If $\{x_n\}$ is a Cauchy sequence in X and if $x_{n_i} \to x$ in norm as $i \to \infty$, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then $x_n \to x$ in norm as $n \to \infty$.

Proof: We want to prove: $\forall \epsilon > 0, \exists N \ni \forall n \ge N$,

$$\|x - x_n\| < \epsilon$$

Then, let $\epsilon > 0$ be given. Since $\{x_n\}$ is Cauchy, $\exists M$ such that for $n, m \ge M$ we have $\|x_n - x_m\| < \epsilon/2$.

Also, since $x_{n_i} \to x$, $\exists I$ such that for $i \ge I$,

$$\|x - x_{n_i}\| < \epsilon/2.$$

Now, let $N \ge M$ and N_I. Let $n \ge N$. Let $n_j \ge M$ and $j \ge I$. Then,

$$\|x - x_n\| \le \|x - x_{n_j}\| + \|x_{n_j} - x_n\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since this is true for all $\epsilon > 0$, we have $\|x - x_n\| \to 0$. //

PROPOSITION: If $x_n \to x$ in norm, then $\{x_n\}$ is Cauchy.

Proof: Let $\epsilon > 0$ be given. Then, $\exists N \ni \forall n \geq N$ we have

$$\|x - x_n\| < \epsilon/2.$$

Hence, if $n, m \geq N$,

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| < \epsilon/2 + \epsilon/2 = \epsilon. //$$

DEFINITION: Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is summable if $\exists\, x \in X$ such that

$$\sum_{i=1}^{N} x_i \longrightarrow x \text{ in norm.} \quad \left(\text{We write: } \sum_{i=1}^{\infty} x_i = x\right)$$

We say that $\{x_n\}$ is absolutely summable if

$$\sum_{i=1}^{\infty} \|x_i\| < \infty$$

We say that $\{x_n\}$ is unconditionally summable if $\exists\, x \in X$ such that for each π permutation of the integers, $\sum_{i=1}^{N} x_{\pi(i)} \to x$ in norm, as $N \to \infty$.

Example: Consider $L_2(\mathbb{R})$.

$$x_n = \frac{1}{n} \chi_{[n, n+1]}$$

Then $\{x_n\}$ is unconditionally summable, for

$$x = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{[n, n+1]}$$

However, $\|x_n\| = 1/n$ and then $\{x_n\}$ is not absolutely summable.

PROPOSITION: A norm space X is complete if and only if each absolutely summable sequence $\{x_n\}$ is summable.

<u>**Proof**</u>: ($\Longrightarrow$) If $\{x_n\}$ is absolutely summable, then $\{\sum_{n=1}^{N} x_n\}$ is a Cauchy sequence.

Let $\epsilon > 0$ be given. Since $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\exists\, p$ such that for $N > p$ we have $\sum_{n=N+1}^{\infty} \|x_n\| < \epsilon$

So, if $N, M \geq p$, $M > N$, we have:

$$\left\|\sum_{n=1}^{N} x_n - \sum_{n=1}^{M} x_n\right\| = \left\|\sum_{n=N+1}^{M} x_n\right\| \leq \sum_{n=N+1}^{M} \|x_n\| \leq \sum_{n=N+1}^{\infty} \|x_n\| < \epsilon .$$

Now, since $\{\sum_{n=1}^{N} x_n\}$ is Cauchy, $\exists\, x \ni \sum_{n=1}^{N} x_n \to x$ in norm.

Hence $\{x_n\}$ is summable.

($\Longleftarrow$) Let $\{x_n\}$ be a Cauchy sequence.

Pick a subsequence: Choose $N_{i+1} > N_i$ such that for $n > N_{i+1}$ we have $\|x_n - x_{N_{i+1}}\| < \frac{1}{2^{i+1}}$.

Our subsequence is $\{x_{N_i}\}$

Now let $y_1 = x_{N_1}$ and $y_i = x_{N_i} - x_{N_{i-1}}$, for $i \geq 2$.

Thus, $\sum_{i=1}^{n} y_i = x_{N_n}$ (telescoping series) and

$\|y_i\| = \|x_{N_i} - x_{N_{i-1}}\| < \frac{1}{2^{i-1}}$ and hence $\sum_{i=1}^{n} \|y_i\| < \infty$

So, by hypothesis, $\exists\, x \in X \ni \sum_{i=1}^{n} y_i \to x$ in norm which means that $x_{N_n} \to x$ in norm and then, by the LEMMA above, $x_n \to x$ in norm.

Hence, X is complete. //

THEOREM: L_p is complete, $1 \le p \le \infty$.

Proof: Let $\{f_n\}$ be absolutely summable, in L_p.

i.e. $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$

We want to show that there is some function $f \in L_p$ with $\|f - \sum_{n=1}^{N} f_n\| \to 0$, as $N \to \infty$.

Proof for $p=1$.: Claim: $\sum_{n=1}^{\infty} |f_n| \in L_1$.

Let $g_N = \sum_{n=1}^{N} |f_n|$. Then.

$0 \le g_1 \le g_2 \le \ldots \le \sum_{n=1}^{\infty} |f_n|$, pointwise.

Thus, by the Monotone Convergence Theorem,

$$\int \sum_{n=1}^{\infty} |f_n| = \lim_{N \to \infty} \int \sum_{n=1}^{N} |f_n| \le \sum_{n=1}^{\infty} \|f_n\|_1$$

Note that $|\sum_{n=1}^{N} f_n| \le \sum_{n=1}^{N} |f_n| \le \sum_{n=1}^{\infty} |f_n|$.

Let $g(x)$ be the pointwise limit of $\{\sum_{n=1}^{N} |f_n|\}$.

Then, $|g| \le \sum_{n=1}^{\infty} |f_n|$

Hence, by the Dominated Convergence Theorem,

$$\lim_{N \to \infty} \int \sum_{n=1}^{N} f_n = \int g.$$

//.

MAA 5306

Lecture: 11/13/78

Note: Consider $\|\cdot\|_p$ on $\mathbb{R}^n$, $0 < p \le \infty$

for $p \ge 1$, $d(x,y) = \|x-y\|_p$ is a metric

for $0 < p < 1$, we can define $\underline{x_n \to x}$ by

$$\|x_n - x\|_p \to 0$$

In any case, they all give the same convergent sequences.

LEMMA: If $\{x_n\}$ is a Cauchy sequence in X and if $x_{n_i} \to x$ in norm as $i \to \infty$, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then $x_n \to x$ in norm as $n \to \infty$.

Proof: We want to prove: $\forall \epsilon > 0, \exists N \ni \forall n \ge N$,

$$\|x - x_n\| < \epsilon$$

Then, let $\epsilon > 0$ be given. Since $\{x_n\}$ is Cauchy, $\exists M$ such that for $n, m \ge M$ we have $\|x_n - x_m\| < \epsilon/2$.

Also, since $x_{n_i} \to x$, $\exists I$ such that for $i \ge I$,

$$\|x - x_{n_i}\| < \epsilon/2.$$

Now, let $N \ge M$ and N_I. Let $n \ge N$. Let $n_j \ge M$ and $j \ge I$. Then,

$$\|x - x_n\| \le \|x - x_{n_j}\| + \|x_{n_j} - x_n\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since this is true for all $\epsilon > 0$, we have $\|x - x_n\| \to 0$. //

PROPOSITION: If $x_n \to x$ in norm, then $\{x_n\}$ is Cauchy.

Proof: Let $\epsilon > 0$ be given. Then, $\exists N \ni \forall n \geq N$ we have

$$\|x - x_n\| < \epsilon/2.$$

Hence, if $n, m \geq N$,

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| < \epsilon/2 + \epsilon/2 = \epsilon. //$$

DEFINITION: Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is summable if $\exists\, x \in X$ such that

$$\sum_{i=1}^{N} x_i \longrightarrow x \text{ in norm.} \quad \left(\text{We write: } \sum_{i=1}^{\infty} x_i = x\right)$$

We say that $\{x_n\}$ is absolutely summable if

$$\sum_{i=1}^{\infty} \|x_i\| < \infty$$

We say that $\{x_n\}$ is unconditionally summable if $\exists\, x \in X$ such that for each π permutation of the integers, $\sum_{i=1}^{N} x_{\pi(i)} \to x$ in norm, as $N \to \infty$.

