Example The Haar basis is not unconditional in $L_1$

$$\| h_1 + h_2 + 2h_3 + 4h_5 + 8h_9 + \dots 2^k h_{2^k+1} \|_1 = 1$$

$$\| h_1 + 2h_3 + 8h_9 + \dots \quad 2^{2k+1} h_{2^{2k+1}+1} + \dots \|_1 = \infty$$

Remark $\Sigma' \alpha_n x_n$ converges unconditionally $\Leftrightarrow$ $\Sigma \pm \alpha_n x_n$ conv all choices of signs $\Leftrightarrow$ $\Sigma_i \alpha_{n_i} x_{n_i}$ conv all subseq $\{n_i\}$ of integers $\Leftrightarrow$ $\Sigma_n \alpha_{\pi(n)} x_{\pi(n)}$ all permutations $\pi : \mathbb{N} \to \mathbb{N}$.

Thm: The Haar basis is an unconditional basis for $L_p$, for $1 < p < \infty$.

Proof: For $p = 2^m$ $m = 1, 2, \dots$. For $p = 2$, Haar basis is orthogonal & orthogonal basis are unconditional.

Let $p = 2^m$, & let $\{\alpha_n\}_{n=1}^k$ be given let $\theta_n = \pm 1$

Let $f_j = \Sigma_1^j \alpha_n h_n$, $g_j = \Sigma_1^j \alpha_n \theta_n h_n$ these are martingales with respect to $\mathcal{A}_j$ = smallest $\sigma$-algebra for which $\{h_n\}_{n=1}^j$ is meas. [same proof as above]

Since $S$ given by Thm D? is the same for $f_j$ & $g_j$

$$C_p^{-2} \| f_k \|_p \le \| g_k \|_p \le C_p^2 \| f_k \|_p$$

Thus the map $T : \Sigma_1' \alpha_n h_n \mapsto \Sigma \alpha_n \theta_n h_n$ has norm at most $C_p^2$ on span $\{h_n\}_1^\infty$ hence on all $L_p$. If $\Sigma_1' \alpha_n h_n$ converges we have $T(\Sigma_1^\infty \alpha_n h_n) = \lim T(\Sigma \alpha_n h_n) = \lim \Sigma_1^N \alpha_n \theta_n h_n$ so $\Sigma \alpha_n \theta_n h_n$ conv.

The proof is completed by using interpolation for $2 \le p < \infty$ and duality for $1 < p \le 2$ [i.e. define $x_n'(\Sigma \alpha_m x_m) = \alpha_n$, if $(x_n)$ is unconditional so is $(x_n')$ unconditional. & for $h_n$, $h_n'$ is $h_n$ in $L_{p'}$.].

Note that the Rademachers are as far from being a martingale as possible w/resp to $\{\mathcal{B}_k\}$ above. Also $r_n = \sum_{i=2^n+1}^{2^{n+1}} h_i$, a "block-basic" sequence of the Haar basis.

This completes section 5. On to interpolation.