MAA 6526 L_p - Spaces Outline (Rough)

§0 Introduction, definitions and standard notations.

§1 Hölder's inequality and consequences

1.1 Other similar inequalities

1.2 Duality, extreme, exposed and strongly exposed pts

1.3 Rademacher functions, L_p is different from ℓ_p $p \neq 2, \infty$

§2 Automatic convergence. [necessary & sufficient conditions for $f_n \to f$ in norm

2.1. necessary conditions, weak topologies

2.2. Sufficient conditions, uniform convexity, uniform integrability

2.3 Examples

§3 Conditional Expectations, contractions, complements

3.1 Martingales

3.2 Haar basis

3.3. RNP ??? (Radon - Nikodým Property for vector-valued measures)

§4 Interpolation

4.1 Rearrangement Invariant spaces, Boyd indices, real interpolation

4.2 The Trigonometric system

4.3 Complex Interpolation ???

§5 Classification and isometries

5.1 Measure algebra's and Stone's theorem

5.2 Banach lattices and abstract L_p-spaces.

§6 Additional Topics depending on time and interest Possibilities:

3.3, 4.3, Embedding L_p in L_q $2 \geq p > q \geq 1$ and negative definite functions, Generalizations to general Banach lattices (like p-convexity, type p etc), The C(K)-spaces, the isomorphism between ℓ_∞ and L_∞, And I am open to suggestions.

The Project:

A goal oriented project aimed at writing a clear and elementary developement of some aspect of L_p-spaces not covered in class. Subject matter must be approved beforehand. (At least ten pages?)

Lecture 1 Introduction

Let (X, Σ_1, μ) be a set X, a σ-algebra Σ_1 on X and a positive measure μ. Reminber

a σ-algebra on X is a nonempty set $\Sigma_1 \subset \mathcal{P}(X)$ so that $A \in \Sigma_1 \Rightarrow X \setminus A \in \Sigma_1$ and $(A_n) \subset \Sigma_1 \Rightarrow \cup A_n \in \Sigma_1$.

and a positive measure on a σ-algebra Σ_1 is a function $\mu : \Sigma_1 \to [0, \infty]$ so that $\mu(\emptyset) = 0$ and if $(A_n) \subset \Sigma_1$ is pairwise disjoint then $\mu(\cup A_n) = \sum \mu(A_n)$

For $0 < p < \infty$, define $\|\cdot\|_p$ on ~~the set of~~ each scalar-valued measurable functions f by

$$\|f\|_p = \left\{ \int_X |f|^p \, d\mu \right\}^{1/p}$$

For $p = \infty$, define

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \inf \left\{ M \geq 0 : \mu\{x : |f(x)| \geq M\} = 0 \right\}$$

The set $L_p(\mu)$ is the collection of functions f with $\|f\|_p < \infty$. Note that $\|f\|_p = 0 \Leftrightarrow f = 0$ a.e. for this reason it is usually convenate to consider L_p the space of equivalence classes of the relation $f = g$ a.e.

Which scalars? Well we will consider both the case when the scalars are the reals $\mathbb{R}$ and when the scalars are the complexes $\mathbb{C}$. However, we will usually only give a formal proof for the real case, leaving the complex case as an exercise. Usually this extension is routine using the following facts

A. $f : X \to \mathbb{C}$ is measurable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f : X \to \mathbb{R}$ are measurable

B. $\int_E f \, d\mu = \int_E \operatorname{Re} f \, d\mu + i \int_E \operatorname{Im} f \, d\mu$

C. $\overline{a+bi} = a - bi$, $|a+bi| = [(a+bi)(\overline{a+bi})]^{1/2} = [a^2 + b^2]^{1/2}$

Useful p, usually we will be concern with the range $1 \leq p \leq \infty$, $1 < p < \infty$ or special p. However a few facts about $0 < p < 1$ come for free or are easy and hence included.

Notation. The most important example will be when (X, Σ', μ) is $X = [0,1]$, Σ' = The Borel or Lebesgue sets and μ is Lebesgue measure. In this case we will write $L_p(\mu)$ as just L_p. Note that either σ-algebra yields the same space L_p since f is Lebesgue measurable if and only if there is a Borel measurable g with $f = g$ a.e.

This is a general situation, if (X, Σ_0', μ_0) is the completion of (X, Σ', μ) then $L_p(\mu)$ is the same as $L_p(\mu_0)$. This is because f is μ_0-measurable iff the is a μ-measurable g with $f = g$ μ-a.e.

Also for future reference note that $(0,1)$, $[0,1)$, $[0,1]$ all yield the same L_p-space.

Other $L_p(\mu)$ spaces that often appear are

$L_p[0,\infty) = L_p(\mu)$ where $(X, \Sigma, \mu) = ([0,\infty)$, Lebesgue sets, Lebesgue meas$)$

$L_p(\mathbb{R}) = L_p(\mu)$ " " $= (\mathbb{R}$, " , " $)$,

Although $\|\cdot\|_p$ behave very differently on the above spaces in some sense they are the same.

<u>Proposition</u> L_p, $L_p[0,\infty)$ and $L_p(\mathbb{R})$ are order isometric.

What does this say? Some definitions

The norm space ~~of~~ X, Y are said to be <u>isometric</u> if there is a 1-1, onto linear map $T: X \to Y$ so that $\|Tx\| = \|x\|$ for each $x \in X$

An ~~isometry~~ linear map $T: X \to Y$ is an order ~~isomorphic iso~~ preserving if $f \le g \Rightarrow Tf \le Tg$.

<u>proof:</u> Let ~~$\varphi: [0,\infty) \to [0,1)$ be~~ $T: L_p \to L_p[0,\infty)$ be defined by $Tf = g$ where

$g(n+t) = 2^{\frac{n+1}{p}} f\left(1 - \frac{1}{2^n} + \frac{t}{2^{n+1}}\right)$ note $\int_0^1 |g(n+t)|^p\, dt$

$n \in N$, $t \in [0,1)$

$= \int_0^{\frac{1}{2^{n+1}}} \left|2^{\frac{n+1}{p}} f\left(1 - \frac{1}{2^n} + t\right)\right|^p dt$

It follows that T is an order isometry. Note that T is different for different p. Also if $p=\infty$, we have to decide what $2^{n+1/\infty}$ means, clearly in this case we want $2^{n+1/\infty}=2^0=1$. Such obvious conventions will often not be explained.

The order isometry between L_p and $L_p(\mathbb{R})$ can be given similarly. Let us consider a more general construction. Let $\{X_\alpha\}_{\alpha\in\Gamma}$ be a collection of normed spaces. Let us define $\oplus_p X_\alpha$ to be the collection of functions $f:\Gamma\to\cup X_\alpha$ with $f(\alpha)\in X_\alpha$ so that $\|f\|<\infty$ where.

$$\|f\| = \Big(\sum_{\alpha\in\Gamma}\|f(\alpha)\|^p\Big)^{1/p} \qquad 0<p<\infty$$

$$\|f\| = \sup_{\alpha\in\Gamma}\|f(\alpha)\| \qquad p=\infty$$

If each X_α is ordered, define $f\le g$ in $\oplus_p X_\alpha$ iff $f(\alpha)\le g(\alpha)$ for each α.

Clearly L_p is order isometric with $L_p\oplus_p L_p$ which is order isometric with $L_p[0,\infty)\oplus_p L_p[0,\infty)$ which is order isometric with $L_p(\mathbb{R})$. Also we have L_p order isometric with $L_p[0,\infty)$ which is order isometric to $\oplus_p X_n$ with each $X_n\equiv L_p$. We can write this last space as $\oplus_p(\Sigma_n L_p)$ or $(L_p\oplus L_p\oplus\ldots)_p$

Other $L_p(\mu)$ spaces of interest

l_p : $(\mathbb{N}, P(\mathbb{N}),$ counting meas$)$
l_p^n : $(\{1,2,\ldots n\}=X, P(X),$ counting meas$)$
L_p^n : $(\{1,2,\ldots n\}=X, P(X),$ ~~equ~~ uniform probability measure.
$l_p(X)$: $(X, \Sigma$ where count & co-count sets $<\Sigma<P(X)$, counting measure$)$
note that $l_\infty(X)$ dependents on the σ-algebra Σ

L_p^n is order isometric to l_p^n which is order isometric to a norm one complemented subspace of l_p which is order isometric to a norm one complemented subspace of $L_p[0,\infty)$ hence L_p.

$l_p(X)$ is not since it is not separable and L_p is separable. One more exotic $L_p(\mu)$ space.

Pg. 4

A Classification of properties about L_p - spaces

1 The weakest : Isomorphic Properties — invariant under isomorphisms [Two Banach spaces X and Y are isomorphic if there are maps $T: X \to Y$, $S: Y \to X$ which are linear and $TS = 1_Y$, $ST = 1_X$ and $K = \|S\|\|T\| < \infty$. K is sometimes called the isomorphism constant.]. Topological properties are of this type, for example continuity.

2. Isometric Properties — invariant under isometries [Two isomorphic Banach spaces X and Y are isometric if T, S can be picked so that $\|T\| = \|S\| = 1$]. Geometric properties are of this type, for examples angles in Hilbert spaces]

A. Order Properties — invariant under order preserving ~~maps~~ bi-jections (usually combined with 1 or 2). If X, Y are partially ordered, ~~they are order~~ $T: X \to Y$ is order preserving if $x \leq y \iff Tx \leq Ty$ (Bi-jections means 1-1 & onto). These are the order properties for example f and g having disjoint supports

3. Representational properties — those which depend on which (X, Σ, μ) you are looking at. For example $L_1 \supset L_2$ but $L_1(\mathbb{R}) \not\supset L_2(\mathbb{R})$ even though L_p is order isometric to $L_p(\mathbb{R})$

Implications

$$\begin{array}{ccccc} 3 & \Rightarrow & 2A & \Rightarrow & 2 \\ & & \Downarrow & & \Downarrow \\ & & 1A & \Rightarrow & 1 . \end{array}$$

From Tuesday's lecture:

$X \oplus_p Y$ is defined to be $\{(x,y) \in X \times Y\}$ with "norm"
$\|(x,y)\|_p = (\|x\|^p \oplus \|y\|^p)^{1/p}$ where $0 < p < \infty$

Note that if $B, A \in \Sigma$ in (X, Σ, μ) with $\mu(A \cap B) = 0$ and $\mu(X \setminus (A \cup B)) = 0$ then $L_p(\mu) = L_p(\mu|_A) \oplus_p L_p(\mu|B)$.

since $\int_X |f|^p d\mu = \int_{X \setminus (A \cup B)} |f|^p d\mu - \int_{A \cap B} |f|^p d\mu + \int_A |f|^p d\mu + \int_B |f|^p d\mu$

$$= \int_A |f|^p d\mu + \int_B |f|^p d\mu.$$

thus $\|f\|^p_{L_p(\mu)} = \|f|_A\|^p_{L_p(\mu|A)} + \|f|_B\|^p_{L_p(\mu|B)}$

For $p = \infty$, $X \oplus_\infty Y = \{(x,y) \in X \times Y\}$ with $\|(x,y)\| = \max \|x\|, \|y\|$.

Now it is easy to see $L_p \oplus_p L_p \equiv L_p$ $\quad L_p([0,\infty)) \oplus_p L_p([0,\infty)) \equiv L_p(\mathbb{R})$
where "$\equiv$" means "order isometric"

Let $\{X_\alpha\}_{\alpha \in \Gamma}$ be a collection of spaces. If $0 < p < \infty$
$\oplus_{\ell_p} (X_\alpha)_{\alpha \in \Gamma}$ is the collection of functions $f: \Gamma \to \cup X_\alpha$
so that $f(\alpha) \in X_\alpha$ and

$$\|f\|_p = \left(\Sigma_\alpha \|f(\alpha)\|^p\right)^{1/p} < \infty$$

If $p = \infty$ $\quad \|f\|_\infty = \sup_\alpha \|f(\alpha)\|_\infty$

If Γ is countably infinite $\oplus_{\ell_p} (X_n)_{n=1}^\infty$ is also written
$(X_1 \oplus X_2 \oplus \ldots)_{\ell_p}$ or $\oplus_{\ell_p} \Sigma_n X_n$ or $(X_1 \oplus X_2 \oplus \ldots)_p$

Note if $\{A_n\} \subset \Sigma$, $\{A_n\}$ pairwise disjoint, $X = \cup_n A_n$,
(X, Σ, μ_n) - measure space $\mu_n = \mu|A_n$. then
$L_p(\mu) = (L_p(\mu_1) \oplus L_p(\mu_2) \oplus \ldots)_{\ell_p}$

proof: if $f \in L_p(\mu)$ then $\int_X |f|^p d\mu \geq \int_{\cup_1^N A_n} |f|^p d\mu = \Sigma_1^N \int_{A_n} |f|^p d\mu$
$= \Sigma_1^N \int_{A_n} |f_n|^p \chi_{A_n} d\mu_n$. Hence $\|f\|_{L_p(\mu)} \geq \left(\Sigma_1^\infty \|f|_{A_n}\|_{L_p(\mu_n)}\right)^{1/p}$

Conversely, $\Sigma_1^N |f|^p \chi_{A_n}$ increases pt-wise to $|f|^p$ so by the monotone convergence theorm the norms are equal.

It follows that $L_p \equiv (L_p \oplus L_p \oplus \ldots)_p$, $L_p([0,\infty)) \equiv (L_p \oplus L_p \oplus \ldots)_p$
hence $L_p \equiv L_p([0,\infty)) \equiv L_p(\mathbb{R})$. So that
$\mu(X)$ finite or infinite is not preserved by order isometries.

Example: A non-separable $L_p(\mu)$ with $\mu(X) < \infty$.
Reminber a $L_p(\mu)$ is separable if there is a sequence $\{f_n\} \subset L_p(\mu)$ which is dense in $L_p(\mu)$ [i.e. $\forall g \in L_p(\mu)$ $\forall \varepsilon > 0$ $\exists n$ s.t. $\|f_n - g\|_p < \varepsilon$].

For $0 < p < \infty$, ℓ_p^n, L_p^n, ℓ_p, L_p, $L_p(\mathbb{R}^n)$ and ℓ_∞^n, L_∞^n are separable where as ℓ_∞ and L_∞ are non-separable. There are several ways to see L_p is separable, the usual way is to take the countable collection of step functions with rational values and rational "jump" points. Another proof can be given by starting with the fact that the continuous functions are dense in L_p.

The usual way of showing that something is non-separable is by producing an uncountable set $\{x_\alpha : \alpha \in \Gamma\}$ such that $\|x_\alpha\| \leq B$ some B and if $\alpha \neq \beta$ $\|x_\alpha - x_\beta\| \geq \varepsilon > 0$ some ε.

Exercise: Show this is equivalent to non-separability in norm spaces. [Hint: use zorn's lemma].

Exercise: Show L_∞ and ℓ_∞ are non-separable.

Back to the example. Let $X = 2^\Gamma$ where Γ is uncountbl. (i.e. the product of Γ-copies of the space $\{0,1\}$ with the discrete topology). Let Σ = Borel sets (i.e the smallest σ-algebra containing the open sets). The measure μ will be the extension of μ_0 defined on the semi-algebra S' where $A \in S' \iff A = \emptyset$ or $A = X$ or $\exists \alpha_1, \ldots, \alpha_n \in \Gamma$ $\exists a_1, \ldots a_n \in \{0,1\}$ s.t.

$A = \{f : \Gamma \to \{0,1\} \mid f(\alpha_i) = a_i \quad i = 1, \ldots, n\}$ in which case $\mu_0(A)$ is defined to be $1/2^n$. [$\mu(X) = 1$, $\mu(\emptyset)$]

Claim S' is a semi-algebra: The intersection of two such sets is $\emptyset$ or such a set. The complement of such a set is a finite union of such sets (For example the complement of A above is $\bigcup_{m=1}^n B_m$ where $B_m = \{f : \Gamma \to \{0,1\} \mid f(\alpha_m) = b_m\}$ where $b_m = \begin{cases} 0 & \text{if } a_m = 1 \\ 1 & \text{if } a_m = 0 \end{cases}$.)