Example: Consider $L_2(\mathbb{R})$.

$$x_n = \frac{1}{n} \chi_{[n,\, n+1]}$$

Then $\{x_n\}$ is unconditionally summable, for

$$x = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{[n,\, n+1]}$$

However, $\|x_n\| = 1/n$ and then $\{x_n\}$ is not absolutely summable.

PROPOSITION: A norm space X is complete if and only if each absolutely summable sequence $\{x_n\}$ is summable.

Proof: ($\Longrightarrow$) If $\{x_n\}$ is absolutely summable, then $\{\sum_{n=1}^{N} x_n\}$ is a Cauchy sequence.

Let $\epsilon > 0$ be given. Since $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\exists\, p$ such that for $N > p$ we have $\sum_{n=N+1}^{\infty} \|x_n\| < \epsilon$

So, if $N, M \geq p$, $M > N$, we have:

$$\left\|\sum_{n=1}^{N} x_n - \sum_{n=1}^{M} x_n\right\| = \left\|\sum_{n=N+1}^{M} x_n\right\| \leq \sum_{n=N+1}^{M} \|x_n\| \leq \sum_{n=N+1}^{\infty} \|x_n\| < \epsilon.$$

Now, since $\{\sum_{n=1}^{N} x_n\}$ is Cauchy, $\exists\, x \ni \sum_{n=1}^{N} x_n \to x$ in norm.

Hence $\{x_n\}$ is summable.

($\Longleftarrow$) Let $\{x_n\}$ be a Cauchy sequence.

Pick a subsequence: Choose $N_{i+1} > N_i$ such that for $n > N_{i+1}$ we have $\|x_n - x_{N_{i+1}}\| < \frac{1}{2^{i+1}}$.

Our subsequence is $\{x_{N_i}\}$

Now let $y_1 = x_{N_1}$ and $y_i = x_{N_i} - x_{N_{i-1}}$, for $i \geq 2$.

Thus, $\sum_{i=1}^{n} y_i = x_{N_n}$ (telescoping series) and

$\|y_i\| = \|x_{N_i} - x_{N_{i-1}}\| < \frac{1}{2^{i-1}}$ and hence $\sum_{i=1}^{n} \|y_i\| < \infty$

So, by hypothesis, $\exists\, x \in X \ni \sum_{i=1}^{n} y_i \to x$ in norm which means that $x_{N_n} \to x$ in norm and then, by the LEMMA above, $x_n \to x$ in norm.

Hence, X is complete. //

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i.e. $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$

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Let $g_N = \sum_{n=1}^{N} |f_n|$. Then.

$0 \leq g_1 \leq g_2 \leq \ldots \leq \sum_{n=1}^{\infty} |f_n|$, pointwise.

Thus, by the Monotone Convergence Theorem,

$$\int \sum_{n=1}^{\infty} |f_n| = \lim_{N\to\infty} \int \sum_{n=1}^{N} |f_n| \leq \sum_{n=1}^{\infty} \|f_n\|_1$$

Note that $|\sum_{n=1}^{N} f_n| \leq \sum_{n=1}^{N} |f_n| \leq \sum_{n=1}^{\infty} |f_n|$.

Let $g(x)$ be the pointwise limit of $\{\sum_{n=1}^{N} |f_n|\}$.

Then, $|g| \leq \sum_{n=1}^{\infty} |f_n|$

Hence, by the Dominated Convergence Theorem,

$$\lim_{N\to\infty} \int \sum_{n=1}^{N} f_n = \int g.$$

//.

Huang / Perzley
Nov. 15, 1978

Thm 1: If $1 \le p < \infty$, L_p is complete.

Proof: We shall show that every absolutely summable sequence is summable.

Let $1 \le p < \infty$ be given, let $\langle f_n \rangle \subset L_p$ s.t. $\{f_n\}$ is absolutely summable.

Define: $s_n = \sum_{i=1}^{n} f_i$; $g_n = \sum_{i=1}^{n} |f_i|$

Note that $0 \le g_1 \le g_2 \le \cdots$

so $g(x) = \lim_{n \to \infty} g_n(x)$ exists for each $x \in [0,1]$, is an extended real valued function.

$g_n^p \to g^p$ pt. wise, and $0 \le g_n^p \le g_{n+1}^p$

so by M.C.T. $\Rightarrow \int g^p = \lim_{n \to \infty} \int g_n^p$.

Now:

$$\|g_n\|_p \le \sum_{i=1}^{n} \|f_i\|_p$$

$$\le \sum_{i=1}^{\infty} \|f_i\|_p = M < \infty.$$

thus

$$\int |g_n|^p < M^p.$$

$\therefore \int g^p \le M^p < +\infty$.

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(this says $g(x) \neq +\infty$ a.e.)

For each $x \in [0,1]$ with $g(x) \neq +\infty$,

$\lim_{n \to \infty} s_n(x) = f(x)$ exists.

and if $g(x) = +\infty$, we define $f(x) = 0$.

We have $s_n \to f$ pt. wise a.e.

Now $|s_n| \le g_n \le g \Rightarrow |f| \le g$.

$\Rightarrow \int |f|^p \le \int |g|^p = m^p < \infty$. hence, $f \in L_p$.

$|s_n(x) - f(x)| \le |s_n(x)| + |f(x)| \le g(x) + g(x)$

Hence;

$$|s_n(x) - f(x)|^p \le 2^p g^p(x)$$

$g^p \in L_1$ and $s_n - f \to 0$ pt. wise a.e.

$\therefore$ By Lebesgue Dom. Conv. Thm.

$$0 = \int 0 = \lim_{n \to \infty} \int |s_n - f|^p$$

i.e. $\|s_n - f\|_p^p \to 0$.

and hence $\|s_n - f\|_p \to 0$ (since $g(x) = x^{1/p}$ is cont. at 0)

Q.E.D.

3/

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Huang / Perzley

Thm 2: If $1 \le p < \infty$, then each of the following sets is dense in L_p.

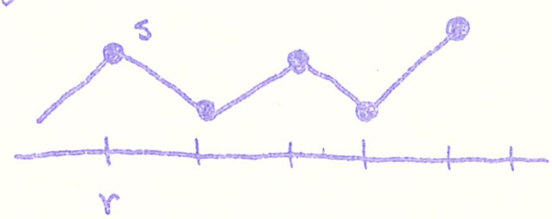
① $\{s : s$ simple $\in L_p\}$

② $\{\varphi : \varphi$ is step $\in L_p\}$

③ $\{f : f$ continuous $\in L_p\}$

④ $\{\varphi : \varphi$ is step $\in L_p$ w/ rational values & rational jump points$\}$

⑤ $\{f : f$ continuous $\in L_p$. f piecewise linear w/ rational corners$\}$.



$(r, s) \in \mathbb{Q} \times \mathbb{Q}$

Proof:

1) If f is meas., $\exists$ simple fctns. $s_n \to f$ pt. wise w/ $|s_n(x)| \le |f(x)|$ $\forall x \in [0,1]$

$$|s_n(x) - f(x)|^p \le 2^p |f|^p \leftarrow \text{in } L_1$$

By Dominated Conv. Thrm.

$$\|s_n - f\|_p^p = \int |s_n - f|^p \to 0 \text{ as } n \to \infty.$$

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HUANG / ~~[illegible]~~ Perzley

$\therefore$ $s_n \to f$ in $\| \cdot \|_p$
(i.e. have shown that every $f \in L_p$ is the limit pt.)

② :

Pf: Let $\epsilon > 0$ be given and suppose s is simple.

Let $M = \max_{x \in [0,1]} |s(x)|$

Let $\delta > 0 \ni \delta 2^p M^p < \epsilon$

Pick $\varphi(x)$ step fcn. s.t. $|\varphi(x)| \le M$
$m\{x : \varphi(x) \neq s(x)\} < \delta$

$$\|\varphi - s\|_p^p = \int |\varphi - s|^p$$

$$= \int_{\{x : \varphi(x) \neq s(x)\}} |\varphi - s|^p$$

$$\le \delta 2^p M^p < \epsilon.$$

DONE.

