Case IV: All others. $1<p<\infty$, $\|f+g\|_p>0$, $\|f\|_p, \|g\|_p<\infty$. Now

$|f+g|^p = |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}$ integrate & use Hölder's inequality

$\|f+g\|_p^p \le \|f\|_p \, \|f+g\|^{p-1}_{p'(p-1)} + \|g\|_p \|f+g\|^{p-1}_{p'(p-1)}$ Since $p'(p-1)=p$

$\|f+g\|_p^p \le (\|f\|_p+\|g\|_p)\,\|f+g\|_p^{p-1}$ dividing by $\|f+g\|_p^{p-1}$ completes the result.

Consider equality. If $\|f+g\|_p=0$, equality occurs iff $f=g=0$ a.e.. Otherwise if $\|f\|_p, \|g\|_p<\infty$ using equality for Hölder's we must have $\{|f|^p, |f+g|^{p'(p-1)}\}$ and $\{|g|^p, |f+g|^{p'(p-1)}\}$ are dependent. This happens $\Leftrightarrow$ $\{|f|, |f+g|\}$ & $\{|g|, |f+g|\}$ are dependent. Since $|f+g|\neq 0$ a.e., we have $|f|=\lambda|f+g|$ & $|g|=\mu|f+g|$ so iff $\{f,g\}$ are dependent and $f\bar{g}\ge 0$ a.e. ~~Either~~ $\exists\,\lambda\ge 0$ so that either $f=\lambda g$ or $g=\lambda f$

Remarks. If $0<p<1$, Minkowski's inequality does not hold. However, if $a,b\ge 0$, since $\cancel{2^{1/p}(a^p+b^p)}$ $(a+b)^p \le (2\max\{a,b\})^p \le 2^p(\max\{a,b\})^p \le 2^p(a^p+b^p)$.

$\|f+g\|_p^p \le 2^p(\|f\|_p^p+\|g\|_p^p)$. Thus $\|f+g\|_p \le 2\,(2\max\{\|f\|_p^p, \|g\|_p^p\})^{1/p}$

$= 2^{1+\frac{1}{p}} \max\{\|f\|_p, \|g\|_p\} \le 2^{1+\frac{1}{p}}(\|f\|_p+\|g\|_p)$. Hence L_p is always a vector space.

$(X, \Sigma, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N})$, counting measure) call it ℓ_p

$(X, \Sigma, \mu) = (\{1,2,\dots n\}, \mathcal{P}(X)$, counting measure) call it ℓ_p^n

$(X, \Sigma, \mu) = (\{1,2,\dots n\}, \mathcal{P}(X)$, $\frac{1}{n}$ counting measure) call it L_p^n

This ~~last~~ notation comes from the fact that it is also $L_p(\mu)$ for $(X, \Sigma, \mu) = ([0,1], \mathcal{B}([0,\frac{1}{n}), [\frac{1}{n},\frac{2}{n}), \dots, [\frac{n-1}{n},1)), m)$.

For L_p^n, $0<p\le q\le\infty \Rightarrow \|\cdot\|_p \le \|\cdot\|_q$ & $L_p^n = L_q^n$

For ℓ_p^n, $0<p\le q\le\infty \Rightarrow \|\cdot\|_p \ge \|\cdot\|_q$ & $\ell_p^n = \ell_q^n$

For ℓ_p, $0<p\le q\le\infty \Rightarrow \|\cdot\|_p \ge \|\cdot\|_q$ & $\ell_p \subset \ell_q$

To see the last two note that $\|f\|_p = (\sum_i |f(i)|^p)^{1/p}$. Hence if $\|f\|_p\le 1$, $\sum_i |f(i)|^p \le 1$, hence $|f(i)|^p\le 1$ for each i. Since $q/p\ge 1$ thus $|f(i)|^q = (|f(i)|^p)^{q/p} \le |f(i)|^p$. So, $\sum_i |f(i)|^q \le \sum |f(i)|^p = 1$ and $\|f\|_q\le 1$. It follows that $\|f/\|f\|_p\|_q = \|f\|_q/\|f\|_p \le 1$ or $\|f\|_q \le \|f\|_p$ (if $\|f\|_p\neq 0$)

(The case $\|f\|_p = 0$ ~~is~~ $\Rightarrow$ $f=0$ a.e. $\Rightarrow \|f\|_q = 0$).

~~are some atomi, i.e. purely atomic measures of interest.~~

r between p & q

I Suppose $f \geq 0$ is measurable, $0 < p, q, r \leq \infty$, so that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$
$0 \leq \theta \leq 1$ [convention $\frac{1}{\infty} = 0$], then $\|f\|_r \leq \|f\|_p^{\theta} \|f\|_q^{1-\theta}$

Hence we always have $p < r < q \Rightarrow L_p(\mu) \cap L_q(\mu) \subset L_r(\mu)$
in which case $\|f\|_r \leq \max\{\|f\|_p, \|f\|_q\}$.

proof: $1/r = 1/\frac{p}{\theta} + 1/\frac{q}{1-\theta}$, $|f| = |f|^{\theta}|f|^{1-\theta}$, by generalized
Hölders $\|f\|_r = \| |f|^{\theta}|f|^{1-\theta}\|_r \leq \||f|^{\theta}\|_{\frac{p}{\theta}} \||f|^{1-\theta}\|_{q/1-\theta}$.
But $\||f|^{\theta}\|_{p/\theta} = \left(\int (|f|^{\theta})^{\frac{p}{\theta}}\right)^{\theta/p} = \left[\left(\int |f|^p\right)^{1/p}\right]^{\theta} = \|f\|_p^{\theta}$
Thus the inequality is true.