Claim μ_0 is countably additive on S'. Certainly $\mu(\emptyset) = 0$ if $A, B, C \in S'$ and $A \cap B = \emptyset$ and $C = A \cup B$ then we can chose $\alpha_1, \alpha_2, \ldots \alpha_n$ $a_2 \ldots a_n$ s.t.

$C = \{f \in 2^\Gamma \mid f(\alpha_i) = a_i \quad 2 \le i \le n\}$
$A = C \cap \{f \in 2^\Gamma \mid f(\alpha_1) = 0\}$
$B = C \cap \{f \in 2^\Gamma \mid f(\alpha_1) = 1\}$
and $\mu_0(A) = \mu_0(B) = 1/2^n$ while $\mu_0(C) = 1/2^{n-1}$
an induction shows μ_0 is finitely additive.

Now let $\{A_n\} \subset S'$, $\{A_n\}$ p.w. disjoint with $\cup A_n \in S'$
We need to show for all such sequences $\mu_0(\cup A_n) = \Sigma \mu(A_n)$
For then, by Royden p. 260 $(2^\Gamma, \Sigma', \mu)$ is a measure space
We show that this is vacously true, there are no such sequences, $\{A_n\}$.
Let $A_n = \{f \in 2^\Gamma \mid f(\alpha_i^n) = a_i^n \quad 1 \le i < m(n)\}$.
$A_n \cap A_m = \emptyset \iff \exists\, \alpha_i^n = \alpha_j^m$ with $a_i^n \ne a_j^m$.
Suppose $\cup A_n = B = \{f \in 2^\Gamma \mid f(\alpha_i^B) = b_i^n \quad 1 \le i < m(B)\} \in S'$
Define $g \in B$ inductively (by diagonalizing) making $g(\alpha_i^B) = b_i^n$ but $g \notin A_n$ each n. This is done by showing that A_n can be re-ordered so that $m(n)$ is strictly increasing [Suppose A_k and A_n have same $m(k) = m(n)$ ~~since~~ $\cancel{\{\alpha_i^B\}_1^{m(B)} \subset \{\alpha_i^n\}_1^{m_0}}$ and $m(k) = \min_{n \ge 1} m(n)$, then $m(k)$ must be $m(B)+1$ and $\{\alpha_i^k\} = \{\alpha_i^n\}$ hence $B = A_k \cup A_n$ and all the rest our void]

To see that this $L_p(\mu)$ is non-separable, for each $\alpha \in \Gamma$ let $A_\alpha = \{f \in 2^\Gamma : f(\alpha) = 1\}$ note that $\mu(A_\alpha) = 1/2$ & if $\alpha \ne \beta$ $\mu(A_\alpha \cap A_\beta) = \mu((X \setminus A_\alpha) \cap A_\beta) = 1/4$. Let $g_\alpha = \chi_{A_\alpha} \in L_p(\mu)$

$$\|g_\alpha\|_p = \left(\int |\chi_{A_\alpha}|^p\right)^{1/p} = \left(\int_{A_\alpha} d\mu\right)^{1/p} = (1/2)^{1/p} \le 1,$$

and if $\alpha \ne \beta$

$$\|g_\alpha - g_\beta\|_p^p = \int |\chi_{A_n} - \chi_{B_n}|^p = \int \chi_{A_n \setminus B_n} d\mu + \int \chi_{B_n \setminus A_n}$$
$$= 1/4 + 1/4 = 1/2 \qquad \text{hence}$$
$$\|g_\alpha - g_\beta\|_p \ge (1/2)^{1/p} > 0.$$

An easier way of doing this is to ~~let~~ let $\{g_\alpha\}$ be uncountably many independent coin-tosses. and let (X, Σ, μ) be the proporbility space they live on.

This example can also be done on the product space I^Γ, $[0,1] = I$, Γ uncountable with $\mu =$ to the product measure, the details are a little harder.

§2. Hölder's Inequality. This is the most important inequalities for us. If $1 \le p \le \infty$, the conjugate exponent of p is p', where $\frac{1}{p} + \frac{1}{p'} = 1$ with the convention $\frac{1}{\infty} = 0$. If f, g are measurable functions on X, then

$$(*) \qquad \int |fg|\, d\mu \le \|f\|_p \|g\|_{p'}$$

with equality holding if and only if $\int |fg|\, d\mu = \infty$; or if $\|f\|_p \|g\|_{p'} < \infty$, and $\{|f|^p, |g|^{p'}\}$ is linearly dependent ($1 < p, p' < \infty$) or if $|g| = \|g\|_\infty$ a.e. on support of f ($p = 1$, $p' = \infty$)

Remarks Note that $(*)$ makes no assumption on the finiteness of any of the numbers. Also we are accepting the usual convention in measure theory that $0 \cdot \infty = 0$.

Proof First some special cases

Case I Either $\|f\|_p$ or $\|g\|_{p'} = 0$. This implies either $f = 0$ a.e. or $g = 0$ a.e. In either case $fg = 0$ a.e. hence both sides of $(*)$ are zero.

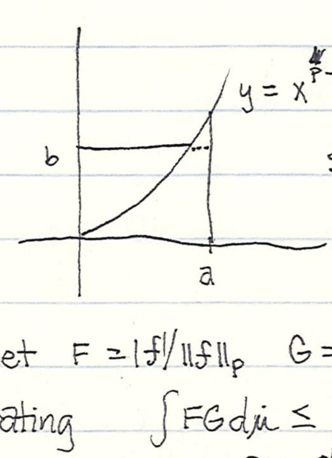
Case II ~~Both~~ When $\|f\|_p \cdot \|g\|_{q'} = \infty$. $(*)$ is obviously true. When does $\int |fg| d\mu = \infty$?

Case III $\{p, p'\} = \{1, \infty\}$. Then $|fg| \le |f|\, \|g\|_\infty$ or $\|f\|_\infty |g|$. Integrating over X we obtain $(*)$. Suppose the right side of $(*)$ is finite and equality holds and $p = 1$. We have $|f|\,\|g\|_\infty - |fg| \le |f|\,\|g\|_\infty$ and $0 = \int |f|\,\|g\|_\infty - |f||g|\, d\mu = \int |f|(\|g\|_\infty - |g|)\, d\mu$ hence $|g| = \|g\|_\infty$ a.e. on the support of f.

Case IV All others. Hence $1 < p, p' < \infty$ and $0 < \|f\|_p, \|g\|_{p'} < \infty$

Lemma If $a, b \ge 0$ $\quad ab \le \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$ with equality holding if and only if $a^p = b^{p'}$

proof An exercise using calculus or consider the picture



$y = x^{p-1}$ or $x = y^{p'-1}$

Since $(p-1)(p'-1) = pp' - p - p' + 1 = 1$ and compute areas. Equality hold if and only if $a^{p-1} = b$ or $a^p = ab = a^p = b^{p'-1} b = b^{p'}$

Now let $F = |f|/\|f\|_p$ $\quad G = |g|/\|g\|_p$. By the lemma $F(x)G(x) \le \frac{1}{p} F^p(x) + \frac{1}{p'} G^{p'}(x)$

integrating $\int FG\, d\mu \le \frac{1}{p} \|F\|_p^p + \frac{1}{p'} \|G\|_{p'}^{p'} = \frac{1}{p} + \frac{1}{p'} = 1 = \|F\|_p \|G\|_{p'}$

with equality iff $F^p = G^{p'}$. Hence $\int |fg|\, d\mu = \|f\|_p \|g\|_{p'} \int FG \le \|f\|_p \|g\|_{p'}$ (1)

with equality iff $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^{p'}}{\|g\|_{p'}^{p'}}$ iff $\{|f|^p, |g|^{p'}\}$ is linearity dependente.

Duality

If $\mathfrak{X}$ is a vector space with a topology making vector addition

Some consequences of Hölders inequality. If f is a measurable function, $\int f \le |\int f| \le \int |f|$. Of course $\int f$ & $|\int f|$ may not be defined, but this can only happen if $\int |f| = \infty$. Thus if $f \in L_p$, $g \in L_{p'}$ we have

II

$$(\$) \quad |\int fg| \le \|f\|_p \|g\|_{p'}$$

and

$$(\divideontimes) \quad \int fg \le \|f\|_p \|g\|_{p'}$$

Let us analyze when equality occurs in $(\$)$ and $(\divideontimes)$. Let h be measurable with $\int |h| < \infty$. Now $\int h = \int \mathrm{Re}\, h + i \int \mathrm{Im}\, h$, hence $\int h$ is real iff $\int \mathrm{Im}\, h = 0$. Futhermore, if $\mathrm{Im}\, h \ne 0$ a.e. Then $|h| - \mathrm{Re}\, h \overset{>}{\ne} 0$ a.e. hence $\int h \ne \int |h|$. Thus if $f \in L_p$, $g \in L_{p'}$

$1 < p, p' < \infty$; $=$ in $(\divideontimes) \iff \{|f|^p, |g|^{p'}\}$ is linearly dependent a.e.

$=$ in $(\$) \iff$ above and $\exists \theta$ $e^{i\theta} fg = |fg|$ a.e.

$=$ in $(\divideontimes) \iff$ above and $fg \ge 0$ a.e.

$p = 1$, $p' = \infty$; $=$ in $(\divideontimes) \iff |g| = \|g\|_\infty$ a.e. on support f.

$=$ in $(\$) \iff$ above and $\exists \theta$ $e^{i\theta} fg = |fg|$ a.e.

$=$ in $(\divideontimes) \iff$ above and $fg \ge 0$ a.e.

generalized Hölder's II

Suppose $0 < r \le \infty$, $r \le p, q \le \infty$, $\frac{1}{r} \le \frac{1}{p} + \frac{1}{q}$, and f, g are measurable then $\|fg\|_r \le \|f\|_p \|g\|_q$.

proof: If $q = \infty$ then $r = p$ and $\begin{cases} r=\infty & |fg| \le \|f\|_\infty \|g\|_\infty \text{ a.e.} \\ r<\infty & |fg|^r \le |f|^p \|g\|_\infty^r \end{cases}$. Integrate & take r^{th} roots

If $p, q < \infty$ then $r < \infty$. Then p/r, q/r are conjugate exp. and

$$\| |fg|^r \|_1 \le \| |f|^r \|_{p/r} \| |g|^r \|_{q/r} = \|f\|_p^r \|g\|_q^r, \text{ take } r^{th} \text{ roots.}$$

I

Suppose $1 \le p \le \infty$, f, g are measurable then $\|f+g\|_p \le \|f\|_p + \|g\|_p$ (Minkowski's inequality) with equality when, $1 < p < \infty$ & the right side is finite iff $\exists \lambda \ge 0$ with either $f = \lambda g$ or $g = \lambda f$

proof Case I Either $\|f\|_p$ or $\|g\|_p = 0$, the $\le$ becomes an equality

Case II Either $\|f\|_p$ or $\|g\|_p = \infty$, the inequality is obviously true with equality when $\|f+g\|_p = \infty$, in particular when the other of $\|f\|_p, \|g\|_p$ is finite

Case III: $p = \infty$. This is easy. Equality iff $\mu(\{x: |f(x)| \ge \|f\|_\infty\} \cap \{x: |g(x)| \ge \|g\|_\infty\}) > 0$ when the right side is finite.

Case IV; $p = 1$: $|f+g| \le |f| + |g|$, integrate, $\|f+g\|_1 \le \int |f| + |g| = \|f\|_1 + \|g\|_1$ equality iff $|f+g| = |f| + |g|$ a.e. iff $f\bar{g} \ge 0$ a.e.

LEMMA If $\mathfrak{X}$ is a normed space, then a linear functional F on $\mathfrak{X}$ is continuous if and only if F is bounded, that is

$$\|F\| = \sup\{|F(x)| : \|x\| \le 1\} = \sup\{|F(x)| : \|x\| = 1\} = \sup\{|F(x)|/\|x\| : x \ne 0\}$$
$$= \inf\{M : \text{s.t. } \forall x \ |F(x)| \le M\|x\|\} < \infty$$

Also we note $\|g\|_q = \|F_g\|$. If $1<p<\infty$ let $f = \operatorname{sgn}(g)\,|g|^{q/p}$ then $\|f\|_p = \|g\|_q$ and $F_g(f) = \|f\|_p \|g\|_q = \|g\|_q^2$, hence $M \ge \|g\|_q$ the reverse comes from Hölder's

If $1 \le p < \infty$, we do the following cases

μ finite: If $F \in [L_p(\mu)]'$ $\nu(E) = F(\chi_E)$ is a measure on $(\mathfrak{X}, \Sigma)$ and $\nu \ll \mu$ $\exists\, g$ s.t. $F(f) = \int f\,d\nu = \int fg\,d\mu$. Hence $F_g = F$.

μ σ-finite: Partition $\mathfrak{X} = \bigcup E_n$ with $\mu(E_n) < \infty$ and piece together g.

μ general $1<p<\infty$. For each $A \subset \mathfrak{X}$ with $\mu(A) < \infty$ we obtain g_A which works for (A, Σ_A, μ). Since if $\{A_n\}$ are p.w. disjoint it follows $\|g_{\cup A_n}\| = \|\Sigma g_{A_n}\| = (\Sigma \|g_{A_n}\|^q)^{1/q} \le \|F\|$. It follows that there is a σ-finite set $Y \subset \mathfrak{X}$ that $A \subset \mathfrak{X} \setminus Y \Rightarrow g_A = 0$ a.e. Thus $g \in L_q(\mu)$

$p=\infty$, μ finite, $L_\infty(\mu)$ finite dimensional. Then $L_1(\mu)$ has the same finite dimension hence $L_1(\mu) = [L_\infty(\mu)]'$

The examples:

(1) Let Γ be uncountable $\Sigma = \mathcal{P}(\Gamma)$, $\Omega = \{A \text{ or } \Gamma \setminus A \text{ is countable}\}$, μ the counting measure

Then $l_1(\Gamma, \Sigma, \mu) \equiv l_1(\Gamma, \Omega, \mu)$ but $l_\infty(\Gamma, \Sigma, \mu) \supsetneq l_\infty(\Gamma, \Omega, \mu)$ and $[l_1(\Gamma)]' = l_\infty(\Gamma, \Sigma, \mu)$.

(2) $\mathfrak{X} = \{1,2\}$, $\Sigma = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ $\mu(\{1\}) = 1$, $\mu(\{2\}) = \infty$ then $L_1(\mu)$ is 1-dimensional, $L_\infty(\mu)$ is 2-dimensional.

The examples for $0<p<1$ will be done latter.

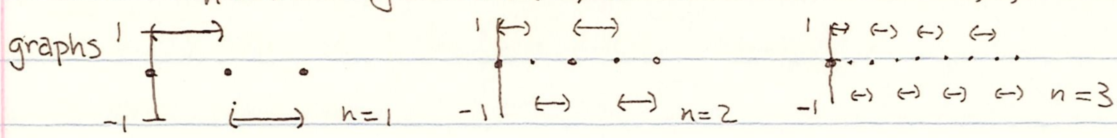
For more examples see the exercises in Royden in Chapter 11.

The Rademacher functions: $r_n(t) : [0,1] \to 1 \quad n = 1, 2, \ldots$

If x is a real number define $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

then $\quad r_n(t) = \operatorname{sgn} \sin (2^n \pi t) \quad n = 1, 2, \ldots \quad 0 \le t \le 1$

graphs



etc.