③ Similar to part ②; or we may use Lusin Thm.

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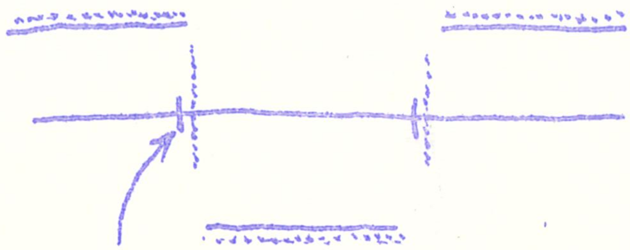
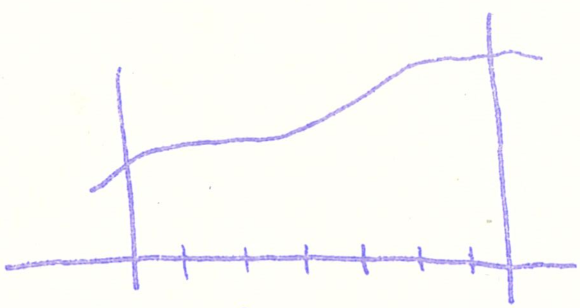
(4) Take any step fcn.



pick rationals so close that area is "small".

(5) pick any cont. fcn.



divide into rational parts, then make smaller & smaller, so that converges uniformly.

11-20-98
D. Niedzwiecki

P.1.
Look at $\mathbb{R}$, $\mathcal{B}$ - Borel sets; study measures μ.
$\mu(B) \geq 0$ $B \in \mathcal{B}$ $\infty > \mu(B) \geq 0$ $\mu(\phi) = 0$.
If $\{B_n\}$ pwd. then $\mu(\bigcup_1^\infty B_n) = \sum_1^\infty \mu(B_n)$

DN1. μ is said to be absolutely continuous with respect to Lebesgue meas. (m) if $\forall B \in \mathcal{B}$ $m(B) = 0 \Rightarrow \mu(B) = 0$ [This is written $\mu \ll m$].

LEMMA 1. $\mu \ll m \Leftrightarrow \forall \varepsilon > 0$ $\exists \delta > 0 \ni m(A) < \delta$ $\Rightarrow \mu(A) < \varepsilon$ $\forall A \in \mathcal{B}$. $[\lim_{m(A) \to 0} \mu(A) = 0$.

Pf. $(\Leftarrow)$ Let $B \in \mathcal{B}$ with $m(B) = 0$ & we want to show that $\mu(B) = 0$. We will show that $\mu(B) < \varepsilon$ for each $\varepsilon > 0$. Let $\varepsilon > 0$ be given find δ s.t. $m(A) < \delta$. $\Rightarrow \mu(A) < \varepsilon$ and since $m(B) = 0 < \delta \Rightarrow \mu(B) < \varepsilon$ $\therefore \mu(B) = 0$
$(\Rightarrow)$ [Note that this requires $\mu(\mathbb{R}) < +\infty$].
Pf. by contrapositive Suppose $\exists \varepsilon > 0$ $\forall \delta > 0$ $\exists A \in \mathcal{B}$ with $m(A) < \delta$ but $\mu(A) \geq \varepsilon$. Choose B_n with $m(B_n) < 2^{-n}$ and $\mu(B_n) \geq \varepsilon$. Let $A_N = \bigcup_{n=N}^\infty B_n$ $A = \bigcap_{N=1}^\infty A_N$. $m(A_N) \leq \sum_N^\infty m(B_n)$ $\leq 2^{-N+1}$; $A_N \supset A_{N+1}$; $m(A) = 0$. However, $\mu(A_n) \geq \varepsilon$. $\mu(A_1) \leq \mu(\mathbb{R}) < +\infty$.

P.2. $\mu(A) = \lim \mu(A_n) \geq \varepsilon$. therefore $\mu \not\ll m$.

// Example: Consider $f \in L_1(\mathbb{R})$ & look at $\mu(E) = \int_E f \, dm$ for $E \in \mathcal{B}$. (1) $\mu(\phi) = \int_\phi f \, dm = \int f \chi_\phi \, dm = \int 0 \, dm = 0$. If A, B are Borel w/ $A \cap B = \phi$ $\mu(A \cup B) = \int_{A \cup B} f \, dm = \int f \chi_{A \cup B} \, dm$

$= \int f(\chi_A + \chi_B) dm = \int_A f \, dm + \int_B f \, dm = \mu(A) + \mu(B)$

By induction we can show that $\{A_n\}_{n=1}^N$ pwd. then $\mu(\bigcup^N A_n) = \sum_1^N \mu(A_n)$. / Show μ is countably additive.

Let $\{B_n\}_1^\infty$ be pwd. Consider $g_n = f \chi_{\bigcup_1^n B_n}$;

$g = f \chi_{\bigcup_1^\infty B_n}$ $g_n \to g$ ptwise

$|g_n| \leq |f|$ & $|g| \leq |f|$ by L.D.C.T.

$\sum_1^\infty \mu(B_n) = \lim \int g_n = \int g = \mu(\bigcup_1^\infty B_n)$; $\mu(B_n)$ must be absolutely convergent. If $f \geq 0$ then $\mu(A) \geq 0$; $\mu(A) < +\infty$

// If μ is a Borel meas & there exists an $f \in L_1(\mathbb{R})$ with $\mu(E) = \int_E f \, dm$. then f is called <u>the</u> Density fcn. for μ. Note that if f, g are both densities for μ then $\forall E \in \mathcal{B}$ $\int_E f - g \, dm = \int_E f \, dm - \int_E g \, dm$.

P3. $= \mu(E) - \mu(E) = 0. \Rightarrow f - g = 0$ a.e. $f = g$ a.e.
If μ has a density then $\mu \ll m$. If $m(E) = 0$
then $\int_E f\, dm = \mu(E) = 0$.

DN2. Two Borel measures μ, ν are said to be singular with respect to each other if $\exists$ A, B with $A \cap B = \phi$
$A \cup B = \mathbb{R}$ and $\mu(A) = \nu(B) = 0$. [Written $\mu \perp \nu$].

Let $\{b_n\} \subset \mathbb{R}$; $\{a_n\}$; $a_n \geq 0$ with $\sum_1^\infty a_n < +\infty$.
Define $\nu(B)$ for $B \in \mathcal{B}$ by $\nu(B) = \sum' a_n$ over those n for which $b_n \in B$.

$= \sum_{\substack{n=1 \\ b_n \in B}}^{\infty} a_n \qquad \nu(\phi) = 0.$

Suppose $\{c_n\}$ is pwd. $\nu(\bigcup_1^\infty c_m) = \sum_{\substack{1 \\ b_n \in \bigcup_1^\infty c_m}}^{\infty} a_n = \sum_{m=1}^{\infty} \sum_{\substack{1 \\ b_n \in c_m}}^{\infty} a_n$.

all numbers are positive so order of summing is irrelevant.

$$= \sum_{m=1}^{\infty} \nu(c_m)$$

Let $D = \{b_n\}$ $\quad \nu(\mathbb{R} \setminus D) = 0$ & $m(D) = 0$ since D is countable. $\therefore$ $\nu \perp m$. In fact we'll say ν is discrete if $\exists$ a countable set D.
s.t. $\nu(\mathbb{R} \setminus D) = 0$.

Wed., 12/29/??   Maloney

LM1: If μ, ν are Borel measures, then so is $\mu+\nu$, where $(\mu+\nu)(E) = \mu(E)+\nu(E)$, for $E \in \mathcal{B}$. Furthermore, if $\mu(E) \geq \nu(E)$ for each $E \in \mathcal{B}$, then $\mu-\nu$ is a Borel measure, where $\mu-\nu$ is also defined setwise.

Proof ① $\mu+\nu : \mathcal{B} \to \mathbb{R}^+$ $\mu-\nu : \mathcal{B} \to \mathbb{R}^+$.