Duality

If X is a vector space with a topology making vector addition and scalar multiplication are continuous. (A normed space is one example. If $0<p<1$, $L_p(\mu)$ the topology of all open sets of the form $\{f+g : \|g\|_p < \varepsilon\}$, $\varepsilon > 0$, $f \in L_p(\mu)$, is another such). We define X^* or X' to be all continuous linear functionals on X.

Theorem:

A. If $1 < p < \infty$, and $1/p + 1/q = 1$, then $[L_p(\mu)]' = L_q(\mu)$ with the duality $g \in L_q(\mu) \sim F_g : L_p(\mu) \to \mathbb{K}$ given by $F_g(f) = \int fg\, d\mu$

B. If μ is σ-finite, then $[L_1(\mu)]' = L_\infty(\mu)$ with duality $g \in L_\infty(\mu) \sim F_g : L_1(\mu) \to \mathbb{K}$ given by $F_g(f) = \int fg\, d\mu$

C. Always $L_\infty(\mu) \subset [L_1(\mu)]'$ and $L_1(\mu) \subset [L_\infty(\mu)]'$ but in general these inclusions are proper

D. If $L_\infty(\mu)$ is finite dimensional, and μ finite $L_1(\mu) = [L_\infty(\mu)]'$

E. For $0<p<1$, the problem is more difficult for examples $[L_p]' = \{0\}$ (i.e no non-constant continuous linear functionals) but $[\ell_p]' = \ell_\infty$ to give two examples

Outline of proof: (Details in Royden). Suppose $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$ then for each $g \in L_q(\mu)$ $F_g : L_p(\mu) \to \mathbb{K}$ given by $F_g(f) = \int fg\, d\mu$ is a continuous linear functional. Now F_g is obviously linear, to see that F_g is continuous we use Hölder's inequality and the following general principle (This shows $L_q(\mu) \subset L_p(\mu)$).

9. Interpolation.

Interpolation is becoming an increasingly important tool in analysis, which got its start on $L_p(\mu)$-spaces. We have already seen examples of operators T which have norm one as operators from $L_1(\mu) \to L_1(\mu)$ and $L_\infty(\mu) \to L_\infty(\mu)$. Later (and sometimes with more work) we have been able to show T has norm one from $L_p(\mu) \to L_p(\mu)$. This is actually a special case of interpolation. Unfortunately this is not a neat subject and there are lots of non-overlapping theorems with wildly differing proofs.

The spaces $X \cap Y$ and $X + Y$:

Let X, Y be Banach spaces which are subspaces (i.e. continuously embedded into V a (Hausdorff) topological vector space. For example $X, Y = L_p(\mu), L_q(\mu)$ and V = space of measurable fcns with the topology of convergence in measure.

$$X + Y = \{ f \in V : \exists\, g \in X\ \exists\, h \in Y \text{ s.t. } f = g + h \} \text{ with norm}$$
$$\|f\| = \inf \{ \|g\|_X + \|h\|_Y : g \in X, h \in Y \text{ with } f = g + h \}$$

$$X \cap Y = \{ f \in V : \|f\| = \max\{ \|f\|_X, \|f\|_Y \} < \infty \}$$

Lemma $X \cap Y$, $X + Y$ are Banach spaces.

pf: Let $\{f_n\}$ be a C.S. in $X \cap Y$, then $\{f_n\}$ is a C.S. in X & in Y. Thus $f_n \to g \in X \subseteq V$ & $f_n \to h \in Y \subseteq V$. But V is T_2 hence $g = h$. Let f be this candidate function now
$\|f_n - f\|_{X \cap Y} = \max\{ \|f_n - f\|_X, \|f_n - f\|_Y \} \longrightarrow 0.$

Now let $\{f_n\}$ be a C.S. in $X + Y$. By passing to a subsequence we may assume $\|f_n - f_m\|_{X+Y} < 2^{-n}$ for $m \geq n$. For $n \geq 1$ let $f_{n+1} - f_n = \varphi_n + \psi_n$, where $\|\varphi_n\|_X + \|\psi_n\|_Y < 2^{-n}$, hence $\|\varphi_n\|_X < 2^{-n}$ in X, $\|\psi_n\|_Y < 2^{-n}$ in Y. Let $f_1 = \varphi + \psi$ with $\varphi \in X, \psi \in Y$.
Now $g = \varphi + \sum_1^\infty \varphi_n \in X$ & $h = \psi + \sum_1^\infty \psi_n \in Y$ let $f = g + h$.
and
$$\|f - f_n\|_{X+Y} \leq \|\varphi + \textstyle\sum_1^\infty \varphi_n - (\varphi + \sum_1^{n-1} \varphi_n)\|_X + \|\psi + \sum_1^\infty \psi_n - \cdots\|_Y$$
$$\leq \textstyle\sum_n^\infty \|\varphi_n\|_X + \sum_n^\infty \|\psi_n\| \leq 2^{-n+1} + 2^{-n+1} \leq 2^{-n+2}$$