$\{r_n\}_{n=1}^{\infty}$ is a example of an orthogonal sequence functions, that is $\int_0^1 r_n(t) r_m(t)\, dt = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \ne m \end{cases}$. In fact something more general is true

LEMMA Let $n(1) < n(2) < \ldots < n(m)$ and $k(1), k(2), \ldots, k(m)$ be positive integers, then

$$\int_0^1 [r_{n(1)}(t)]^{k(1)} [r_{n(2)}(t)]^{k(2)} \ldots [r_{n(m)}(t)]^{k(m)}\, dt = \begin{cases} 1 & \text{if all } k(i) \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

proof: If k is even, $[r_n(t)]^k = 1$ a.e. hence it suffices to show that if $n(1) < n(2) < \ldots < n(m)$ then

$$\int_0^1 r_{n(1)} r_{n(2)} \ldots r_{n(m)}\, dt = 0.$$

Let $1 \le i \le 2^{n(m-1)}$ on the interval $[(i-1)/2^{n(m-1)}, i/2^{n(m-1)}]$ the product $r_{n(1)} r_{n(2)} \ldots r_{n(m-1)} = 1$ or -1 a.e. hence

$$\int_{(i-1)/2^{n(m-1)}}^{i/2^{n(m-1)}} r_{n(1)} r_{n(2)} \ldots r_{n(m-1)} r_{n(m)}\, dt = \pm \int_{(i-1)/2^{n(m-1)}}^{i/2^{n(m-1)}} r_{n(m)}(t)\, dt = 0$$

since $1/2^{n(m-1)}$ is several times the period of $\sin(2^{n(m)} \pi t)$. Adding from $i = 1$ to $2^{n(m-1)}$ we obtain the result.

Theorem Let $1 \le p < \infty$, then ~~for all scalars~~ there are constants A_p, B_p so that for all sequences of scalars $\{\alpha_n\}_{n=1}^{\infty}$

$$A_p \left(\sum_n |\alpha_n|^2\right)^{1/2} \le \left(\int_0^1 \left(\sum_{n=1}^{\infty} \alpha_n r_n(t)\right)^p dt\right)^{1/p} \le B_p \left(\sum_n |\alpha_n|^2\right)^{1/2}$$

That is, if we restrict the L_p-norm to the closed linear span of the Rademacher functions is isomorphic to the Hilbert space ℓ_2.

proof: If $p = 2$: The sequence $\{r_n(t)\}_{n=1}^{\infty}$ is an orthonormal sequence with respect to the inner product $\langle f, g \rangle = \int f \bar{g}\, dt$. It follows that we can take $A_2 = B_2 = 1$ from the following general

Lemma: If $\{e_n\}$ is an orthonormal sequence in a Hilbert space Then $\| \Sigma \alpha_n e_n \| = (\Sigma |\alpha_n|^2)^{1/2}$.

First half: Since we are working on $[0,1]$, if $1 \le p \le 2 \le q \le \infty$

$$\|f\|_p \le \|f\|_2 \le \|f\|_q \quad \text{hence}$$

for $1 \le p \le 2$

$$\| \Sigma_1' \alpha_n r_n \|_p \le \| \Sigma \alpha_n r_n \|_2 = (\Sigma_1' |\alpha_n|^2)^{1/2} \quad (\text{i.e. } B_p = 1)$$

and for $2 \le q \le \infty$

$$\| \Sigma \alpha_n r_n \|_q \ge \| \Sigma \alpha_n r_n \|_2 = (\Sigma |\alpha_n|^2)^{1/2} \quad (\text{i.e. } A_q = 1)$$

Second half: we will do the special case $q = 2k$ k an integer ≥ 1

$$\int_0^1 |\Sigma_1^m \alpha_n r_n(t)|^{2k} dt = \int_0^1 [(\Sigma_1^m \alpha_n r_n(t))(\Sigma_1^m \bar{\alpha}_n r_n(t))]^k dt$$

Since $(\Sigma_1^m \alpha_n r_n(t))^k (\Sigma_1^m \bar{\alpha}_n r_n(t))^k = \sum_{\substack{i(1)+i(2)+\dots i(m)=k \\ j(1)+j(2)+\dots j(m)=k}} \left(\prod_{\ell=1}^m \alpha_\ell^{i(\ell)} \bar{\alpha}_\ell^{j(\ell)} r_\ell(t)^{i(\ell)+j(\ell)} \right)$

integrating this last sum term by term, by the lemma we must have $i(\ell)+j(\ell)$ must be even for $1 \le \ell \le m$ to get a non-zero contribution, and if $i(\ell)+j(\ell)$ is even for $1 \le \ell \le m$ then

$$\prod_{\ell=1}^m \alpha_\ell^{i(\ell)} \bar{\alpha}_\ell^{j(\ell)} r_\ell(t)^{i(\ell)+j(\ell)} = \prod_{\ell=1}^m \alpha_\ell^{i(\ell)} \bar{\alpha}_\ell^{j(\ell)} \quad \text{a.e.}$$

~~Consider the terms with all $\alpha_1^{i(1)} \bar{\alpha}_1^{j(1)} \dots \alpha_m^{i(m)} \bar{\alpha}_m^{j(m)}$ fixed but the ℓ-th ones, but $i(\ell)+j(\ell) = 2n$ for fix n. There are equal numbers of~~ Thus by the triangle inequality

$$\int_0^1 |\Sigma_1^m \alpha_n r_n(t)|^{2k} dt \le \int_0^1 \sum_{\substack{i(1)+i(2)+\dots+i(m)=k \\ j(1)+j(2)+\dots+j(m)=k \\ \& \ i(\ell)+j(\ell) \text{ is even}}}' \prod_{\ell=1}^m |\alpha_\ell|^{i(\ell)+j(\ell)} dt$$

On the otherhand, $(\Sigma_1^m |\alpha_n|^2)^k = \sum_{i(1)+i(2)+\dots i(m)=k} \prod_{\ell=1}^m |\alpha_\ell|^{2i(\ell)}$

these look somewhat similar, we need to how many times each particular term is chosen in each product.

$$|a_1|^{i(1)+j(1)} |a_2|^{i(2)+j(2)} \dots |a_m|^{i(m)+j(m)} \quad \text{occurs} \quad \frac{(2k)!}{(i(1)+j(1))!\,(i(2)+j(2))! \dots (i(m)+j(m))!}$$

$$|a_1|^{2i(1)} |a_2|^{2i(2)} \dots |a_m|^{2i(m)} \quad \text{occurs} \quad \frac{k!}{i(1)!\, i(2)! \dots i(m)!}$$

(number of ways of combinations with repeatitions)

The expression $(a+b+c+\dots+z)^k$ as $a^{k_1}b^{k_2}\dots z^{k_m}$ as term $k_1+k_2+\dots k_m=k$ how many times? we must chose k_1 integers from $\{1,\dots k\}$ to be a's then chose k_2 integers from the remaining $k-k_1$ integers to be b's and so on til we get to the z's which are those remaining hence the number ways it is a term

$$\frac{k!}{k_1!\,(k-k_1)!}\;\frac{(k-k_1)!}{k_2!\,(k-k_1-k_2)!}\;\cdots\;\frac{(k-k_1-\dots-k_{m-1})!}{(k_{m-1})!\;k_m!}$$

$$= \frac{k!}{k_1!\,k_2!\cdots k_m!}$$

Since $\frac{(2k)!}{k!} \le 2^k k^k$ and $\frac{i!}{(2i)!} \ge \frac{1}{2^i}$ we have

$$\sup_{k_1+k_2+\dots k_m=k} \frac{(2k)!}{(2k_1)!\,(2k_2)!\cdots(2k_m)!}\;\frac{k_1!\,k_2!\cdots k_m!}{k!} \le \frac{2^k k^k}{2^{k_1}2^{k_2}\cdots 2^{k_m}} = k^k$$

Hence

$$\int_0^1 \Big|\sum \alpha_n r_n(t)\Big|^{2k} dt \le k^k \int_0^1 \Big(\sum |\alpha_n|^2\Big)^k dt = k^k\Big(\sum|\alpha_n|^2\Big)^k$$

and

$$\Big\|\sum \alpha_n r_n(t)\Big\|_{2k} \le \sqrt{k}\,\Big(\sum|\alpha_n|^2\Big)^{1/2} \quad \text{so} \quad B_{2k} \le \sqrt{k}$$

Now if $2 \le q \le \infty$ Let $2k$ be a even integer $\ge q$

$$\Big\|\sum \alpha_n r_n(t)\Big\|_q \le \Big\|\sum \alpha_n r_n(t)\Big\|_{2k} \le \sqrt{k}\,\Big(\sum|\alpha_n|^2\Big)^{1/2}$$

This completes the theorem for $2 \le q < \infty$.

Now for $p=1$, Let $\sum_1^m \alpha_n r_n(t) = f(t)$. Since $\frac{1}{3}+\frac{1}{3/2}=1$ by Hölder's

$$\int_0^1 |f(t)|^2 dt = \int |f|^{2/3}|f|^{4/3} \le \Big(\int |f(t)|^{\frac{2}{3}\frac{3}{2}}\Big)^{2/3}\Big(\int |f|^{\frac{4}{3}3}\Big)^{1/3}$$
$$= \Big(\int|f|\Big)^{2/3}\Big[\Big(\int |f|^4\Big)^{1/4}\Big]^{4/3} \le \Big(\int|f|\Big)^{2/3}\Big(\sqrt{2}\Big(\int|f|^2\Big)^{1/2}\Big)^{4/3}$$

$$\|f\|_2^2 \le \|f\|_1^{2/3}\, 2^{2/3}\, \|f\|_2^{4/3} \quad \text{or}$$

$$\|f\|_2^{2/3} \le 2^{2/3}\|f\|_1^{2/3}$$

$$\frac{1}{2}\|f\|_2 \le \|f\|_1$$

Thus $A_p = \frac{1}{2}$ works for $1 \le p \le 2$. This completes the proof of The Theorem

If X is a Banach space & $Y \subseteq X$ is a closed subspace Y is said to be <u>complemented</u> in X if there is a projection $P: X \to X$ (i.e. $P^2 = P \circ P = P$) so that range $P = Y$ and $\|P\| = \sup\{\|Px\| : \|x\| \le 1\} < \infty$.

Each closed subspace of a Hilbert space is complemented but every other ∞-dimensional Banach space has a subspace which is <u>NOT</u> complemented.

<u>Theorem</u> For $1 < p < \infty$, the closed linear span of the Rademacher functions are complemented in L_p.

<u>proof</u>: This follows from a more general principle

<u>Proposition</u>: If X is a Banach space with dual X' and suppose $\{x_n\} \subset X$ & $\{x_n'\} \subset X'$ so that $x_j'(x_i) = \delta_{ij}$ [such a sequence is called biorthogonal]. Suppose there are constants A & B so that

(1) $\quad \|\sum_1 \alpha_n x_n\| \le A\left(\sum_1 |\alpha_n|^2\right)^{1/2}$

(2) $\quad \|\sum \beta_n x_n'\| \le B\left(\sum |\beta_n|^2\right)^{1/2}$

for each sequence of scalars $\{\alpha_i\}$, $\{\beta_i\}$ (which are finitely non-zero) then $P: X \to X$ defined by $P(x) = \sum_n x_n'(x) x_n$ is a projection onto the closed linear span of $\{x_n\}$, and $\|P\| \le B^{-1}A$.

<u>proof</u>: Let $x \in X$ with $\|x\| \le 1$, The first claim is that the sequence $\{x_n'(x)\}$ belongs to ℓ_2. Suppose not, then $\left(\sum_1 |x_n'(x)|^2\right)^{1/2} = \infty$. By the equality part of Hölder's inequality for ℓ_2, there is a sequence $\{\beta_n\}_{n=1}^N$ so that $\left(\sum_1^N |\beta_n|^2\right)^{1/2} = 1$ and

$$\sum_1^N \beta_n x_n'(x) = \|\{\beta_n\}\|_2 \, \|\{x_n'(x)\}_{n=1}^N\|_2 = \left(\sum_1^N |x_n'(x)|^2\right)^{1/2}.$$

Furthermore, $\|\sum_1^N \beta_n x_n'\| \le B\left(\sum_1 |\beta_n|^2\right)^{1/2} = B$

Thus $\quad \sum_1^N \beta_n x_n'(x) = \left(\sum_1^N \beta_n x_n'\right)(x) \le \|\sum_1 \beta_n x_n'\| \, \|x\| \le B.$

$$\infty = \lim_N \left(\sum_1^N |x_n'(x)|^2\right)^{1/2} = \lim_N \left(\sum_1^N \beta_n x_n'(x)\right) \le B$$

a contraction.

Let $(\Sigma_1^N |\alpha_n|^2)^{1/2} = 1$ and chose $\{\beta_i\}_1^N$ so that $(\Sigma_1^N |\beta_n|^2)^{1/2} = 1$ and $\Sigma \alpha_i \beta_i = 1$, again we can do this by Hölders inequality. Now $(\Sigma_1^N \beta_i x_i')(\Sigma_1^N \alpha_i x_i) = \Sigma \alpha_i \beta_i = 1$, thus

$\|\Sigma_1^N \beta_i x_i'\| \, \|\Sigma \alpha_i x_i\| \geq 1$ and $\|\Sigma_i \alpha_i x_i\| \geq 1/\|\Sigma_1^N \beta_i x_i'\| \geq 1/B(\Sigma|\beta_n|^2)^{1/2} \geq 1/B$. Hence $\|\Sigma_1^N \alpha_i x_i\| \geq B^{-1} (\Sigma_1^N |\alpha_i|^2)^{1/2}$.

Also $(\Sigma_1^N \alpha_i x_i')(\Sigma_1^N \beta_i x_i) = \Sigma \alpha_i \beta_i = 1$ thus

$\|\Sigma_1^N \alpha_i x_i'\| \geq 1/\|\Sigma_1^N \beta_i x_i\| \geq 1/A(\Sigma|\beta_i|^2)^{1/2} \geq A^{-1}$.

Hence $\|\Sigma_1^N \alpha_i x_i'\| \geq A^{-1} (\Sigma_1 |\alpha_i|^2)^{1/2}$.

This show $\{x_i\}$ and $\{x_i'\}$ are "equivalent" to l_2's usual basis.

Claim: closed linear span $\{x_i\} = \{\Sigma \alpha_i x_i : \{\alpha_i\} \in l_2\}$. $\|\Sigma_N^M \alpha_i x_i\| \leq A(\Sigma_N^M |\alpha_i|^2)^{1/2} \to 0$ hence $\Sigma_1^N \alpha_i x_i$ is a Cauchy sequence and hence it converges. Thus each $\Sigma \alpha_i x_i$ with $\{\alpha_i\} \in l_2$ is in the closed linear span $\{x_i\}$. Let $y \in$ closed linear span $\{x_i\}$. Now by above $\{x_n'(y)\} \in l_2$ hence $(\Sigma_N^M |x_n'(y)|^2)^{1/2} \to 0$ and $\Sigma_1^N x_n'(y) x_n$ converges to some element z in the closed linear span $\{x_n\}$. Now since x_m' is continuous $x_m'(z) = \lim x_m'(\Sigma_1^N x_n'(y) x_n) = x_m'(y)$ and $x_m'(y-z) = 0$ for all m.

Suppose $w \neq 0$ ~~$w \in$ closed span $\{x_n\}$. Let $\|\Sigma_1^N \alpha_i x_i - w\| < \frac{1}{2B}\|w\|$. If $(\Sigma_1^N |\beta_i|^2)^{1/2} = 1$ s.t. $\Sigma_1 \beta_i \alpha_i = (\Sigma|\alpha_i|^2)^{1/2}$ $\Sigma \beta_i x_i' (\Sigma_1^N \alpha_i x_i - w)$ $\leq \|\Sigma \beta_i x_i'\| \, \|\Sigma_1^N \alpha_i x_i - w\| \leq B/_{2B} \|w\|$. So~~

Claim $y - z = 0$. Suppose not, let $w = y-z/\|y-z\|$ then $\|w\| = 1$ and $x_n'(w) = 0$ and $n = 1, 2, \ldots$. Let $\Sigma_1^N \alpha_i x_i$ so that $\|\Sigma_1^N \alpha_i x_i - w\| < 1/100AB$ so $\|\Sigma_1^N \alpha_i x_i\| \geq 1 - 1/100$ and $(\Sigma|\alpha_i|^2)^{1/2} \geq \frac{1}{A}(1 - \frac{1}{100AB})$.
If $(\Sigma_1^N |\beta_i|^2)^{1/2} = 1$, $\Sigma \alpha_i \beta_i = (\Sigma_1 |\alpha_i|^2)^{1/2}$ $\|\Sigma_1 \beta_i x_i'\| \leq B$
$\left(\frac{1}{A}(1 - \frac{1}{100AB}) - \Sigma \beta_i x_i'(w)\right) \leq \Sigma_1 \beta_i x_i' (\Sigma_1 \alpha_i x_i - w) \leq B/100AB$ hence $x_i'(w) \neq 0$ for all i.
This mess completes both claims. There is an easier way by the open mapping theorem.