② ϕ: $(\mu+\nu)(\phi) = \mu(\phi) + \nu(\phi) = 0+0=0.$
$(\mu-\nu)(\phi) = \mu(\phi) - \nu(\phi) = 0-0=0.$

③ Let $\{A_n\}$ be a pwd sequence of Borel sets.

Then $(\mu+\nu)(\cup A_n) = \mu(\cup A_n) + \nu(\cup A_n)$
$= \sum \mu(A_n) + \sum \nu(A_n)$
$= \sum [\mu(A_n) + \nu(A_n)]$
$= \sum (\mu+\nu)(A_n)$

$(\mu-\nu)(\cup A_n)$ works similarly since we've got an absolutely convergent seq.

Note: If $\mu \geq \nu$ setwise then $\mu = (\mu-\nu)+\nu$.

DN2 If μ is a Borel meas. the distribution of μ is a fn.
$F = F_\mu : \mathbb{R} \to \mathbb{R}$ defined by $F_\mu(x) = \mu((-\infty, x])$.

Properties of $F_\mu(x)$

① $F(x) \geq 0$ because $\mu \geq 0$ setwise.

② $F(x)$ is increasing   because μ is monotone.

③ $F(x)$ is continuous on the right. I.e., $\forall a$, $\lim_{x \to a^+} F(x) = F(a)$.

Claim that it's enough to show that $\lim_{n\to\infty} F(a+\frac{1}{n}) = F(a)$. This follows from defn of $\lim_{x\to a^+} F(x)$.

Now $\bigcap_{n=1}^{\infty} (-\infty, a+\frac{1}{n}] = (-\infty, a]$.

Since $\mu(\mathbb{R}) < +\infty$, $F(a) = \mu((-\infty, a]) = \lim_{n\to\infty} \mu((-\infty, a+\frac{1}{n}]) = \lim_{n\to\infty} F(a+\frac{1}{n})$.

11/29 p2

④ $\lim_{x\to-\infty} F(x)=0$. Since $\bigcap_{n=1}^{\infty}(-\infty,-n]=\phi$, $\lim_{n\to\infty} F(-n)=\mu(\phi)=0$, by claim in ③.

⑤ $\lim_{x\to+\infty} F(x)=\mu(\mathbb{R})<+\infty$

This follows from $\bigcup_{n=1}^{\infty}(-\infty,n]=\mathbb{R}$.

⑥ $F_{\mu+\nu}=F_{\mu}+F_{\nu}$ and $F_{\mu-\nu}=F_{\mu}-F_{\nu}$

Follows from defn's of $\mu+\nu$.

⑦ F_μ is continuous at a iff $\mu(\{a\})=0$.

Proof

$$\lim_{x\to a^-} F(x)=\lim_{n\to\infty}\mu((-\infty,a-\tfrac{1}{n}])$$
$$=\mu((-\infty,a)).$$

So $F_\mu(x)$ is cont. at a iff $\lim_{x\to a^-}F(x)=\lim_{x\to a^+}F(x)$

iff $\mu((-\infty,a))=\mu((-\infty,a])$ (*).

Since $\mu((-\infty,a])=\mu(\{a\})+\mu((-\infty,a))$, (*) is true iff $\mu(\{a\})=0$.

⑧ $F_\mu=F_\nu \Leftrightarrow \mu=\nu$.

($\Leftarrow$) $F_\mu(b)=\mu((-\infty,b])=\nu((-\infty,b])=F_\nu(b)$.

($\Rightarrow$) Let $\mathcal{M}=\{E\in\mathcal{B}:\mu(E)=\nu(E)\}$.

The following are true about $\mathcal{M}$:

ⓐ $(-\infty,b]\in\mathcal{M}$.

ⓑ $(a,b]\in\mathcal{M}$.

ⓒ $(a,+\infty)\in\mathcal{M}$.

ⓓ Finite disjoint unions of things of form ⓐ ⓑ or ⓒ are in $\mathcal{M}$.
This is because both μ and ν are finitely additive.

ⓔ The collection in ⓓ (i.e., {finite disjoint unions of things of form ⓐ ⓑ or ⓒ})

11/29 p3

is an algebra, call it $\mathcal{A}$.

(f) $\mathfrak{M}$ is a monotone class.

(g) $\mathfrak{M} \supseteq m(\mathcal{A}) = \sigma(\mathcal{A}) = \mathcal{B}$

$\therefore \mu(E) = \nu(E), \ \forall E \in \mathcal{B}$.

DN 3 A Borel measure μ is said to be non-atomic if F_μ is continuous everywhere (which is true only if $\mu(\{x\}) = 0, \ \forall x \in \mathbb{R}$).

DN 4 let $G: \mathbb{R} \to \mathbb{R}$ with $G(x)$ increasing.
$\forall a \in \mathbb{R}$ define $G(a^-) = \lim_{x \to a^-} G(x)$ and $G(a^+) = \lim_{x \to a^+} G(x)$.

Note that $G(a^-) \le G(a) \le G(a^+)$, and that if $b < a < c$,
$G(b) \le G(a^-)$ and $G(a^+) \le G(c)$.

LM 5 There are at most countably many points x in $[a,b]$ for which $G(x^+) \neq G(x^-)$.

Proof Consider the set $\{x_\alpha : \alpha \in \Lambda\}$ of distinct points in $[a,b]$ with $G(x_\alpha^-) \neq G(x_\alpha^+)$, $\forall \alpha \in \Lambda$. It's enough to show that Λ must be countable. Note that for $\alpha_1, \ldots, \alpha_n \in \Lambda$, if
$\sum_{i=1}^n [G(x_{\alpha_i}^+) - G(x_{\alpha_i}^-)] \le G(b) - G(a) < +\infty$, then by problem #1, TP set #5, only countably many of the α_i's will be such that $G(x_{\alpha_i}^+) - G(x_{\alpha_i}^-) \neq 0$.

Let $a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b$ have the property that $\xi_{i-1} < x_{\alpha_i} < \xi_i$. Then $G(\xi_{i-1}) \le G(x_{\alpha_i}^-) \le G(x_{\alpha_i}^+) \le G(\xi_i)$.

So $G(x_{\alpha_i}^+) - G(x_{\alpha_i}^-) \le G(\xi_i) - G(\xi_{i-1})$.

Thus
$$\sum_{i=1}^n [G(x_{\alpha_i}^+) - G(x_{\alpha_i}^-)] \le \sum_{i=1}^n [G(\xi_i) - G(\xi_{i-1})]$$
$$= G(\xi_n) - G(\xi_0)$$
$$= G(b) - G(a).$$

11/29 p.4

COR 6 F_μ has at most countably many pts of discontinuity and there are at most countably many pts. b_n with $\mu(\{b_n\}) \neq 0$.

Given μ let b_n be a list of pts. with $\mu(\{b_n\}) = a_n > 0$.
Define the Borel meas. ν by $\nu(E) = \sum_{\substack{n=1 \\ b_n \in E}}^{\infty} a_n$.
Then $\nu(E) = \sum_{\substack{n=1 \\ b_n \in E}}^{\infty} a_n = \sum_{\substack{n=1 \\ b_n \in E}}^{\infty} \mu(\{b_n\}) = \mu(\{b_n\}_{n \in \mathbb{N}} \cap E) \leq \mu(E)$.

So $\mu \geq \nu$ setwise. Consider $\mu - \nu$. Note that $F_{\mu-\nu}(x)$ is continuous.
If $a \in \mathbb{R}$, $(\mu-\nu)(\{a\}) = \mu(\{a\}) - \nu(\{a\}) = \begin{cases} 0 - 0 & \text{if } a \neq b_n, \forall n. \\ \mu(\{b_n\}) - a_n & \text{if } a = b_n, \text{ for some } n. \end{cases}$

So any Borel meas. μ can be written as $\mu = \lambda + \nu$,
where ν is discrete and λ is non-atomic. $\lambda = \mu - \nu$.

Friday Dec. 1, 1978   Frank Guess

Thm If $G(x)$ is an increasing fcn of $\mathbb{R} \to \mathbb{R}$, then $G'(x)$ exist a.e.

i.e. $\lim_{h\to 0} \frac{G(x+h) - G(x)}{h}$ exist as a Real number a.e x

(will show later)

Lemma: $G: \mathbb{R} \to \mathbb{R}$ increasing then $G(a) \le \liminf_{n\to\infty} n\cdot\int_a^{a+\frac{1}{n}} G(x)dx \le \limsup_{n\to\infty} n\cdot\int_a^{a+\frac{1}{n}} G(x)dx \le G(a^+)$

pf note: $G(a) \le G(x)$ for $a \le x \le a+\frac{1}{n}$, hence $G(a)\cdot\frac{1}{n} \le \int_a^{a+\frac{1}{n}} G(x)dx$ and thus $G(a) \le n\int_a^{a+\frac{1}{n}} G(x)dx$. If $b > a$, then for large n: $a+\frac{1}{n} < b$; so $G(x) \le G(b)$ for $a \le x \le a+\frac{1}{n}$. Hence, $n\int_a^{a+\frac{1}{n}} G(x)dx \le G(b)$. Thus for $b > a$, we have $\limsup_{n\to\infty} n\int_a^{a+\frac{1}{n}} G(x)dx \le G(b)$ and $\limsup_{n\to\infty} n\int_a^{a+\frac{1}{n}} G(x)dx \le \lim_{b\to a^+} G(b) = G(a^+)$ $\boxtimes$ (lemma)

Corr.: If $G(x)$ is continuous at a, all 4 number in lemma are the same.

Lemma: If $G: \mathbb{R} \to \mathbb{R}$ increasing, then $\int_a^b G'(x)dx \le G(b) - G(a)$

before proving this note

Corr. $G'(x) \ne +\infty$ a.e. (by $G(b) - G(a) < \infty$)

Corr If μ is a Borel measure, then $F_\mu' \in L_1(\mathbb{R})$

(note: fcn increasing so $G'(x) \ge 0$, so $F_\mu' \ge 0$ + $\int_a^b F_\mu'\, dm \le F(b) - F(a)$

and thus $\int_{-\infty}^{\infty} F_\mu'\, dm \le \lim_{\substack{b\to+\infty\\ a\to-\infty}} \int_a^b F_\mu'\, dm \le \lim_{b\to\infty} F(b) - \lim_{a\to-\infty} F(a) = \mu(\mathbb{R}) - \mu(\emptyset) < \infty$)

pf (of second lemma) It suffices to show $\int_a^b G'(x)dx \le G(b^+) - G(a)$, because $\int_a^b G'(x)dx = \lim_{c\to b^-} \int_a^c G'(x)dx \le \lim_{c\to b^-} (G(c^+) - G(a)) \le G(b) - G(a)$ since $c < b \Rightarrow G(c^+) \le G(b)$.

$g_n(x) = \frac{G(x+\frac{1}{n}) - G(x)}{1/n} = n[G(x+\frac{1}{n}) - G(x)]$. Since $G'(x)$ exists a.e. $g_n \to G'$ pt.wise a.e. By $g_n \ge 0$ and <u>Fatou's Lemma</u> $\int_a^b G'(x)dx \le \liminf_{n\to\infty} \int_a^b g_n$

$\int_a^b g_n = n\left[\int_a^b G(x+\frac{1}{n}) - \int_a^b G(x)\right] = n\left[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} G(x) - \int_a^b G(x)\right] = n\cdot\int_b^{b+\frac{1}{n}} G(x) - n\int_a^{a+\frac{1}{n}} G(x)$

hence $\int_a^b G'(x)dx \le \liminf_{n\to\infty}\left[n\cdot\int_b^{b+\frac{1}{n}} G(x) - n\int_a^{a+\frac{1}{n}} G(x)\right] \le G(b^+) - G(a)$ as desired $\boxtimes$ (lemma)

Lemma: Suppose $f \in L_1([a,b])$ and for each c, $a \le c \le b$, $\int_a^c f\,dm = 0$, then $f = 0$ a.e.

pf spse $f \ne 0$ a.e., wlog $m\{x: f(x) > 0\} > 0$, then $\exists$ closed set $F \subseteq \{x: f(x) > 0\}$ with $m(F) > 0$. Now $\int_F f\,dm \ne 0$ by $\int_{[a,b]/F} f\,dm \ne 0$. We can write $[a,b]/F = \bigcup_{n=1}^{\infty} (a_n, b_n)$ where (a_n, b_n) are p.w.d. $\int_{[a,b]/F} f\,dm = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f\,dm$ (L.D.C.T). There is an (a_n, b_n)

pf. such that $\int_{a_n}^{b_n} f \neq 0 \Rightarrow \int_a^{a_n} f\,dm \neq 0$ or $\int_a^{b_n} f\,dm \neq 0$ a contradiction ⊠ (lemma) ②

Lemma: $f \geq 0$, f bounded and measurable on $[a,b]$. Let $F(x) = \int_a^x f\,dm$, then $F'(x) = f(x)$ a.e.

pf. Note: $F(x)$ decreasing so $F'(x)$ exist a.e., furthermore $F(x)$ is continuous.

$F_n(x) = n\left(F(x+\frac{1}{n}) - F(x)\right)$. $F_n \xrightarrow[(ptws)]{} F'$ a.e. Also, $0 \leq F_n(x) \leq B$ where

$B \geq \sup_{x\in[a,b]} |f(x)|$ by $F_n(x) = n\int_x^{x+\frac{1}{n}} f\,dm \leq n(\frac{1}{n})B = B$. L.D.C.T. where

$a \leq c \leq b$ gives: $\int_a^c F'(x)\,dx = \lim_{n\to\infty} \int_a^c F_n(x)$. $\int_a^c F_n(x) = n\int_c^{c+\frac{1}{n}} F(x) - n\int_a^{a+\frac{1}{n}} F(x)$

$\lim_{n\to+\infty} n\int_c^{c+\frac{1}{n}} F(x)\,dx = F(c)$ since F is continuous at c. Thus

$\int_a^c F'(x)\,dm = F(c) - F(a) = \int_a^c f\,dm$ by def. of $F(c)$ and above. Now,

$\int_a^c \left(F'(x) - f(x)\right) dm = 0$ for each c, $a \leq c \leq b$; which implies

$F'(x) - f(x) = 0$ a.e. i.e. $F'(x) = f(x)$ a.e. as claimed ⊠ (lemma)

Proposition: $f \geq 0$, $f \in L_1([a,b])$. Let $F(x) = \int_a^x f\,dm$, then $F'(x) = f(x)$ a.e.

pf. let c be $a \leq c \leq b$; $\int_a^c F'(x)\,dx \leq F(c) - F(a)$. let $f_n(x) = \inf[f(x), n]$

so f_n is bded, $0 \leq f_n \uparrow f$. $G_n(x) = \int_a^x (f - f_n)$, then $F(x) = G_n(x) + \int_a^x f_n\,dm$. G_n

increasing; so $F'(x) = G_n'(x) + \frac{d}{dx}\left(\int_a^x f_n\,dm\right) = G_n'(x) + f_n(x)$ a.e.

$F'(x) \geq f_n(x)$ a.e., thus $F'(x) \geq f(x)$ a.e. $\therefore \int_a^c F'(x) \geq \int_a^c f(x) = F(c) - F(a)$

Hence, $\int_a^c (F'(x) - f(x))\,dm = 0$ for each c, $a \leq c \leq b$ $\therefore$ $F'(x) = f(x)$ a.e. ⊠ (Proposition)

Corr. If the Borel measure μ has a density, then it is F_μ' a.e.

pf. if f is the density $F_\mu(x) = \int_{-\infty}^x f\,dm = \int_a^x f\,dm + F(a)$, $\therefore F_\mu'(x) = f$ a.e.

since $f \in L_1(\mathbb{R})$ ⊠ (corr)

(Coming) (going)

μ - Borel Measure

F_μ - Distribution

$F'_\mu \in L_1(\mathbb{R})$

ν Define by $\nu(E) = \int_E F'_\mu\,dm$, then ν a.c. and can show $\nu < \mu$

and $\mu = (\mu - \nu) + \nu$. $\mu - \nu$ is singular; ν as said a.c.