Next we show $P^2 = P$. If $x \in \mathfrak{X}$ $\{x_n'(x)\} \in l_2$ $\Sigma_1 x_n'(x) x_n \in$ closed linear span. Now by continity $x_m'(\Sigma x_n'(x) x_n) = x_m'(x) \cdot x_m'(x_m) = x_m'(x)$ hence $P^2 = P$ and range $P =$ closed linear span $\{x_n\}$.

Believe it or not the above is all standard stuff. Let us compute the norm of P. Let x so that $\|Px\| = \|\Sigma x_n'(x) x_n\| = 1$ hence $(\Sigma_i' |x_n'(x)|^2)^{1/2} \geq A^{-1}$

Chose $\{\beta_i\}$ $(\Sigma \|\beta_i|^2)^{1/2} = 1$ ~~$\Sigma \alpha_i \beta_i = (\Sigma_i' |$~~

$\Sigma \beta_i \{x_n'(x)\} = (\Sigma |x_n'(x)|^2)^{1/2}$ $\|\Sigma \beta_i x_i'\| \leq B$

Hence $\|x\|/B \geq \|\Sigma \beta_i x_i'\| \|x\| \geq (\Sigma \beta_i x_i')(x) = \Sigma \beta_i x_n'(x) = (\Sigma |x_n'(x)|^2)^{1/2}$

and $\|x\| \geq BA^{-1}$ so $\|x\| \geq BA^{-1} \|Px\|$

or $\|Px\| \leq B^{-1} A \|x\|$. Which completes the proof.

Proof: ~~Of~~ Theorem follows taking $\{r_n\} = \{x_n\} \subset L_p$ and $\{r_n\} = \{x_n'\} \subset L_q$ where $1/q + 1/p = 1$

$x_n'(x_m) = \int_0^1 r_n r_m \, dt = \delta_{nm}$.

The Rademacher's are not complemented in either L_1 or L_∞. Clearly the proposition does not apply. But I know no easy* proof of this. One must check all possible projections not just the one above. We will bypass this for the moment.

Possible Future Topics or Projects

(1) Type and Co-Type in general Banach Spaces

(2) Absolutely summing operators (to show ℓ_2 is not complemented in L_1)

(3) Gaussian Random Variables in L_p

(4) Every ℓ_2 isomorph in L_p $(p \geq 2)$ is complemented

(5) Negative definite functions and embeddings of L_p into L_r $2 \geq p \geq r \geq 1$

(6) When can you embed ℓ_p into L_q?

(7) $L_p \approx L_p \oplus \ell_2$, in fact, if X is complemented in L_p, $L_p \approx L_p \oplus X$

(8) Each complemented subspace of ℓ_p is isomorphic to ℓ_p

* nor any hard proof. Actually this is not a difficult result we just need some preliminaries that we will not use it rest of the quarter.

§3. Automatic Convergence

Let $\{f_n\} \subset L_p$ $f_n \to f$ in L_p-norm if $\|f_n - f\|_p \to 0$. This section ~~is~~ produces weaker conditions on $\{f_n\}$ and f which imply that $f_n \to f$ in the L_p-norm. First some conditions implied by $f_n \to f$ in norm.

Lemma If $\{f_n\}, f \subset L_p(\mu)$ and $f_n \to f$ in $L_p(\mu)$ norm, then
(1) $\|f_n\|_p \to \|f\|_p$
(2) $\forall g \in L_{p'}(\mu)$ $(1/p' + 1/p = 1)$ $\int g f_n \, d\mu \to \int g f \, d\mu$.
(3) $f_n \to f$ in measure, ~~if $p<\infty$~~

proof: (1) This is true for general Banach space since
$\|x\| \le \|x-y\| + \|y\|$ & $\|y\| \le \|x-y\| + \|x\|$ hence
$|\,\|x\| - \|y\|\,| \le \|x-y\|$. Thus $|\,\|f_n\| - \|f\|\,| \le \|f_n - f\| \to 0$.
and $\|f_n\|_p \to \|f\|$.
(2) This is also true for general Banach spaces X, if $x' \in X'$, then x' is continuous. So $f_n \to f$ implies $x'(f_n) \to x'(f)$.
(3) If $f_n \to f$ in norm then $g_n = f - f_n$ and $g_n \to 0$ in L_p-norm. Consider $A_\varepsilon = \{x : |g_n(x)| \ge \varepsilon\}$ then $\|g_n\|_p \ge \varepsilon[\mu(A_\varepsilon)]^{1/p}$ hence if $\|g_n\|_p \to 0$ $\mu\{x : |g_n(x)| \ge \varepsilon\} \to 0$ and $g_n \to 0$ in measure. So that $f_n \to f$ in measure.

Remarks. A. ~~Of course (3) is false for $L_\infty(\mu)$ if $f_n = \frac{1}{n}\chi_X$, where X is the whole space does not converge in measure but $\|f_n\|_\infty = 1/n \to 0$.~~ nonsense
B. (2) is called weak convergence (for $p<\infty$ or $p=1$ & μ is σ-finite) (when $p=\infty$ it is called weak-star convergence if μ is σ-finite.
C. (3) of course is are tie in with measure theory.

Examples: (1) Let $f_n = \sqrt{2^n}\,\chi_{[1/2^n,\, 1/2^{n-1}]}$, $\|f_n\|_2 = 1$, but $\|f_n - f_m\|_2 \ge \sqrt{2}$. So (1) is not sufficient.
(2) Let $g \in L_2[0,\infty)$ Then $(\int_0^\infty |g|^2)^{1/2} = (\sum_1^\infty (\int_n^{n+1} |g_n|^2))^{1/2} < \infty$ so $f_n = \chi_{[n,n+1]}$ so $\int f_n g \to 0$. Hence $f_n \to 0$ weakly but $\|f_n\| = 1 \not\to 0$. Since $L_p \approx L_p([0,\infty))$ it can be done in L_2

(3) $f_n = 2^n \chi_{[0,\frac{1}{n}]}$, $\|f_n\|_p = \left(\frac{1}{n^{1/p}} 2^{n/p}\right) \to \infty$ but $f_n \to 0$ in measure.

Condition (1) is particularly easy to use. Let $\|f_n\| \to \|f\|$.

Case I. $\|f\| = 0$, then $f = 0$ and $\|f_n\| \to 0$ so that $f_n \to 0$ in norm.

Case II. $\|f\| \neq 0$. Let $g_n = f_n \|f\| / \|f_n\|$, now $\|g_n\| = \|f\|$ and since $g_n \to f$ in norm $\iff f_n \to f$ in norm

Hence the condition $\|f_n\| \to \|f\|$, can be replaced by $\|f_n\| = \|f\|$.

Conditional (2) almost implies (1), but not quite but we have if $\int g f_n \, d\mu \to \int g f \, d\mu \ \forall g \in L_{p'}$ then $\liminf_n \|f_n\| \geq \|f\|$.

proof: Suppose $p < \infty$ then $\exists g \in L_{p'}$ s.t. $\|g\| = 1$ and $\int g f \, d\mu = \|f\|$ (This follows from the Hahn-Banach Theorem or for $1 \leq p < \infty$ from Hölder's inequality). Hence $\|f_n\|_p \|g\|_{p'} \geq \int g f_n \, d\mu \longrightarrow \int g f \, d\mu = \|f\|$ hence $\varliminf \|f_n\|_p \geq \|f\|_p$.

If $p = \infty$ $\exists g \in L_1(\mu)$ $\|g\|_1 = 1$ s.t. $\|f\| - \varepsilon \leq \int g f \, d\mu$ by definition of norm of $L_\infty(\mu)$ as $[L_1(\mu)]'$ this gives $\liminf \|f_n\| \geq \|f\| - \varepsilon$ (from the above). Since ε is arbitrary we are done.

~~Condition (3) does not imply anything about (2), But reminber your results from measure theory~~

Condition (1) has the following interesting example. Let X be a separable Banach space and let $\{x_n\}$ be dense in $X \setminus \{0\}$. Define $y_n = x_n / \|x_n\|$. We have $\{y_n\}$ is a collection of norm one elements dense in the unit sphere $\{x : \|x\| = 1\}$. Clearly $\{y_n\}$ does not converge but for each x in the unit sphere there is a subsequence of $\{y_n\}$ converging to x.

<u>Theorem</u> ~~E~~ Let f_n, ~~f~~ be in $L_p(\mu)$, and f a measurable function. Each of the following conditions imply $\|f_n - f\|_p \to 0$

(A) Results from measure theory. These are by far the most useful results. Let's go through these ~~measurablee~~ measure theory Theorems

(A1) Monotone Convergence Theorem says if
$0 \le f_1 \le f_2 \le \dots$ and f = pt-wise limit f_n a.e.
then $\lim \int f_n = \int f$.
$\boxed{\text{If } 0 < p < \infty}$, $0 \le f_1^p \le f_2^p \le \dots$ and f^p = pt-wise limit of f_n^p a.e., hence $\|f_n\|_p^p \to \|f\|_p^p$.
Since $g(x) = x^{1/p}$ is continuous, we have

$$\|f_n\|_p \to \|f\|_p$$

Now if $f \notin L_p(\mu)$ then $\|f_n\|_p \nearrow +\infty$ and hence f_n cannot converge ~~to~~ in L_p-norm.

However, $\boxed{\text{if } f \in L_p(\mu)}$ then $g_n = f - f_n$ satisfies $0 \le g_n \le f$ $g_n \to 0$ pt-wise a.e.
Thus $\lim \int |g_n|^p = 0$, since $|g_n|^p$ is dominated by $|f|^p \in L_1(\mu)$ ~~and~~ by the dominated conv Thm.

$$\|f - f_n\|_p^p \to 0 \quad \text{or} \quad \|f - f_n\|_p \to 0$$

Now there is no result for $p = \infty$, in fact in L_∞ let $f_n = \chi_{[1/n, 1]}$, clearly $0 \le f_n \le f_{n+1}$
$f_n \to \chi_{[0,1]}$ p.t. wise a.e. let $f = \chi_{[0,1]}$
$\|f - f_n\|_\infty = 1$ for each n.

However there is a Condition (1) result for $p = \infty$. Since $0 \le f_n \le f$, $\limsup \|f_n\|_\infty \le \|f\|_\infty$.
Now let $A = \{x : f(x) \ge \alpha\}$ and suppose $\mu(A) > 0$.
Define $A_{n,\varepsilon} = \{x : f_n(x) \ge \alpha - \varepsilon$ and $x \in A\}$. Note
$n < m \Rightarrow A_{n,\varepsilon} \subset A_{m,\varepsilon}$. And for almost every $x \in A$
$\exists n$ s.t. $x \in A_{n,\varepsilon}$ since $f_n \to f$ pt-wise a.e.
Thus $\forall_m$ $A \le \bigcup_{n \ge m} A_{n,\varepsilon}$ except for set of measure zero.
Therefore $\forall_\varepsilon \forall_m \exists_n$ s.t. $\{x : f_n(x) \ge \alpha - \varepsilon\}$ has positive μ measure so $\|f_n\|_\infty \ge \alpha - \varepsilon$.
It follows $\liminf \|f_n\|_\infty \ge \|f\|_\infty$ and so

$$\|f_n\|_\infty \to \|f\|_\infty$$

We restate this as:

→ (A1) If $0 \le f_n \le f_{n+1}$ and $f_n \to f$ pt-wise a.e, then $\forall p$ $0<p\le\infty$ $\|f_n\|_p \to \|f\|_p$. Furthermore if $f \in L_p(\mu)$ and $0<p<\infty$ then $\|f-f_n\|_p \to 0$.

(A2) Dominated convergence theorems. These always start with the hypothesis $f_n \to f$ pt-wise a.e. From (A1) clearly we should start trying to prove a condition (1) type result for pt-wise a.e. convergence sequences. However there is no such result. Consider $\alpha_n(p) = 2^{n/p}$ if $0<p<\infty$ and 1 if $p=\infty$. Let $f_n = \alpha_n(p)\chi_{[2^{-n},2^{-n+1}]}$. Now $\|f_n\|_p = 1$ all n,p but $f_n \to 0$ pt-wise a.e. and $\|0\|_p = 0$. [Note that this is all done in L_p.]

Of course, this is why the measure theory results require some form of domination. Perhaps the most general result is the following
If $f_n \to f$ & $g_n \to g$ pt-wise a.e. $g_n, g \in L_1(\mu)$ and $\|g_n\|_1 \to \|g\|_1$ and $\forall n$ $|f_n| \le g_n$. Then $\|f_n\|_1 \to \|f\|_1$
This includes the Lebesgue Convergence Theorem as a special case where $\forall n$ $g_n \equiv g$.

First we will show that there is no result for $p=\infty$. The example above and the example in (A1) for $p=\infty$ are dominated by the function $\chi_{[0,1]}$.

We state ~~our~~ most general result as follows

→ (A2) If ~~f_n~~ $0<p<\infty$, $f_n, f \in L_p(\mu)$ ~~then~~ and $f_n \to f$ pt-wise a.e. Then: $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$.

proof: ($\Rightarrow$) is just condition (1).
($\Leftarrow$) Each of following sequences converge pt-wise a.e to the given limit $|f_n|^p \to |f|^p$, $|f_n - f|^p \to 0$, $|f_n|^p + |f|^p \to 2|f|^p$
Since $f_n, f \in L_p(\mu)$; $|f_n|^p$, $|f|^p$ and $|f_n|^p + |f|^p \in L_1(\mu)$. Furthermore $\int |f_n|^p + |f|^p = \|f_n\|_p^p + \|f\|_p^p \to 2\|f\|_p^p = \int 2|f|^p$. (Linearity of $\int$ & Cond (1)).
Finally, since $|f_n - f|^p \le (|f_n| + |f|)^p \le 2^p(|f_n|^p + |f|^p)$ the generalized

dominate convergence theorem quoted above yields

$$\|f_n - f\|_p^p \to 0 \quad \text{or} \quad \|f - f_n\|_p \to 0 \quad \text{done.}$$

<u>Remarks</u>: 1. A slightly more general result could be obtained by just assuming $\exists\, g_n, g \in L_p(\mu)$ s.t. $g_n \to g$ pt-wise a.e. $|f_n| \le g_n$ and $\|g_n\|_p \to \|g\|_p$. However, in practise, it seems that g_n is always chosen to be $|f_n|$.

2. Convergence in L_p-norm does not require or imply pt-wise a.e. convergence. For each n there are unique k, ξ with $0 \le \xi < 2^k$; $k = 0, 1, \ldots$ so that $n = 2^k + \xi$. Define $f_n = \chi_{[\xi/2^k, (\xi+1)/2^k]}$. Then $f_n \to 0$ in meas and $\|f_n\|_p \to 0$ for all p, $0 < p < \infty$. But $f_n \not\to 0$ pt-wise a.e.

(A3) Convergence in Measure. We need the following general Lemma.

<u>Lemma</u> If X is topological space and $x_n, x \in X$, then $x_n \to x$ if and only if every subsequence $\{x_{n'}\}$ of $\{x_n\}$ has a further subsequence $\{x_{n''}\}$ so that $x_{n''} \to x$.

proof: ($\Rightarrow$) easy way ($\Leftarrow$) straightforward proof by contradiction.

Reminber, if $f_n \to f$ in measure, then there is a subsequence $\{f_{n'}\}$ of $\{f_n\}$ so that $f_{n'} \to f$ pt-wise a.e. [See Royden p.92].