We have shown how to take $f \in L_1(\mathbb{R})$, $f \geq 0$ and use it to construct μ a Borel measure with density f. We have shown that $\mu \ll m$ and that if F_μ is the ~~distribution function of μ then $F_\mu' = f$ a.e. Thus~~ if we know that μ has a density function we can construct it.

The next problem we will consider is how can you tell (from μ alone) if μ has a density function? We will solve this problem by looking at F_μ, and the answer has to do with absolute continuity. The following definition is needed.

<u>DN 1.</u> An increasing function $G: \mathbb{R} \to \mathbb{R}$ is said to be <u>absolutely continuous</u> on $[a,b]$ if for each $\varepsilon > 0$ there is a $\delta > 0$ so that for each integer n and pts $x_1 \ldots x_n$ & $y_1, \ldots, y_n$ satisfying

(1) $a \leq x_1 \leq y_1 < x_2 \leq y_2 < x_3 \leq y_3 < \ldots < x_n \leq y_n \leq b$

(2) $\sum_{i=1}^{n} (y_i - x_i) < \delta$

then we have $\sum_{i=1}^{n} [G(y_i) - G(x_i)] < \varepsilon$.

<u>LEMMA 2</u> If the increasing function G is absolutely continuous on $[a,b]$ then G is continuous on $[a,b]$.

<u>Note</u>: The converse of Lemma 2 is False!

<u>pf of Lemma 2</u>: Suppose for some $c \in [a,b]$, G is not

continuous at c then $G(c^+) > G(c^-)$. Let $\varepsilon = \frac{1}{2}(G(c^+) - G(c^-))$ then for each $\delta > 0$ $x_1 = c - \delta/3$, $y_1 = c + \delta/3$ satisfies (1) and (2) of DN 1 but $G(y_1) - G(x_1) \geq G(c^+) - G(c^-) > \varepsilon$. Thus G is not absolutely continuous.

Proposition 3: The Borel measure μ is absolute continuous, if and only if F_μ is absolutely continuous on $(-\infty, \infty)$.

proof: ($\Rightarrow$) If $\mu << m$, then F_μ is continuous, and for each $\varepsilon > 0$ $\exists \delta > 0$ so that for $E \in \mathcal{B}$, $m(E) < \delta \Rightarrow \mu(E) < \varepsilon$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be the resulting δ in $\mu << m$. Condition (1) & (2) can be restated as $m(\bigcup_{i=1}^{n} (x_i, y_i]) < \delta$ and $\sum_{i=1}^{n} [F_\mu(y_i) - F_\mu(x_i)] = \sum_{i=1}^{n} \mu((x_i, y_i]) = \mu(\bigcup_{i=1}^{n} (x_i, y_i]) < \varepsilon$.

($\Leftarrow$) If F_μ is absolute continuous and let $E \in \mathcal{B}$ so that $m(E) = 0$; we want to show $\mu(E) = 0$, we will show that for each $\varepsilon > 0$, $\mu(E) \leq \varepsilon$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be the resulting δ from DN1. There are (a_n, b_n) pairwise disjoint open intervals so that $E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{\infty} (b_n - a_n) < \delta$. Thus for each m, $\mu(\bigcup_{n=1}^{m} (a_n, b_n]) = \sum_{n=1}^{m} \mu((a_n, b_n]) = \sum_{n=1}^{m} (F_\mu(b_n) - F_\mu(a_n)) < \varepsilon$ [note that $\{(a_n, b_n)\}_{n=1}^{m}$ may have to be re-ordered to satisfy DN1, but this will not effect the sum]. Therefore $\mu(E) \leq \mu(\bigcup_{m=1}^{\infty} (\bigcup_{n=1}^{m} (a_n, b_n]) = \lim_{m \to \infty} \mu(\bigcup_{n=1}^{m} (a_n, b_n]) \leq \varepsilon$, which completes the proof.

Now let us consider a general Borel measure μ. We know that if $f = F_\mu'$ then $f \geq 0$ and $f \in L_1(\mathbb{R})$. Thus we can define a Borel measure ν with density f.

Lemma 4: With ν, μ as above, then $\forall E \in \mathcal{B}$ $\nu(E) \le \mu(E)$.

proof: Let $\mathcal{M} = \{E \in \mathcal{B} : \nu(E) \le \mu(E)\}$. If $a < b$ then $(a,b] \in \mathcal{M}$ because

$$\nu((a,b]) = \int_a^b f = \int_a^b F_\mu' \le F_\mu(b) - F_\mu(a) = \mu((a,b])$$

similarly $(-\infty, b]$ and $(a, +\infty) \in \mathcal{M}$. Since both μ, ν are finitely additive, each finite disjoint union of the above sets are in $\mathcal{M}$. Call this collection $\mathcal{A}$. We have shown that $\mathcal{A}$ is an algebra. If we can show $\mathcal{M}$ is a monotone class then $\mathcal{M} \supset \mathcal{M}(\mathcal{A}) = S(\mathcal{A}) = \mathcal{B}$ and ~~then~~ Lemma would be proved.

Suppose $A_1 \supset A_2 \supset \dots$ are each in $\mathcal{M}$ since $\nu(A_1), \mu(A_1) < \infty$ we have $\nu(\cap A_n) = \lim \nu(A_n) \le \lim \mu(A_n) = \mu(\cap A_n)$ and thus $\cap A_n \in \mathcal{M}$. Increasing chains are handled similarly. Therefore $\mathcal{M}$ is a monotone class.

Continuing, now we can form the Borel measure $\mu - \nu$. Since $F_{\mu-\nu} = F_\mu - F_\nu$, and $F_{\mu-\nu}' = F_\mu' - F_\nu' = F_\mu' - F_\mu' = 0$ a.e. and since for $E \in \mathcal{B}$ $(\mu-\nu)(E) \le \mu(E)$ we have:

Proposition 5. Each Borel measure μ can be written as the sum of two Borel measures $\lambda + \nu$ where $F_\mu' = F_\nu'$ a.e and $F_\lambda' = 0$ a.e.

Suppose we can prove the following

Proposition 6: If $\mu \ll m$ and $F_\mu' = 0$ a.e. then $\mu(E) = 0$ for each $E \in \mathcal{B}$

Theorem 7. The Borel meas μ has a density function, if and only if $\mu << m$

proof: ~~($\Rightarrow$) has already been done~~
($\Leftarrow$): If λ is as in the notation of Proposition 5, we have $F_\lambda' = 0$ a.e. Further more if $E \in \mathcal{B}$ with $m(E)=0$ then $\mu(E)=0$ by hypothesis; and since $\forall E \in \mathcal{B}$ $0 \le \lambda(E) \le \mu(E)$ we have $m(E)=0 \Rightarrow \lambda(E)=0$. Therefore, F_λ is absolutely continuous and by Prop 6 it must be constant. Therefore $F_\lambda \equiv 0$ & $\lambda \equiv 0$ so that $\mu \equiv \nu$, and ν has density F_μ'.

We will now start to prove Prop 6 & the fact that increasing functions are differentiable a.e.. Several preliminary thoughts are in order

DN 8: If $E \subset \mathbb{R}$ and $\mathcal{J}$ is a collection of non-trivial intervals of $\mathbb{R}$ $\mathcal{J}$ is said to be a Vatali covering of E if $\forall x \in E$ and $\forall \varepsilon > 0$ $\exists I \in \mathcal{J}$ with $x \in I$ & $\ell(I) < \varepsilon$

Lemma 9 (Vatali): If $E \subset \mathbb{R}$ with $m(E) < \infty$ and $\mathcal{J}$ is a Vatali covering of E then for each $\varepsilon > 0$ there is a finite p.w. disjoint collection $I_1, \dots I_N$ of elements of $\mathcal{J}$ with $m\left(E \setminus \left(\cup_{i=1}^N I_i\right)\right) < \varepsilon$.

proof: We may assume each $I \in \mathcal{J}$ is closed (since the endpts have measure zero). Let U be an open set with $U \supset E$ and $m(U) < \infty$. Note that $\mathcal{J}' = \{I \in \mathcal{J} : I \subset U\}$ is also a Vatali covering of E (since $e \in E$ implies that e is a positive distance from the closed set $\mathbb{R} \setminus U$) Therefore we may assume that each $I \in \mathcal{J}$ is contained in U.

The idea of the proof is simple, we will inductively choose $\{I_n\}$ to be p.w. disjoint elements of $\mathcal{J}$, so that they take up as much "room" as "possible". Let $I_1 \in \mathcal{J}$ be arbitrary, and suppose $I_1, \ldots, I_n$ have been choosen so that

(1) $\{I_i\}_{i=1}^{n}$ are p.w.d elements of $\mathcal{J}$

(2) $\ell(I_{j+1}) \geq \frac{1}{2} \sup\{\ell(I) : I \in \mathcal{J}, I \cap I_i = \emptyset \text{ for } i=1,\ldots j\}$.

(and let us define $k_j = \sup\{\ell(I) : I \in \mathcal{J}, I \cap I_i = \emptyset \text{ for } i=1,\ldots j\}$)

As long as $E \setminus (\bigcup_{i=1}^{n} I_i) \neq \emptyset$, we can choose such an I_{n+1}. (If $E \setminus (\bigcup_{i=1}^{n} I_i) = \emptyset$ the Lemma is proved.)

Since $\bigcup_{i=1}^{\infty} I_i \subset U \Rightarrow \sum_{i=1}^{\infty} \ell(I_i) \leq m(U) < \infty$ and thus $\sum_{i=j}^{\infty} \ell(I_i) \to 0$ and so does $k_j \to 0$, as $j \to \infty$. Pick N so that $\sum_{i=N+1}^{\infty} \ell(I_i) < \varepsilon/5$ and let $R = E \setminus (\bigcup_{i=1}^{N} I_i)$. we will complete the prove by showing $m(R) < \varepsilon$. We will do this by show $R \subset \bigcup_{i=N+1}^{\infty} J_i$, where J_i is the interval with the same midpoint as I_i and $\ell(J_i) = 5\ell(I_i)$. this will follow since then $m(R) \leq \sum_{i=N+1}^{\infty} \ell(J_i) = \sum_{i=N+1}^{\infty} 5\ell(I_i) < 5(\frac{\varepsilon}{5}) = \varepsilon$

Let $x \in R$ since $\bigcup_{i=1}^{N} I_i$ is closed, x must be a positive distance from the set $\bigcup_{i=1}^{N} I_i$, hence there is $A \in \mathcal{J}$, $\ell(A) > 0$ with $x \in A$ and $A \cap (\bigcup_{i=1}^{N} I_i) = \emptyset$. Since $k_j \to 0$ there must be some k_n with $\ell(A) > k_n$ and hence $A \cap I_i \neq \emptyset$ for some $i \leq n$ (by the definition of k_n). Let I_m be the first index with $A \cap I_m \neq \emptyset$. We have $m \leq n$, $m > N$ and $k_{m-1} \geq \ell(A)$ thus $\ell(A) \leq 2I_m$ and $x \in A \subset J_m$ which completes the proof.

If $f : [a,b] \to \mathbb{R}$ define $V(f) = V_a^b(f)$ to be sup of the sums $\sum_{i=1}^{n} |f(a_i) - f(a_{i-1})|$ where $a = a_0 \leq a_1 \leq \ldots \leq a_n = b$. If $V_a^b(f) < \infty$ we will say f is of bounded variation and $V_a^b(f)$ is called the variation of f.

Note that if f is increasing (decreasing) $V_a^b(f) = f(b) - f(a)$ (resp. $f(a) - f(b)$) and it is easy to check that if f and g are increasing then $V_a^b(f \pm g) \leq V_a^b(f) + V_a^b(g)$. Also if f is of bounded variation then $V(\alpha f) = |\alpha| V(f)$. The only thing keeping V from being a norm is that $V(f) = 0$ is equivalent to $f \equiv$ constant. Thus the space of functions f with Bounded variation ~~f~~ and $f(a) = 0$ is a norm space with norm $V(\cdot)$.

<u>Lemma</u>: f is of bounded variation on $[a,b]$ if and only if f is the difference of two increasing functions

pf ($\Leftarrow$) was done above

($\Rightarrow$) Suppose f is of bounded variation define two functions on $[a,b]$ $I(c)$ and $D(c)$ via $I(c)$ (resp $D(c)$) is the sup of the sums like in $V_a^c(f)$ except $|f(a_i) - f(a_{i-1})|$ is replaced with $(f(a_i) - f(a_{i-1}))^+$ (resp $(f(a_i) - f(a_{i-1}))^-$). Clearly $I(c)$ and $D(c)$ are increasing functions.

I Claim $V_a^c(f) = I(c) + D(c)$ & $f(c) = I(c) - D(c) + f(a)$

Since (1) $(f(a_i) - f(a_{i-1})) = (f(a_i) - f(a_{i-1}))^+ - (f(a_i) - f(a_{i-1}))^-$

We use (1) by

$$f(b) - f(a) = \sum (f(a_i) - f(a_{i-1}))^+ - \sum (f(a_i) - f(a_{i-1}))^-$$

$$f(c) - f(a) + D(c) \geq \sum (f(a_i) - f(a_{i-1}))^+$$

$$f(c) - f(a) + D(c) \geq I(c)$$

so $\quad f(c) \geq I(c) - D(c) + f(a)$

also $\quad f(a) - f(c) + \sum (f(a_i) - f(a_{i-1}))^+ = \sum (f(a_i) - f(a_{i-1}))^-$

so $\quad f(a) - f(c) + I(c) \geq D(c)$

similarly $\quad V_a^c(f) \leq I(c) + D(c)$ (although $V_a^c(f) \geq I(c) + D(c)$ is a little trickier)

Cor 12: If f is of bdd variation, f' exist a.e.

Cor 13: If $f \in L_1(\mathbb{R})$ $\quad F(x) = \int_{-\infty}^{x} f\,dx$, then F is of bounded variation

DN 14: Redefining abs cont of $F(x)$ by replacing $\sum F(x_i) - F(x_{i-1})$ by $\sum |F(x_i) - F(x_{i-1})|$

Lemma 15. F abs cont on $[a,b] \Rightarrow F$ is of bdd variation on $[a,b]$.

proof Let $\varepsilon = 1$, δ the resulting δ $\quad K$ an integer so that $K\delta \geq b - a + 1$. Any sum used to find $V_a^b(f)$ can be refined by add the division pts $a+\delta, a+2\delta, \ldots$ and this increases the sum thus $V_a^b(f) \leq K$.

Note that both I & D defined Lemma 11 are abs cont if f is absolutely cont.

Prop. 10. If G is increasing and absolutely continuous on $[a,b]$ with $G' = 0$ a.e on $[a,b]$ then G is constant on $[a,b]$.

Proof: Before ~~proof~~ proving this we note that this will also prove Prop 6. Let $c \in [a,b]$, it suffices to show that for each $\omega > 0$ that $G(c) - G(a) < \omega$. So let $\omega > 0$ be given and let $\varepsilon_1 (b-a) < \omega/2$ and $\varepsilon_2 = \omega/2$.

Since $G' = 0$ a.e. on $[a,c]$, let E be the set with $m(E) = c-a$, $G' = 0$ on E & $a, c \notin E$. For each $\delta > 0$ and each $e \in E$ there is $e' \in (a,c)$ with $|e - e'| < \delta$ and $G(e) - G(e') < \varepsilon_1 (e - e')$ since $G'(e) = 0$. Thus the collection of such intervals $[e, e']$ (or $[e', e]$) form a Vitali Covering of E. Let $\delta > 0$ be the number resulting from G being absolutely continuous with $\varepsilon = \varepsilon_2$.