(A3) If $0 < p < \infty$, $f_n, f \in L_p(\mu)$, $f_n \to f$ in measure; then $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$

<u>proof</u>: ($\Rightarrow$) is condition (1)

($\Leftarrow$) Consider a subsequence $\{f_{n'}\}$ of $\{f_n\}$, since $f_{n'} \to f$ in measure there is a subsequence $\{f_{n''}\}$ of $\{f_{n'}\}$ so that $f_{n''} \to f$ pt-wise a.e. Clearly $\|f_{n''}\|_p \to \|f\|_p$ so by (A2) $\|f_{n''} - f\|_p \to 0$. Hence by the Lemma we are done.

<u>Remarks</u> 1. There is no result for $p=\infty$. If $f_n = \chi_{[\xi/2^k, (\xi+1)/2^k]}$, $n = 2^k + \xi$, then $f_n \to 0$ in measure. Hence $g_n = \chi_{[0,1]} - f_n \to g = \chi_{[0,1]}$ in measure. $\|g_n\|_\infty = \|g\|_\infty = 1$ for $n > 1$ but $\|g - g_n\|_\infty = \|f_n\|_\infty = 1$.

2. We could summarize (A3) by saying that Condition 1 and Condition 3 together imply convergence in L_p-norm for $0<p<\infty$.

(B) The weak topology (i.e. Condition (2) - like properties)

These results will take awhile to produce, however the are the most important in the following sense, if $f_n \not\to f$ pt-wise, then the weak conditions are the easiest to check. There are also other reasons why this is important.

First some examples.

1. Let $f : [0,1] \to \mathbb{K}$ be measurable and $0<p<\infty$. There is an s, $0<s<1$ so that (s depends on p)

$$\int_0^s |f|^p = \int_s^1 |f|^p = \frac{1}{2}\|f\|_p^p$$

Let $g = 2f\chi_{[0,s]}$ and $h = 2f\chi_{(s,1]}$. We have $f = \frac{1}{2}(g+h)$ and $\|g\|_p = \|h\|_p = \|f\|_p 2^{1-1/p}$

if $1<p<\infty$, $\|g\|_p, \|h\|_p > \|f\|_p$
if $p=1$ $\quad \|g\|_p, \|h\|_p = \|f\|_p$
if $0<p<1$ $\quad \|g\|_p, \|h\|_p < \|f\|_p$

This construction has a couple of interesting consequences

(i) If $0<p<1$ $(L_p)' = \{0\}$

(ii) If U is the unit ball of L_1 $(= \{f : \|f\|_1 \le 1\})$ has no extreme points.

<u>proof:</u> (i) If $F : L_p \to \mathbb{K}$ is continuous and linear then $\exists \varepsilon > 0$ s.t. $F(\{x \in L_p : \|x\|_p \le \varepsilon\}) \subseteq \{\lambda \in K : |\lambda| \le 1\}$. Suppose f, $\cancel{\|f\|_p \le \varepsilon}$ so that $F(f) \ne 0$. We can chose f so that $|F(f)| \ge 2$. Let $M = \|f\|_p$. Consider g, h as

as above and $\lambda = 2^{1-1/p} < 1$. $\|g\|_p = \|h\|_p = \lambda M$ and $2 \le |F(f)| = |F(\frac{1}{2}(g+h))| \le \frac{1}{2}(|F(g)| + |F(h)|)$ thus at least one of g, h; call it f_1; satisfies $|F(f_1)| \geq 2$. Repeat obtaining f_n with $\|f_n\|_p = \lambda^n M$ and $|F(f_n)| \geq 2$. Eventually $\lambda^n M < \varepsilon$ contradicting the assumption that F is continuous. Hence $F = \{0\}$

(ii) FOLLOWS from the definition of an extreme point

<u>Definition</u> ξ in a vector space X is said to be an extreme point of $A \subseteq X$ if $\xi \in A$ and $\xi \notin co(A \setminus \{\xi\})$ where $co(B) = \{\sum \lambda_i b_i : b_i \in B_i, \lambda_i \geq 0, \sum \lambda_i = 1$ & $\{\lambda_i\}$ is finitely non-zero $\}$. [co ≡ convex hull]

<u>Remarks</u> 1. (i) implies that for $0<p<1$ and any sequence $\{f_n\} \subset L_p$ then for all $f \in L_p$ & $F \in (L_p)'$ $F(f_n) \to F(f)$. Hence any Condition (2) - like result could imply absolutely nothing for the range $0<p<1$
2. The construction above requires the measure to be purely non-atomic. If μ is purely atomic $L_p(\mu)$ has lots of non-constant continuous linear functionals even when $0<p<1$.
3. The construction for $1<p<\infty$ yields no information about the extreme points (there are many other convex combinations to check) However we do have

<u>Lemma</u> If $1<p<\infty$, each element of norm one in $L_p(\mu)$ is an extreme point of the unit ball in $L_p(\mu)$.

The proof is a collection of side tracks.

<u>Definition</u> A pt $\xi \in A \subset X$ (vector space with top so that vector add & scalar mult are cont) is said to be an exposed point of A if there is a cont. lin functional F on X so that $\mathrm{Re}F(\xi) = \sup\{\mathrm{Re}F(a) : a \in A\}$ and $a \in A$ and $\mathrm{Re}F(\xi) = \mathrm{Re}F(a) \Rightarrow a = \xi$.

<u>Remarks</u>. 1 If $\mathfrak{X}$ is complex vector space & $\mathfrak{X}_{\mathbb{R}}$ is the same space over the reals (just forget how to multiply by non-real scalars) then f is lin on $\mathfrak{X} \Leftrightarrow \mathrm{Re} f$ lin on $\mathfrak{X}_{\mathbb{R}}$ (& f is cont on $\mathfrak{X} \Leftrightarrow \mathrm{Re} f$ is cont on $\mathfrak{X}_{\mathbb{R}}$) Hence we can just work with real vector spaces for exposed points & the sequel.

2. Multipling F by a suitable scalar, we may assume $F(\xi) = 1$ & $a \in A \setminus \{\xi\} \Rightarrow F(a) < 1$.

3. Exposed points are extreme. If $\xi = \sum' \lambda_i a_i$, then $1 = F(\xi) = F(\sum \lambda_i a_i) = \sum' \lambda_i F(a_i) < \sum' \lambda_i = 1$ if the $\{a_i\} \subset A \setminus \{\xi\}$.

4. Hölder's inequality implies if $1 < p < \infty$ and $\|f\|_p = 1$ then $g = (\mathrm{sgn} f)\, |f|^{p/q}$ exposes f from the rest of the unit ball of $L_p(\mu)$ [where $1/p + 1/q = 1$].

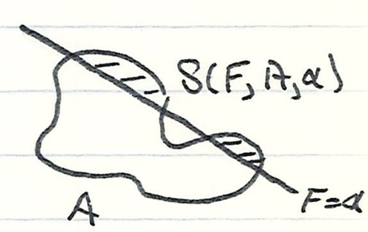
5. Extreme points need not be exposed. The unit ball of $L_2 \subset L_{1/2}$. The points on the unit sphere are still extreme points by there are not continuous linear functionals to expose them in $L_{1/2}$.

6. The notion of extreme & expose agree in $\mathbb{C}^n$.

7. Note A is not assumed to be convex.

Actually there are stronger conditions that agree with extreme points in $\mathbb{C}^N$.

<u>Definition</u>. If $A \subset \mathfrak{X}$, F cont lin functional on $\mathfrak{X}$ a slice is any set $S(F, A, \alpha)$ where $\alpha < \sup_{a \in A} F(a)$. and $S(F, A, \alpha) = \{a \in A;\ F(a) \geq \alpha\}$.



<u>Definition</u> A point $\xi \in A \subset \mathfrak{X}$, $\mathfrak{X}$ normed is said to be <u>strongly exposed</u> if $\exists$ continuous linear functional F on $\mathfrak{X}$ so that ~~F exp~~ diameter $(S(F, A, \alpha)) \to 0$ as $\alpha \nearrow F(\xi)$

Where diameter $B = \sup\{\|x - y\|;\ x, y \in B\}$

It is easy to see that F must expose ξ if it strongly exposes ξ.

Why this sidetrack? Consider the following

<u>Proposition</u> Suppose X is normed, ξ a strongly exposed point of the unit ball of X (by F say) then $x_n \to \xi$ in norm if and only if $\|x_n\| \to \|\xi\|$ and $F(x_n) \to F(\xi)$

<u>proof</u>: ($\Rightarrow$) is cond (1) & cont of F
($\Leftarrow$) Since $\|\xi\| = 1$ we may and do assume $\|x_n\| \equiv 1$. Let $\varepsilon > 0$ be given, let $\alpha < F(\xi)$ be so that $\mathrm{dia}(S(F,U,\alpha)) < \varepsilon$ (U = unit ball of X). Since $F(x_n) \to F(\xi)$ $\exists N$ s.t. $n \geq N$ implies $F(x_n) \geq \alpha$. So if $n \geq N$, $\|x_n - \xi\| \leq \mathrm{dia}(S(F,U,\alpha)) < \varepsilon$ hence $x_n \to \xi$ in norm.

<u>Remark</u> This is a remarkable result (even if it is easy) it says with 2 "1-dimensional" estimates we can determine an "infinite dimensional" estimate.

Obviously, our goal is to show that each norm one element of $L_p(\mu)$ $1 < p < \infty$ is a strongly exposed point of the unit ball. We will do this by yet a stronger property.

<u>Definition</u> A normed space X $\|\cdot\|$ is said to be uniformly convex if $\forall \varepsilon > 0$ $\exists \delta > 0$ so that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq 2\varepsilon$ imply $\|x + y\| \leq 2(1 - \delta)$

<u>Proposition</u> If ξ is a norm one element of a uniformly convex space then ξ is a strongly exposed point of the unit ball.

<u>Proof</u>: Let $F : [\xi] \to \mathbb{R}$ be define by $F(\lambda\xi) = \lambda$. F has norm one on $[\xi]$. Use Hahn-Banach to extend F to a norm one ~~element~~ functional defined on all of X. If $\|x\| = 1$ and $\|x - \xi\| \geq 2\varepsilon$ then $F(x) + 1 = F(x+y) \leq \|F\|\|x+y\| \leq 2 - 2\delta$ hence $F(x) \leq 1 - 2\delta$ and $x \notin S(F, U, 1-2\delta)$ this indicates that the diameter is small.

To Complete the proof we need an equivalent definition of uniform convexity. namely

(*) $\forall \varepsilon > 0\ \exists \delta > 0$ so that $\|x\|, \|y\| \le 1$ $\|x-y\| \ge 2\varepsilon \Rightarrow \|x+y\| \le 2(1+\delta)$

Clearly (*) $\Rightarrow$ uniform convexity to see the converse note that if (*) is false then for some $\varepsilon > 0$ there are x_n, y_n of norm ≤ 1 $\|x_n - y_n\| \ge 2\varepsilon$ & $\|x_n + y_n\| \ge 2(1-\frac{1}{n})$

Thus $\|x_n\| + \|y_n\| \ge 2 - \frac{2}{n}$ so that

$$1 \ge \|x_n\|, \|y_n\| \ge 1 - \frac{2}{n} = \frac{n-2}{n}$$

Let $w_n = x_n / \|x_n\|$, $z_n = y_n / \|y_n\|$.

$$w_n - z_n = (w_n - x_n/\|y_n\|) - (z_n - x_n/\|y_n\|)$$

$$\|w_n - z_n\| \ge \big| \|w_n - x_n/\|y_n\|\| - \|z_n - x_n/\|y_n\|\| \big|$$

$$\|w_n - x_n/\|y_n\|\| = (1/\|x_n\| - 1/\|y_n\|)\|x_n\| \le \frac{n}{n-2} - 1 = \frac{2}{n-2}$$

$$\|z_n - x_n/\|y_n\|\| = (1/\|y_n\|)\|y_n - x_n\| \ge 2\varepsilon$$

Hence $\|w_n - z_n\| \ge 2(\varepsilon - \frac{1}{n-2})$

$$w_n + z_n = w_n - x_n/\|y_n\| - (-z_n - x_n/\|y_n\|)$$

$$\|z_n + x_n/\|y_n\|\| = 1/\|y_n\| \, \|y_n + x_n\| \ge 2(1 - 1/n)$$

Hence $\|w_n + z_n\| \ge 2(1 - \frac{1}{n} - \frac{1}{n-2})$

Therefore for each $\varepsilon' < \varepsilon$ the definition of uniform convex is false.

The condition $\|x\| = 1$ can be reduced to $\|x\| \le 1$ and the proof of proposition now works $\square$

<u>Remark</u> 1. Suppose $\{x_n\}, \{y_n\} \subset$ uniformly convex space $\mathfrak{X}$ and $\lim \|x_n\| = A$ $\lim \|y_n\| = \cancel{A}$ and $\lim \|x_n + y_n\| = A + \cancel{A}$ then $\lim \|x_n - y_n\| = 0$. <u>Proof</u>. Let $A = B \le 1$, Assume $\|x_n\| = \|y_n\| = 1$ then $\forall \varepsilon$ n large enuff so that $\|x_n + y_n\| \ge 2(1-\delta)$ $\Rightarrow \|x_n - y_n\| < 2\varepsilon$. ~~Now suppose~~

2. $\mathfrak{X}$ $\|\cdot\|$ is said to be strictly convex if $\|x\| = \|y\| = 1$ $x \ne y \Rightarrow \|(x+y)/2\| < 1$. Uniform convexity implies strictly convex but not conversely [uniform convexity implies reflexivity but each separable space can be renormed to be strictly convex.] Exercises 1. show that each element of norm one is an exposed point of the unit ball of a strictly convex space. 2. Show that if $\mathfrak{X}$ is strictly convex $Y \subset \mathfrak{X}$, $x \in \mathfrak{X} \setminus Y$ then if there is $y \in Y$ s.t. $\|x - y\| = \inf\{\|x - w\| \ w \in Y\}$ it is unique & if $\mathfrak{X}$ is uniformly convex such a y exists.

<u>Thm</u>: If $1<p<\infty$, Then $L_p(\mu)$ is uniformly convex.

<u>Lemma</u> Let ξ, η be scalars

$$|\xi+\eta|^p + |\xi-\eta|^p \geq 2(|\xi|^p+|\eta|^p) \quad \text{if } 2\leq p<\infty$$

$$|\xi+\eta|^p + |\xi-\eta|^p \leq 2(|\xi|^p+|\eta|^p) \quad \text{if } 1<p\leq 2$$

if $p\neq 2$ then equality occurs iff $\xi\cdot\eta=0$.

<u>pf</u>: Let $p=2$

(0) $|\xi+\eta|^2+|\xi-\eta|^2 = (\xi+\eta)(\bar\xi+\bar\eta) + (\xi-\eta)(\bar\xi-\bar\eta)$

$= \xi\bar\xi + \eta\bar\xi + \bar\eta\xi + \eta\bar\eta + \xi\bar\xi - \eta\bar\xi - \xi\bar\eta - \eta\bar\eta = 2(|\xi|^2+|\eta|^2)$

Now let α, β be ≥ 0.

$2\leq p<\infty$ $\quad 1\leq p/2$ & $p/p-2<\infty$ are congugate exponents

so $\quad (\alpha^2, \beta^2)(1, 1) \leq \|\alpha^2, \beta^2\|_{p/2} \, \|(1,1)\|_{p/p-2}$ in $\mathbb{R}^2$

or $\quad \alpha^2+\beta^2 \leq (\alpha^p+\beta^p)^{2/p}(2)^{\cancel{p/(p-2)}}$ $\frac{p-2}{p}$

so

(1) $\quad \alpha^p+\beta^p \geq 2^{(2-p)/2}(\alpha^2+\beta^2)^{p/2} \qquad 2<p<\infty$

$1<p<2$ $\quad$ so $\quad 4/p>2$ & $2-4/p/2$ = $\cancel{\ldots}$ $\to 1-\frac{2}{p} = \frac{p-2}{p}$

$\alpha^{4/p}+\beta^{4/p} \geq 2^{(p-2)/p}(\alpha^2+\beta^2)^{2/p}$

replace α by $\alpha^{p/2}$ β by $\beta^{p/2}$ yields

$\alpha^2+\beta^2 \geq 2^{(p-2)/p}(\alpha^p+\beta^p)^{2/p}$

so

(2) $\quad \alpha^p+\beta^p \leq 2^{(2-p)/2}(\alpha^2+\beta^2)^{p/2} \qquad 1<p<2$

Note equality occurs in (1) or (2) iff $\alpha=\beta$.