By Lemma 9, there are $a_1, \ldots a_N$, $b_1, \ldots b_N$ with $a \le a_1 < b_1 < a_2 < b_2 \ldots < a_N < b_N \le c$ with

(1) $G(b_i) - G(a_i) \le \varepsilon_1 (b_i - a_i)$ and

(2) $m(E \setminus (\bigcup_{i=1}^{n} [a_i, b_i])) < \delta$

Hence (3) $(a_1 - a) + \sum_{i=1}^{n-1} (a_{i+1} - b_i) + (c - b_n) < \delta$

and (4) $G(c) - G(b_n) + \sum_{i=1}^{n-1} [G(a_{i+1}) - G(b_i)] + G(a_1) - G(a) < \varepsilon_2$ (by abs cont)

By (1): (5) $\sum_{i=1}^{n} (G(b_i) - G(a_i)) \le \varepsilon_1 \sum_{i=1}^{n} (b_i - a_i) \le \varepsilon_1 (c - a) < \omega/2$

thus by summing (4) & (5) $G(c) - G(a) < \varepsilon_2 + \omega/2 < \omega$, which completes the proof.

Our final result we have been postoning for weeks is

Theorem ??: If f is increasing $: \mathbb{R} \to \mathbb{R}$ then f' exists a.e.

proof: To say take $f'(x)$ exists says that $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists. since limits can fail to exist we can rewrite this condition in terms of things that do always exist as follows define

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \qquad D^- f(x) = \liminf_{h \to 0^+} \frac{f(x+h)-f(x)}{h}$$

$$D_+ f(x) = \limsup_{h \to 0^-} \frac{f(x+h)-f(x)}{h} \qquad D_- f(x) = \liminf_{h \to 0^-} \frac{f(x+h)-f(x)}{h}$$

clearly $D^+ f(x) \geq D^- f(x)$ and they are equal if and only if f has a right-hand derivative at x. Similarly $D_+ f(x) \geq D_- f(x)$ with $=$ equivalent to f having a left-hand derivative at x. And $f'(x)$ exists exactly if the 4 numbers $D^{\pm} f(x)$, $D_{\pm} f(x)$ are are equal. [we do not worry about $\pm\infty$ for a value of $f'(x)$ since we have shown (assuming that the limit exist a.e, as an extended real) $f'(x) = \pm\infty$ at most on a set of measure zero.]

Thus the set $B = \{x : \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ fails to exist as an extended real$\}$ can be written as the union of three sets $B = \{x : D^+ f(x) > D^- f(x)\} \cup \{x : D_+ f(x) > D_- f(x)\} \cup \{x : D^+ f(x) \neq D_- f(x)\}$. It suffices to show that each of these three sets as measure zero. We will show that the first set has measure zero, the others are similar (~~the~~ Compare with Rogden which shows that the last set has measure zero.)

We first reduce the problem even further, let

$W = \{(u,v) : u, v \in \mathbb{Q}, u > v\}$. As a subset of $\mathbb{Q}\times\mathbb{Q}$, W is countable. We claim that

$$\{x : D^+f(x) > D^-f(x)\} = \bigcup_{(u,v)\in W} \{x : D^+f(x) > u > v > D^-f(x)\}$$

It is easy to see that the right hand side $\subseteq$ left hand side.
Conversely, the density of the rationals implies that if $D^+f(x) > D^-f(x)$ then there are rationals u, v with $D^+f(x) > u > v > D^-f(x)$. Since W is countable it suffices to show for each $(u,v) \in W$, $E = \{x : D^+f(x) > u > v > D^-f(x)\}$ has measure zero.

Suppose $m(E) = \lambda > 0$ and let $\varepsilon > 0$, let U open $\supset E$ with $m(U) < \lambda + \varepsilon$ (assume $2\varepsilon < \lambda$ without loss of generality). Since $x \in E$ implies $D^-f(x) < v$ for each $\delta > 0$ $\exists h$, $0 < h \le \delta$ with $f(x+h) - f(x) < vh$ and $[x, x+h] \subset U$. Thus the set of such $[x, x+h]$ form a Vitali covering for E. By the Lemma $\exists\, x_1, \dots x_N, h_1, \dots, h_N$ with $\{[x_i, x_i+h_i]\}_{i=1}^N$ and
$m\left(E \setminus \left(\bigcup_{i=1}^N [x_i, x_i+h_i]\right)\right) < \varepsilon$. Let $F = E \cap \bigcup_{i=1}^N [x_i, x_i+h_i)$
clearly $m(F) > \lambda - \varepsilon$ and $\sum_{i=1}^N h_i > \lambda - \varepsilon$ and we have
$\sum_{i=1}^N (f(x_i+h_i) - f(x_i)) \le v\sum_{i=1}^N h_i < v(\lambda+\varepsilon)$ since $\bigcup_{i=1}^N [x_i, x_i+h_i] \subset U$.

For each $y \in F$, $\delta > 0$ $\exists k$, $0 < k < \delta$ with $[y, y+k] \subset \bigcup_{i=1}^N [x_i, x_i+h_i]$ so that $f(y+k) - f(y) > vk$. ~~where~~ Thus the set of all such $[y, y+k]$ form a Vitali covering of F. so there are $y_1 \cdots y_M$, $k_1 \dots k_M$ with $\{[y_j, y_j+k_j]\}_{j=1}^M$ p.w.d. $\bigcup_{j=1}^M [y_j, y_j+k_j] \subset \bigcup_{i=1}^N [x_i, x_i+h_i]$ and $m(F \setminus (\bigcup_{j=1}^M [y_j, y_j+k_j]) < \varepsilon$. Thus
$\lambda - 2\varepsilon < m(\bigcup_{j=1}^M [y_j, y_j+k_j]) \leqq \sum_{j=1}^M k_j$. we have
$(\lambda - 2\varepsilon)u < u\sum_{j=1}^M k_j < \sum_{j=1}^M (f(y_j+k_j) - f(y_j)) \le \sum_{i=1}^N (f(x_i+h_i) - f(x_i))$
$\le v\sum_{i=1}^N h_i < v(\lambda - \varepsilon)$ or $(\lambda - 2\varepsilon)u < v(\lambda - \varepsilon)$ since this is true $\forall \varepsilon > 0$ $u \le v$ a contradiction.

TEST DUE AT HIGH NOON MONDAY 11 DEC 1978

I. Let $U = [0,1] \times [0,1] \subseteq \mathbb{R}^2$ (the unit square), for each $c \in U$ and for each $r \geq 0$ define $B_c(r) = \{x \in U;\ \text{distance}(x,c) \leq r\}$ For each closed set $A \subset U$ define

$$R(A) = \inf\{r : \exists c \in U,\ A \subset B_c(r)\}$$

Show there is a $C(A) \in U$ so that $A \subset B_{C(A)}(R(A))$

II, Suppose $1 \leq p < r < q < \infty$ show there is an α $1 > \alpha > 0$ with $\frac{1}{r} = \alpha\frac{1}{p} + (1-\alpha)\frac{1}{q}$ and then show for each $f : \mathbb{R} \to \mathbb{R}$ $\|f\|_r \leq (\|f\|_p)^{\alpha}(\|f\|_q)^{1-\alpha}$ and finally show $L_p(\mathbb{R}) \cap L_q(\mathbb{R}) \subsetneqq L_r(\mathbb{R})$

III, Let $\{r_n\}_{n=1}^{\infty}$ be a listing of the rationals and for each $x \in \mathbb{R}$ define $a_n(x) = \begin{cases} 1/2^n & \text{if } r_n \leq x \\ 0 & \text{if } r_n > x \end{cases}$ and then define $g(x) = \sum_{n=1}^{\infty} a_n(x)$. show that $g(x)$ is strictly increasing, continuous on the right and $g'(x) = 0$ a.e.

IV, Let $f(x)$ be the Cantor ternary function and let C be the Cantor set. Define $\mu : \mathcal{B} \to \mathbb{R}$ by for each $E \in \mathcal{B}$ $\mu(E) = m(f(E \cap C))$. Show that μ is a Borel measure. with μ non-atomic, $\mu \perp m$. Show that $F_\mu = f$ and $F_\mu' = 0$ a.e.

V. Show that if $f \in L_1(\mathbb{R})$, $\lim_{n \to +\infty} \int_{-\infty}^{\infty} f(x) \cos nx \, dx = 0$