Since $\quad 0\leq |\xi|^2/(|\xi|^2+|\eta|^2) \leq 1$

(3) $\quad |\xi|^p/(|\xi|^2+|\eta|^2)^{p/2} \leq |\xi|^2/(|\xi|^2+|\eta|^2) \qquad 2<p<\infty$

(4) $\quad |\xi|^p/(|\xi|^2+|\eta|^2)^{p/2} \geq |\xi|^2/(|\xi|^2+|\eta|^2) \qquad 1<p<2$

Forming similar $\neq$ with η and adding yields

(5) $\quad |\xi|^p+|\eta|^p \leq (|\xi|^2+|\eta|^2)^{p/2} \qquad 2<p<\infty$

(6) $\quad |\xi|^p+|\eta|^p \geq (|\xi|^2+|\eta|^2)^{p/2} \qquad 1<p<2$

Note equality occurs in (5) or (6) iff $|\xi|\cdot|\eta|=0$

Now if $p>2$ replace $\alpha=|\xi+\eta|$ $\beta=|\xi-\eta|$ in (1)

$|\xi+\eta|^p+|\xi-\eta|^p \geq 2^{(2-p)/2}(|\xi+\eta|^2+|\xi-\eta|^2)^{p/2}$ by (1)

$\geq 2^{(2-p)/2}(2[|\xi|^2+|\eta|^2])^{p/2}$ by (0)

$\geq 2^{(2-p)/2+p/2}(|\xi|^p+|\eta|^p) = 2(|\xi|^p+|\eta|^p)$ by (5)

Note $\xi=\eta \Rightarrow =$ & by (5) $= \Rightarrow$ ~~$|\xi+\eta|^p = |\xi-\eta|$~~ $|\xi|\cdot|\eta|=0$

using (2) & (6) yields the inequality for $1<p<2$.

~~Proposition (Para~~

<u>Sublemma</u>: Do (1) & (5) ((2) & (6)) in opposite order yields

$2 \le p < \infty \qquad |\xi+\eta|^p + |\xi-\eta|^p \le 2^{p-1}(|\xi|^p + |\eta|^p)$

$1 < p \le 2 \qquad |\xi+\eta|^p + |\xi-\eta|^p \ge 2^{p-1}(|\xi|^p + |\eta|^p)$

<u>pf:</u> $1<p\le 2 \quad |\xi+\eta|^p + |\xi-\eta|^p \ge (|\xi+\eta|^2 + |\xi-\eta|^2)^{p/2}$ (6)

$= 2^{p/2}(|\xi|^2+|\eta|^2)^{p/2} \ge 2^{p/2}\, 2^{p-2/2}(|\xi|^p + |\eta|^p)$ (2)

Similar for $2\le p<\infty$

<u>Proposition</u> (Parallalogram Laws) If $f,g \in L_p(\mu)$

$2\le p<\infty \qquad 2^{p-1}(\|f\|_p^p + \|g\|_p^p) \ge \|f-g\|_p^p + \|f+g\|_p^p \ge 2(\|f\|_p^p + \|g\|_p^p)$

$1<p\le 2 \qquad 2^{p-1}(\|f\|_p^p + \|g\|_p^p) \le \|f-g\|^p + \|f+g\|^p \le 2(\|f\|_p^p + \|g\|_p^p)$

~~if~~ $p \ne 2$ the right hand inequalities are equalities iff f and g have disjoint supports

<u>pf:</u> Integrate the lemma or sublemma.

<u>Cor</u> If $T: L_p(\mu) \to L_p(\nu)$ is isometry & $1<p<\infty$ then T maps disjointly supported functions to disjointly support functions. (use equality in Proposition)

<u>Remark</u> Thus isometries preserve some of the order properties (but $f \mapsto -f$ shows isometries need not preserve order)

Uniform Convexity of $L_p(\mu)$

$2\le p<\infty$. Let $\|x\|=\|y\|=1 \quad \|x-y\| \ge 2\varepsilon$

Let $f=(x+y)/2 \quad g=(x-y)/2 \qquad f+g=x \quad f-g=y$

$\|f-g\|^p + \|f+g\|^p \ge 2(\|f\|^p + \|g\|^p)$

$1 + 1 \ge 2(\|x+y\|^p/2^p + \|x-y\|^p/2^p)$

$2^p - \|x-y\|^p \ge \|x+y\|^p$

$2(1-\varepsilon^p)^{1/p} \ge \|x+y\|$ ✓

nope.

$1 \le p \le 2$, ~~$\|f\|^2 + \|g\|^2 \ldots$~~ Let x,y,f,g as above

$2^{p-1}(\|f\|^p + \|g\|^p) \le \|f-g\|^p + \|f+g\|^p$

~~then~~ $2^{p-1}\left(\frac{\|x+y\|^p}{2^p} + \frac{\|x-y\|^p}{2^p}\right) \le$ ~~$\|f-g\|^p + \|f+g\|^p$~~ $1+1=2$

$\frac{2^{p-1}}{2^p}\{\|x+y\|^p + \|x-y\|^p \le$ ~~$\|f+g\|^p$~~ 2

$\|x+y\|^p \le 4 - 2^p\varepsilon^p$ Opps

LEMMA: $[\cancel{\tfrac{1}{2}}(1+c)]^p \le \overset{2^{p-1}}{\cancel{\tfrac{1}{2}}}(1+c^p)$ for $c \ge 0$, $1<p<\infty$

pf: $\dfrac{1+t^p}{(1+t)^p} = f(t)$ $\quad f(1) = 2^{1-p}$ $\quad f'(t) = \dfrac{pt^{p-1}(1+t)^p - (1+t^p)p(1+t)^{p-1}}{(1+t)^{2p}}$

but $t^{p-1}(1+t)^p - (1+t^p)(1+t)^{p-1} = [t^{p-1}+t^p-1-t^p](1+t)^{p-1}$

is positive for $t>1$ negative for $t<1$

hence $1+t^p \ge 2^{1-p}(1+t)^p$ $\quad 0 \le t < \infty$ with equality iff $t=1$

~~LEMMA: For $\forall \varepsilon>0$, $\forall 1<p<\infty$ $\exists \delta_p>0$ s.t. a, b scalars $|a-b| \ge \varepsilon$ $|a|,|b| \le 1$~~

~~$|\tfrac{1}{2}(a+b)|^p \le (1-\delta_p)\left(\dfrac{|a|^p+|b|^p}{2}\right)$~~

~~pf: It suffices to prove this when a, b are both positive since $|a|+|b| \ge |a+b|$. Furthermore we may assume $a=1 \ge b \ge 0$~~

LEMMA: If X is uniformly convex, $1<p<\infty$ then $\forall \varepsilon>0$ $\exists \delta_p = \delta_p(\varepsilon) > 0$ s.t. $\|x\|, \|y\| \le 1$ $\|x-y\| \ge 2\varepsilon$ then

$$\|(x+y)\|^p \le (1-\delta_p)(\|x\|^p + \|y\|^p)\,2^{p-1}$$

pf: Suppose not then $\exists$ $\|x_n\|, \|y_n\| \le 1$ $\|x_n - y_n\| \ge 2\varepsilon$ s.t.

$$\frac{\|(x_n+y_n)\|^p}{2^{p-1}(\|x_n\|^p + \|y_n\|^p)} \longrightarrow 1$$

Suppose $\|y_n\|$ uniformly $\le \alpha < 1$ and $\|x_n\| \equiv 1$

$\|x_n+y_n\|^p \le (1+\|y_n\|)^p$ $\quad (\Delta \ne)$

$\le 2^{p-1}(1+\|y_n\|^p) \le 2^{p-1}(1+\alpha^p) \le \rho\, 2^{p-1}(\|x_n\|^p + \|y_n\|^p)$

a contradiction. Since it suffices to assume $\|x_n\| \equiv 1 \ge \|y_n\|$ we have $\|y_n\| \to 1$

Thus $\|x_n+y_n\|^p \longrightarrow 2^{p-1}\,2 = 2^p$

but this contradicts the uniform convexity of X

In general $\quad \|x+y\|^p \le \left(1-\delta_p\left(\dfrac{2\|x-y\|}{2\max\|x\|\,\|y\|}\right)\right)(\|x\|^p+\|y\|^p)\,2^{p-1}$

Now to prove uniform convexity Let $1<p<\infty$

$\|f\|_p \le 1$ $\quad \|g_p\| \le 1$ $\quad \|f-g\|_p \ge 2\varepsilon$

Let

$$M = \left\{t : |f(t)-g(t)|^p \ge \frac{\varepsilon^p}{2}(|f(t)|^p + |g(t)|^p) \ge \frac{\varepsilon^p}{2}\max |f(t)|^p, |g(t)|^p\right\}$$

Using the last lemma on M (using the fact $\mathbb{R}, \mathbb{C}$ unif conv)

$$|f(t)+g(t)|^p \le \left(1-\delta_p\left(\frac{\varepsilon}{4^{1/p}}\right)\right)\left(|f(t)|^p+|g(t)|^p\right)2^{p-1}$$

on $N = M^c$ $\quad \int_N |f-g|^p \le \frac{\varepsilon^p}{2}\int |f|^p+|g|^p \le \varepsilon^p$

hence $\quad \int_M |f-g|^p \ge \varepsilon^p \quad$ since $\|f-g\| \ge 2\varepsilon$

Let $f_M = f\chi_M \quad g_M = g\chi_M \quad \|f_M - g_M\| \ge \varepsilon$

so $\quad \max \|f_M\|, \|g_M\| \ge \varepsilon/2$

Now $\int 2^{p-1}(|f|^p+|g|^p) - |f+g|^p \ge \int_M$ same

$$\ge \int_M \delta_p\left(\frac{\varepsilon}{4^{1/p}}\right)2^{p-1}(|f|^p+|g|^p) \ge \delta_p\left(\frac{\varepsilon}{4^{1/p}}\right)2^{p-1}\frac{\varepsilon^p}{2^p}$$

$$= \delta_p(\varepsilon/4^{1/p})\,\varepsilon^p/2 = \alpha$$

OR

$$2^{p-1}\left(\|f\|^p+\|g\|^p\right) - \|f+g\|^p \ge \alpha > 0$$

$$2(1-\delta) \ge (2^p-\alpha)^{1/p} \ge \|f+g\|. \quad \text{Done.}$$

<u>Remarks</u> 1. $f = 2\chi_{[0,\frac12]}$ $\quad g = 2\chi_{[\frac12,1]}$, $\|f\|_1 = \|g\|_1 = 1$
$\|f-g\|_1 = \|f+g\|_1 = 2$. Thus L_1 is not uniformly convex

2. $f = \chi_{[0,1]}$ $\quad g = \chi_{[0,\frac12)} - \chi_{[\frac12,1]}$ $\quad \|f\|_\infty = \|g\|_\infty = 1$
$\|f-g\|_\infty = \|f+g\|_\infty = 2$. Thus L_∞ is not uniformly convex.

3. Actually since L_1 has no extreme points in its unit ball, it has no strongly exposed points so it cannot be uniformly convex.

4. L_∞'s unit ball has lot's of extreme points i.e. ~~all~~ meas. functions f so that $|f| = 1$ a.e., But not every such function is strong exposed (exercise are there any) (~~Ther~~ Try doing in for ℓ_∞).

5. One can show that

$$\delta(\varepsilon) = (p-1)\varepsilon^2/8 + o(\varepsilon^2) \qquad 1<p<2$$

$$\delta(\varepsilon) = \varepsilon^p/p2^p + o(\varepsilon^p) \qquad 2 \le p < \infty$$

(Note $f(\varepsilon) = o(\varepsilon^2)$ if $\lim_{\varepsilon\to 0^+} f(\varepsilon)/\varepsilon^2 = 0$)

(B1) **Thm:** Let $1<p<\infty$ and suppose $\{f_n\}, f$ are in $L_p(\mu)$ and assume that $\|f_n\| \to \|f\|$

Then any of the following imply that $\|f-f_n\| \to 0$

(i) $\|f_n+f\| \to 2\|f\|$

(ii) $\int g f_n \to \int g f$ for $g = \operatorname{sgn} f\, |f|^{p/q}$

(iii) $\forall g \in L_q(\mu)$ $(1/p+1/q=1)$ $\int g f_n \to \int g f$

(iv) $\forall A \in \Sigma$ with $\mu(A)<\infty$ $\int_A f_n \to \int_A f$.

Proof Note that each of the conditions are necessary. Also note that (iii) $\Rightarrow$ (ii) & (iv) since $\chi_A \in L_q(\mu)$ & $\operatorname{sgn} f\, |f|^{p/q} \in L_q(\mu)$

To start the proof, note that if $\|f\|=0$, we are done. Since multiplicatio by a scalar sequence $\{\lambda_n\}$ which converges to λ_∞ to the sequence $\{f_n\}$ & f respectively does not change any of the conditions we may and do assume $\|f_n\| = 1 = \|f\|$.

(i): Now following since $L_p(\mu)$ is uniformly convex & by the remark on p 3.10

(ii): Followin since g strongly exposes f & by the Propositions on Pg 3.9

(iii): Follows since (iii) $\Rightarrow$ (ii)

That leaves us with (iv). To complete the proof we will give a proof that (iv) $\Rightarrow$ (iii) in more general cases. First we note that if $\varphi = \sum_1^n \alpha_i \chi_{A_i}$ is a simple fun with $\{A_i\}$ pwd. & $\mu(A_i)<\infty$ then $\int \varphi f_n = \sum_1^n \alpha_i \int_{A_i} f_n \to \sum_1^n \alpha_i \int_{A_i} f = \int \varphi f$. Thus (iv) implies the existence of a dense subset $S \subset L_q(\mu)$ s.t. $g \in S$ implies $\int g f_n \to \int g f$. (The simple functions are dense in $L_p(\mu)$ for $0<p<\infty$.) Thus to complete the proof it suffices to proof:

Lemma. If X is B-space with dual X' and suppose $Y < X'$ is norm dense in X', $\{x_n\}, x < X$ with $\|x_n\| = \|x\| = 1$ then $\forall x' \in Y$ $x'(x_n) \to x'(x)$ $\Rightarrow$ $\forall x' \in X'$ $x'(x_n) \to x'(x)$

Proof: Suppose not, then $\exists x' \in X'$, $\varepsilon>0$ and a subsequence $\{x_{n'}\}$ of $\{x_n\}$ so that $\|x'\|=1$, & $\forall n'$ $|x'(x_{n'}) - x'(x)| \ge \varepsilon$. Since we can assume Y is a subspace, $\exists y' \in Y$ $\|y'\|=1$ & $\|y'-x'\|<\delta$. Now

$y'(x_{n'}) - y'(x) = y'(x_{n'}) - x'(x_{n'}) + x'(x_{n'}) - x'(x) + x'(x) - y'(x)$ hence

$|y'(x_{n'}) - y'(x)| \ge |x'(x_{n'}) - x'(x)| - |y'(x_{n'}) - x'(x_{n'})| - |x'(x) - y'(x)| \ge \varepsilon - 2\delta > 0$

a contradiction.

Remark 1. Condition (iv) is often the easiest to check.

2. A second way to proof that (iv) $\Rightarrow$ (iii) for $L_p(\mu)$ $1<p<\infty$ is to notice that the unit ball of $L_p(\mu)$ is compact & Π_2 under both the $\sigma(X, X')$ & $\sigma(X, Y)$-topology (notation of lemma) [where the $\sigma(X, \Gamma)$-topology $\Gamma \subset X'$, is the weakest topology on X making each of the functionals in Γ continuous] and then using a result from topology which says since $\sigma(X, X') \geq \sigma(X, Y)$ the topologies are the same.

3. The lemma is more general, since the above proof does not work for L_1, by since the simple fcns are dense in L_∞, the lemma does,

(B2) The Case $P=\infty$, there is not such a Thm. For example in L_∞. Let $f_n = \chi_{[0, 1/2 + 1/n]}$. then $f_n \to f = \chi_{[0, 1/2]}$ pt-wise, in meas, and $\|f_n\|_\infty = 1 = \|f\|_\infty$ and if $g \in L_1$ we have

$\int g f_n = \int_0^{1/2 + 1/n} g \to \int_0^{1/2} g = \int gf$ so f_n, f satisfy all the above conditions and yet $\|f_n - f\|_\infty = 1$.

[this is not so bad, convergence in L_∞-norm is not hard to check]

(B3) The Case $P=1$, Surprizing enough things can even be better in some $L_1(\mu)$-spaces then in the case $1<p<\infty$, For example

Proposition: If f_n, f are in l_1 then $\|f_n - f\| \to 0$ iff $\forall g \in l_\infty$ $\int g f_n \to \int gf$.

proof: ($\Rightarrow$) is given before (i.e. g is continuous on l_1)

($\Leftarrow$): First we show that $\{f_n\}$ is "small at infinitely" that is for all $\varepsilon > 0$ $\exists N$ s.t. $\sum_{i=N}^{\infty} |f_n(i)| < \varepsilon$ for each n. Suppose not then $\exists \varepsilon > 0$ s.t. the statement is false. Inductively choose

$0 = N(0) < N(1) < \dots$ $\quad n(1) < n(2) < \dots$ so that

$$\sum_{i=N(j-1)+1}^{N(j)} |f_{n(j)}(i)| \geq \varepsilon/2, \quad \sum_{i=N(j)+1}^{\infty} |f_{n(j)}(i)| < \delta/2^j$$

[Opps, forgot to normalize (by replacing f_n with $f_n - f$ we can assume $\int f_n g \to 0$ $\forall g \in l_\infty$]

and $\quad \sum_{i=1}^{N(j-1)} |f_{n(j)}(i)| < \delta/2^j$

To see pick $f_{n(1)}$ so that $\|f_{n(1)}\| \geq \varepsilon$ choose $N(1)$ s.t. $\sum_{i=N(1)+1}^{\infty} |f_{n(1)}(i)| < \delta/2 < \varepsilon/2$ hence $\sum_{i=1}^{N(1)} |f_{n(1)}(i)| \geq \varepsilon/2$

We illustrate the induction step by picking $n(2)$, $N(2)$. Since $g_j(i) = \delta_{ij} \in \ell_\infty$ we may assume (by passing to a subsequence if ~~necessary~~ necessary) that $m > n(1)$ $i \le N(1) \Rightarrow |f_m(i)| < \delta/4\,(1/N(1))$ thus if $n(2) > n(1)$ $\sum_1^{N(1)} |f_{n(2)}(i)| < \delta/2^2$. Now use the negation of small at ∞ to pick $n(2)$ s.t. $\sum_{i=N(1)+1}^{\infty} |f_{n(2)}(i)| \ge \varepsilon$. Then choose $N(2)$ s.t. $\sum_{i=N(2)+1}^{\infty} |f_{n(2)}(i)| > \delta/2^2$ & again $\sum_{N(1)+1}^{N(2)} |f_{n(2)}(i)| \ge \varepsilon/2$. So much for the induction. (This is called the gliding hump)

Now let $g(i) = \operatorname{sgn} f_{n(j)}(i)$ for $N(j-1) < i \le N(j)$ we have $\|g\|_\infty = 1$. And $\int g f_{n(j)} = \sum_1^{N(j-1)} g(i) f_{n(j)} + \sum_{N(j-1)+1}^{N(j)} g(i) f_{n(j)} + \sum_{N(j)+1}^{\infty} f_{n(j)}$

$$\ge -\|g\|_\infty \delta/2^j + \varepsilon/2 - \|g_\infty\| \delta/2^j \ge \varepsilon/2 - \delta/2^j \not\to 0$$

a contradiction.

Now to prove $f_n \to f$ in norm. (Again we can and do assume $f = 0$)
Let $\varepsilon > 0$ be given find N s.t. $\forall n$ $\sum_{N+1}^{\infty} |f_n(i)| < \varepsilon/2$
Find M s.t. $n \ge M$ $i \le N \Rightarrow |f_n(i)| < \varepsilon/2N$ then
$n \ge M \Rightarrow \|f_n\| = \sum_i |f_n(i)| = \sum_1^N |f_n(i)| + \sum_{N+1}^{\infty} |f_n(i)| \le N\left(\frac{\varepsilon}{2N}\right) + \frac{\varepsilon}{2} = \varepsilon.$
Done.

Remarks: 1. This is false in ℓ_p $1 < p < \infty$ as $e_n(i) = \delta_{ni}$ shows (it converges weakly to zero but not in norm) Furthermore since ℓ_2 is isomorphic to a subspace of L_p, $1 \le p < \infty$ it is false for these L_p's.
2. The proof can be modified to show that weak precompact sets in ℓ_1 are norm pre-compact. Hence for convex sets in ℓ_1 weak and norm compactness are equivalent
3. The proof also works for the spaces $\ell_1(\Gamma)$ when small at ∞ is replaced with $\forall \varepsilon > 0$ $\exists F \subset \Gamma$ F finite s.t. $\forall n$ $\sum_{i \notin F} |f_n(i)| < \varepsilon$.

Now for the last result of this section.

(B3) Theorem: If f_n, f are in $L_1(\mu)$ and $\|f_n\| \to \|f\|$ then $\|f_n - f\| \to 0$ if ~~either~~ any of these conditions hold.
(i) $\forall g \in L_\infty$ $\int g f_n \to \int g f$
(ii) if μ is finite & $\forall A \in \Sigma$ $\mu(A) < \infty$ $\int_A f_n \to \int_A f$
(iii) if ~~μ is~~ $\forall A \in \Sigma$ $\int_A f_n \to \int_A f$.

→ Well This Thm is No Thm.

→ Counter example: in L_1

Let $f = \chi_{[0,1]}$ and let $r_n(t)$ be the rademacher functions and let $f_n = r_n + f$. $\|f\| = 1$ & $\|f_n\| = 1$ since the rademachers are 1 ½ the time and −1 the other half. Now $\|f_n - f\| = \|r_n\| = 1$ so that $f_n \not\to f$ in norm. To provide the counter example it suffices to show that $g \in L_\infty$ implies $\int g r_n \to 0$. To see this let $g \in L_\infty$ and $F_g : L_1 \to \mathbb{K}$ be defined by $F_g(f) = \int fg$. Now restrict F_g to $[r_n]$. We have that this is a bounded linear functional on $[r_n] \cong \ell_2$. Hence it suffices to check that the unit vectors u_n in ℓ_2 are weakly null. If $(a_n) \in \ell_2' = \ell_2$ $(a_n)u_m = a_m$ but since $(\sum |a_n|^2)^{1/2}$ converges $a_m \to 0$ and we are done.

(B3) Change to the following are equivalent for $f_n, f \in L_1(\mu)$ if $\|f_n\|, \|f\|$ are uniformly bdd. The proof is the lemma on p 3.15. However the following result is true

(B3') Thm: If $f_n, f \subset L_1(\mu)$ $\forall g \in L_\infty(\mu)$ $\int g f_n \to \int g f$ and $f_n \to f$ in meas on each set A with $\mu(A) < \infty$ then $\|f_n - f\|_1 \to 0$

We need the following result which is also of interest:

(C1) Thm: Let $1 \le p < \infty$, $f_n \subset L_p(\mu)$ f a measurable fcn then $f \in L_p(\mu)$ and $\|f_n - f\|_p \to 0$ iff the following three conditions hold.

(i) $f_n \to f$ in meas

(ii) $\lim\limits_{\varepsilon \to 0^+} \sup\limits_{\substack{A \in \Sigma \\ \mu(A) < \varepsilon}} \sup\limits_n \int_A |f_n|^p \, d\mu = 0$

(iii) $\forall \varepsilon > 0$ $\exists E = E_\varepsilon$ with $\mu(E) < \infty$ s.t. $\sup\limits_n \int_{X \setminus E} |f_n|^p < \varepsilon$.

Proof: If $\|f_n - f\|_p \to 0$ then we already know (i) is true. To see that norm conv implies (iii) Let $\delta > 0$ be given since $f \in L_p$ $\exists A$ s.t. $\int_{X \setminus A} |f|^p < \delta$ [f is supported on σ-finite set] $\exists N$ s.t. $n \ge N$ implies $\|f_n - f\| < \delta$ hence $(\int_{X \setminus A} |f_n|^p)^{1/p} \le (\int_{X \setminus A} |f_n - f|^p)^{1/p} + (\int_{X \setminus A} |f|^p)^{1/p}$ $\le \delta + \delta^{1/p} \le 2\delta < \varepsilon$ for $n \ge N$. Now for $n = 1, \dots, N-1$ pick A_n s.t. $\int_{X \setminus A_n} |f_n|^p < \varepsilon$ then $E = A \cup (\bigcup_1^{N-1} A_n)$ works

Now we show that norm conv implies (ii). First we show $\exists \varepsilon\ \exists \delta$ s.t. $\mu(A) < \delta \Rightarrow \int_A |f|^p < \varepsilon$.

First assume f is simple $= \sum a_i \chi_{A_i}$, let $\delta = \varepsilon / \max |a_i|^p$, it works.

Now suppose $\int |g-f|^p < \eta$, & $\delta_{\varepsilon,\delta}$ works for f. Let $\mu(A) < \delta$

$$\left(\int_A |g|^p\right)^{1/p} \le \left(\int_A |g-f|^p\right)^{1/p} + \left(\int_A |f|^p\right)^{1/p} \le \eta^{1/p} + \varepsilon^{1/p}$$

[we could of also used the absolute continuity of meas $\nu(A) = \int_A |f|^p d\mu$.]

Now since $f_n \to f$ in norm $\forall \varepsilon\ \exists N$ s.t. $n \ge N$ $\mu(A) < \delta$

$\Rightarrow \int_A |f_n|^p, \int_A |f|^p < \varepsilon$. For each $n = 1, \dots, N-1$ $\exists\, \delta_n$ s.t. $\mu(A) < \delta$

$\Rightarrow \int_A |f_n|^p < \varepsilon$ then $\delta' = \min_{1 \le n < N}\{\delta, \delta_n\}$ works.

note that this is not needed if we assume $f \in L_p(\mu)$ and the missing parts are also obviously true if $p=1$.

Now for the converse. Suppose (i), (ii) & (iii) are true. Let $g_n = |f_n|^p$ $g = |f|^p$. First we show $\{g_n\}$ is a C.S. in L_1. Since by (iii) $\forall \varepsilon\ \exists E$ s.t.

$\|g_n - g\|_1 \le 2\varepsilon + \int_E |g_n - g|$ it suffices to show $\{g_n\}$ is C.S. in $L_1(E)$. I.E. we may assume $\mu(X) < \infty$. Since $g_n \to g$ in meas [proof: exercise] & cond (ii) $\forall \varepsilon > 0$ $\exists N$ s.t. $n \ge N \Rightarrow$

$\mu(\{x : |(g_n - g_m)(x)| \ge \varepsilon\}) < \delta$ & $\mu(A) < \delta\ \int_A |g_n|, \int |g| < \varepsilon$. Hence

$$\|g_n - g_m\|_1 \le \int_{\{x : |g_n - g| \ge \varepsilon\}} |g_n - g_m| + \int_{\{x : |g_n - g| < \varepsilon\}} |g_n - g_m| \le 2\varepsilon + \varepsilon\mu(X) \to 0.$$

So $\{g_n\}$ is C.S. in $L_1(\mu)$ Let h_n simple s.t. $\int |h_n - g_n| < \frac{1}{n}$ and so that. $\mu\{x : |h_n - g_n| \ge \frac{1}{n}\} < \frac{1}{n}$. It follows that h_n is C.S. in L_1 & $h_n \to g$ in meas. Thus $g \in L_1(\mu)$ & $f \in L_p(\mu)$. [proof: exercise]

~~For each $\varepsilon > 0, n$ $\exists A_n$ s.t. $\int_{A_n} |f_n - f|^p(s) < \varepsilon$ $s \notin A_n$.~~

$f_n \to f$ meas $F_n = \{x : |f - f_n| \ge \varepsilon\} \Rightarrow$ ~~$\mu(X \setminus F_n)$~~ $\mu(F_n) \to 0$.

Hence by (ii) $\exists \delta\ \exists N$ $n \ge N \Rightarrow \mu(F_n) < \delta \Rightarrow \int_{F_n} |f_n|^p < \varepsilon$

By (i) $\exists E$ with $\varepsilon > 0$ $\mu(E) < \infty$ thus

$$\|f_n - f\|_p \le \left\{\int_{X \setminus (E \cup F_n)} |f_n|^p\right\}^{1/p} + \left\{\int_{X \setminus (E \cup F_n)} |f|^p\right\}^{1/p} + \left\{\int_{E \setminus F_n} |f_n - f|^p\right\}^{1/p}$$

$$+ \left\{\int_{F_n} |f_n|^p\right\}^{1/p} + \left\{\int_{F_n} |f|^p\right\}^{1/p}$$

$$\le \varepsilon^{1/p} + \varepsilon^{1/p} + \varepsilon\mu(E) + \varepsilon^{1/p} + \varepsilon^{1/p}.$$

$\varepsilon \to$ ~~δ~~ $E \to$ to F_n. [Correct order]. Done.

<u>Lemma</u>: $f_n, f \in L_1(\mu)$ $\forall g \in L_\infty(\mu)$ $\int g f_n \to \int g f$ then $\lim_{\varepsilon \to 0^+} \sup_{\substack{A \in E \\ \mu(A) < \varepsilon}} \sup_n \int_A |f_n| = 0$.

pf: Let λ_n be the measure on Σ $\lambda_n(E) = \int_E f_n\, d\mu$ by hyp $\lambda(E) = \int_E f\, d\mu$ satisfies $\forall E \in \Sigma$ $\lambda_n(E) \to \lambda(E)$. If the conclusion is false $\exists E_n$ $\mu(E_n) \to 0$ but $|\lambda_n(E_n)| \ge \varepsilon > 0$. This contradicts the Vitali-Hahn-Saks Lemma (see below) more detail latter

(<u>Vitali – Hahn – Saks Lemma</u>) If λ_n are (complex) measures on Σ_1 ~~$\lambda_n \ll \mu$~~ and if $\forall E \in \Sigma_1$ $\lambda_n(E) \to \lambda(E)$ then $\lim_{\mu(E)\to 0} \sup_n \lambda_n(E) = 0$.

$\forall \varepsilon \exists \delta_n$ $\mu(E) < \delta_n \Rightarrow \lambda_n(E) < \varepsilon$

<u>Proof</u>: Let $\Sigma_1' = \Sigma_1/\eta$ = Equivalence classes of measurable sets with $A \sim B \iff \mu(A \triangle B) = 0$ define a metric on Σ_1' via $d(A,B) = \arctan \mu(A\triangle B)$ $[\mu \geq 0]$. Σ_1' with this distance is a complete metric space and λ_n is continuous on this space. Thus $\Sigma_{n,m} = \{E \in \Sigma_1' \ \ |\lambda_n(E) - \lambda_m(E)| < \varepsilon\}$ are closed for m, n, ε and so is $\Sigma_N = \bigcap_{m,n \geq N} \Sigma_{n,m}$. Since $\lim \lambda_n(E)$ exists $\Sigma' = \bigcup_N \Sigma_N$.

Hence by Baire Category $\exists N$ s.t. Σ_N has interior point. Thus $\exists N, A, \delta$ s.t. $\mu(A \triangle B) < \delta$, $m,n \geq N \Rightarrow |\lambda_n(B) - \lambda_m(B)| < \varepsilon$. Choose δ s.t. $|\lambda_n(B)| < \varepsilon$ for $\mu(B) < \delta$ & $1 \leq n < N$. If $\mu(B) < \delta$ then $E = A \cup B$ or $A \setminus B$ satisfy $\mu(E \triangle A) < \delta$ so $(A\cup B) \setminus (A \setminus B) = B$ $\quad n \geq N$

$$\lambda_n(B) = \lambda_N(B) + (\lambda_n(B) - \lambda_N(B)) = \lambda_N(B) + \{\lambda_n(A\cup B) - \lambda_N(A \cup B)\} - \{\lambda_n(A\setminus B) - \lambda_N(A\setminus B)\}$$
$\leq 3\varepsilon$. done.

<u>back to the lemma</u>. Note that if the conclusion is false either $\lim_{\mu(E)\to 0} \sup_n \int_E |\mathrm{Re} f_n| \neq 0$ or $\lim_{\mu(E)\to 0} \sup_n \int_E |\mathrm{Im} f| = 0$ we may and do assume the former is false. Since

$$\int_E |\mathrm{Re} f| = \int_E (\mathrm{Re} f)^+ - \int_E (\mathrm{Re} f)^-$$ Thus $\exists A_n \subset E_n$ $\mu(E_n) \to 0$ s.t. $\left| \int_{A_n} f_n \right| \geq \varepsilon/2$. ~~Let $\nu(E) = \Sigma_1' \lambda_n(E)/2^n \chi(x)$ then $\lambda_n \ll \nu$~~

but this is impossible.

<u>Proof of (B3)'</u> we have (ii) of (C1); (i) & (iii) are true $\forall E \in \Sigma_1$ with $\mu(E) < \infty$ hence $\int_E |f_n - f| \to 0$ all such E. Since $f_n, f \in L_1(\mu)$ they have σ-finite supports so it suffices to prove this for X - μ-σ-finite, Let $E_m \nearrow$ $\bigcup E_m = X$ with $\mu(E_m) < \infty$. For each n, ε $\exists m$ s.t. $\int_{X\setminus E_m} |f - f_m| < \varepsilon/2$

hence

$$\| f_n - f\| \leq \int_{X \setminus E_m} |f - f_n| + \int_{E_m} |f_n - f| \leq \varepsilon/2 + \|(f_n - f)|_{E_m}\|$$

opps need another lemma.

Lemma: If $f_n \to f$ wky then $\forall E_n \supset E_{n+1}$ with $\bigcap E_n = \emptyset$ then $\lim_n \sup_m \int_{E_n} f_m = 0$

(now m can be picked independently of n, since $\bigcap (X \setminus E_m) = \emptyset$

Need another result for this.

We shall return to these questions in Section 5 or 6.

<u>Exercise</u>: Show $f_n \to f$, wky & in measure on μ-finite sets & $\|f_n\| \to \|f\|$ $\quad f_n, f \in L_1 \Rightarrow f_n \to f$.

§4 Measures.

I need this again, so we might as well do it now. Let (X, Σ, μ) be a measure space. We will construct a metric space $(\Sigma/\eta, d)$ as follows

(1) The points in Σ/η are equivalence classes of Σ measurable sets with the equivalence relation $A \sim B$ iff $\mu(A \Delta B) = 0$. [where $A \Delta B = (A \setminus B) \cup (B \setminus A) = B \Delta A$. Since $A \Delta A = \phi$ and $A \Delta C \subset (A \Delta B) \cup (B \Delta C)$ this is an equivalence relation] The η stands for null sets.

(2) We define the metric $d(A,B)$ in many different values

(i) if $\mu(X) < \infty$ $\quad d(A,B) = \mu(A \Delta B)$

(ii) if $\mu(X) = \infty$ $\quad d(A,B) = \arctan \mu(A \Delta B)$

Our first step is to show this are both metrics. We require the convention $\arctan \infty = \pi/2$. Clearly each d satisfies $d(A,B) \geq 0$; $d(A,B) = 0 \Leftrightarrow A \sim B$; $d(A,B) = d(B,A)$. To see the $\Delta \neq$ note that $\mu(A \Delta C) \leq \mu(A \Delta B) + \mu(B \Delta C)$ which shows (i) is a metric. To see that (ii) is a metric we use that arctan is increasing &

Lemma: $s, t \geq 0$ then $\arctan(s+t) \leq \arctan s + \arctan t$

pf: Clearly true if $s = 0$, $\forall t$. $f(s) = \arctan s + \arctan t - \arctan(s+t)$ so $f'(s) = \frac{1}{1+s^2} - \frac{1}{1+(s+t)^2} \geq 0$ for $s, t \geq 0$, so $f(s)$ is non-decreasing and lemma is true.

Remarks. Note that (ii) yields the same topology on X as (i) if $\mu(X) < \infty$ since arctan is strictly increasing. The only reason for (ii) is that (i) fails to be a metric for the trival reason that $d(A,B)$ could be infinite. In any case, the open balls of (i) are open balls of (ii) whereas the open balls of ii are either the whole space X or open balls of (i).

Lemma $(\Sigma/\eta, d)$ is complete

pf: Suppose $\{A_n\}$ is a C.S. ie. $\mu(A_n \Delta A_m) \to 0$ as $n, m \to \infty$. If $\mu(A_n) \to 0$ then $d(A_n, \phi) = \mu(A_n) \to 0$. Suppose $A_n \subset A_{n+1} \subset \ldots$ Then $A = \cup_n A_n$ is the limit, Since $A \Delta A_m = \cup_k (A_{k+m+1} \setminus A_m)$ and we can pass to a subsequence with $\mu(A_{m+1} \setminus A_m) \leq 2^{-m}$. Now consider the case $\{A_n\}$ where $\mu(A_n) \geq \varepsilon$ for each n. Again pick a subsequence so that $\mu(A_n \Delta A_{n+1}) < 2^{-n}$ and let $B_n = \bigcap_{m \geq n} A_m$. Clearly $B_n \subset B_{n+1}$

Further more $B_{n+1}\setminus B_n = (\bigcap_{m\ge n+1} A_m)\setminus(\bigcap_{m\ge n} A_m) \subset \bigcup_{m\ge n}(A_{m+1}\setminus A_m) \subset \bigcup_{m\ge n}(A_{m+1}\triangle A_m)$

hence $\mu(B_{n+1}\triangle B_n) \le \sum_{m\ge n}' 2^{-m} = 1/2^{m-1}$. Therefore

$$\mu(B_{n+k}\triangle B_n) \le \sum_{i=0}^{k-1}{}' \mu(B_{n+i+1}\triangle B_{n+i}) \le \sum_{i=0}^{\infty} 2^{-n-i-1} = 1/2^{n-2} \to 0$$

so that $\{B_n\}$ converges to $\bigcup B_n$. To complete the proof we will show that $A_n \to \bigcup B_n$ as well by showing $\mu(A_n\triangle B_n) \to 0$. Now

$$A_n\triangle B_n = A_n\setminus B_n = A_n \setminus \bigcap_{m\ge n} A_m \subset \bigcup_{m\ge n} A_n\setminus A_m \subset \bigcup_{m\ge n} A_n\triangle A_m$$

hence $\mu(A_n\triangle B_n) \le \sum_{m\ge n}' 2^{-m} = 1/2^{n-1} \to 0$.

Lemma. μ is continuous on $(\Sigma/\eta, d)$

pf: if $\mu(A\triangle B)<\varepsilon$ $B\subset A \uplus (A\triangle B)$ so $\mu(B)\le\mu(A)+\varepsilon$

similarly $\mu(A)\le\mu(B)+\varepsilon$.

Remark 1. Note that if $\mu(A)=\infty$ and $d(A,B) = \arctan\mu(A\triangle B) < \pi/2$ then $\mu(B)=\infty$.

2. If $A_n = \{t : r_n(t) = 1\}$ $A_n\subset[0,1]$ then $m(A_n\triangle A_m) = m(A_n\cap A_m) + m(A_n\cap \tilde{A}_m) = 1/4 + 1/4 = 1/2$ ${}_{m\ne n}$ since the sets $\{A_n\}$ are independent. Thus $\{A_n\}$ has no Cauchy subsequence and so $(\mathcal{B}/\eta, m)$ is not compact.

Lemma The operations : $\Sigma/\eta \times \Sigma/\eta \to \Sigma/\eta$ given by $(A,B)\to A\cup B$, $(A,B)\to A\cap B$ and $(A,B)\to A\triangle B$ are continuous, as is $:\Sigma/\eta\to\Sigma/\eta$ given by $A\to X\setminus A$.

pf: $(A\cup B)\triangle(A'\cup B') \le (A\triangle A')\cup(B\triangle B')$; $(A\cap B)\triangle(A'\cap B') \le (A\triangle A')\cup(B\triangle B')$;
$(A\triangle B)\triangle(A'\triangle B') \le (A\triangle A')\cup(B\triangle B')$; $(X\setminus A)\triangle(X\setminus B) = A\triangle B$.

Lemma Suppose λ is (signed, complex) measure on Σ and ~~…~~
i.e. $\forall\varepsilon>0\ \exists\delta>0$ $\mu(A)<\delta \Rightarrow |\lambda(B)|<\varepsilon$ ~~$\mu(A)=0 \Rightarrow \lambda(B)=0$~~ then λ is cont on $(\Sigma/\eta, d)$.

pf: This condition is required so that λ is well-defined on Σ/η. (i.e. it implies that $\mu(A)=0 \Rightarrow \lambda(A)=0$). ~~Thus $\lambda(A)\le\lambda(B)+\lambda(A\triangle B)$~~

$\lambda(E)-\lambda(F) = \lambda(E\setminus E\cap F) - \lambda(F\setminus E\cap F)$ which is "small" when $|\mu(E\triangle F)|$ is small

Remark $[\mu(A)=0 \Rightarrow \lambda(B)=0] \not\Rightarrow [\forall\varepsilon>0\ \exists\delta>0\ \mu(A)<\delta \Rightarrow |\lambda(B)|<\varepsilon]$ in general. ~~if $\mu(X)<\infty$ but not if X is just σ-finite~~ [ie. $X=\mathbb{N}$ λ = counting $\mu(i)=2^{-i}$].

Lemma: The map $(\Sigma/\eta) \to$ Meas fns on X is a homeomorphism $(A\to\chi_A)$ if the latter space is given the metric "conv in meas"

pf $A_n\to A$ iff $\mu(A_n\triangle A)\to 0$ iff $\mu\{x : |\chi_{A_n}-\chi_A| \ge 1/2\} \to 0$

In $(\Sigma/\eta, \mu)$ we can define a partial order by $A \le B$ if $A \subseteq B$. Passing to equivalence classes $A \le B$ iff $\mu(A \setminus B) = 0$ note that $\max(A,B) = A \cup B$, $\min(A,B) = A \cap B$ and that $X \setminus A = B$ is the unique element s.t. $A \cap B = \emptyset$ & $A \cup B = X$.

A lattice is a set X, with a partial order $\le$ so that $a \wedge b = \min\{a,b\}$; and $a \vee b = \max\{a,b\}$ are defined for each $a,b \in X$. All our function spaces $L_p(\mu)$ are lattices.

$(\Sigma/\eta, \mu)$ can be also viewed as a Boolean ~~Algebra~~ Ring with unit. with $A \odot B = A \cap B$ and $A \oplus B = A \Delta B$. Clearly $A \odot A = A$, $A \oplus A = X = \text{unit}$. There is a Theorem (due to Stone) which we may need.

Thm Each Boolean ring with unit is isomorphic with the collection of open and closed subsets of a totally disconnected compact T_2-space

pf: Dunford & Schwartz pp41–43 (see also 43–44)

Totally Disconnected means it has a base for the topology which is both open and closed. Examples include the Cantor set, 2^{ω}, 2, 2^{Γ},

A map $\Phi : (\Sigma/\eta, \mu) \to (\Omega/\eta, \nu)$ is said to be an isomorphism if $\Phi(X \setminus A) = Y \setminus \Phi(A)$; $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ and $\mu(A) = \nu(\Phi(A))$. (Σ/η) is said to be separable if the metric has a countable dense subset.

Thm If $\mu(X) = 1$ & μ is separable then $\exists\, \Phi\ (\Sigma/\eta, \mu) \xrightarrow{\text{into}} (\mathcal{B}|[0,1], m)$ which is onto if μ has no atoms.

pf: Let $\{A_n\}$ be dense in Σ/η, Let Σ_n = σ-algebra generate $\{A_1 \cdots A_n\}$. Let $\Sigma_\infty = \bigcup \Sigma_n$ it is easy to check Σ_∞ is an algebra. We will define Φ inductively on Σ_n. Now $\Sigma_1 = \{\emptyset, A_1, X \setminus A_1, X\}$ define $\Phi(\emptyset) = \emptyset$, $\phi(X) = [0,1]$, $\phi(A_1) = [0,s]$ where $s = \mu(A_1)$ & $\phi(X \setminus A_1) = (s,1]$. Clearly this is an isomorphism. Suppose Φ has been defined to be an isomorphism on Σ_m. Now we may assume Σ_m = σ-algebra generated by $\{B_1^m, \dots, B_{i(m)}^m\}$ where the B_i^m's are in Σ_m, are p.w.d & whose union is X. We may also assume by induction that $\Phi(B_i^m) = (b_{i-1}^m, b_i^m]$ where $b_0^m = 0$ $b_{i(m)}^m = 1$ and $\{b_i^m\}$ is a partition of $[0,1]$. For Σ_{m+1} we get at most $\{A_{m+1} \cap B_1^m, A_{m+1}^c \cap B_1^m, \dots,\}$ $2i(m)$ p.d. sets whose union is all of X. map $\Phi(A_{m+1} \cap B_i^m)$ in left part of interval (b_{i-1}^m, b_i^m)

and $\Phi(A^c_{m+1} \cap B^m_i)$ into remaining right half. It seems reasonable and it is true that this extends the isomorphism to Σ_{m+1}.

Now Φ is an isomorphism on Σ'_∞ since $A, B \in \Sigma_\infty \Rightarrow \exists m$ s.t. $A, B \in \Sigma_m$ hence $A \cap B$, $A \cup B$, $X \setminus A \in \Sigma_m$. Thus $\Phi : \Sigma_\infty/\eta \subset \Sigma/\eta$ into $\mathcal{B}([0,1])/m$ is an isometry defined on a dense subset of Σ/η, We can use continuity to extended to all of Σ/η (since $\mathcal{B}([0,1])/m$ is complete.) Since $\{A : \mu(A) = \nu\Phi(A)\}$ is closed $= (\mu - \nu\varphi)^{-1}(0)$ and contains a dense set it is all of Σ/η, so that $\mu(A) = \nu(\Phi(A))$. If $A, B \in \Sigma'$ and $A_n, B_n \subset \Sigma_\infty$ s.t. $A_n \to A$, $B_n \to B$ then $\Phi(A_n) \to \Phi(A)$ & $\Phi(B_n) \to \Phi(B)$ also $A_n \cup B_n \to A \cup B$, $A_n \cap B_n \to A \cap B$, $X \setminus A_n \to X \setminus A$. so that

$\Phi(A \cup B) = \lim \Phi(A_n \cup B_n) = \lim \Phi(A_n) \cup \Phi(B_n) = \Phi(A) \cup \Phi(B)$ etc hence Φ is an isomorphism.

Suppose (Σ'/η) has no atoms, ie. If $\mu(A) > 0$ $\exists$ $B \subset A$ s.t. $\mu(A) > \mu(B) > 0$. We want to show $\Phi(\Sigma'_\infty)$ is dense in $\mathcal{B}_{[0,1]}/\eta_\lambda$. It suffices to show $\lim_m \underbrace{\max_{0 \le i \le i(m)} b^m_i - b^m_{i-1}}_{\alpha_m} = 0$ For then this collection of intervals generates the Borel sets. Since B^m_i is not an atom $\exists C$ s.t. $\mu(B^m_i) > \mu(C) > 0$. Since $\{A_i\}$ is dense $\exists$ A_j s.t. $\mu(A_j \triangle C) < \frac{1}{2}\mu(C), \frac{1}{2}\mu(B^m_i \setminus C)$ hence $\mu(B^m_i) > \mu(A_j \cap B^m_i) > 0$ and so α_m decreases and $\forall m \exists n$ s.t. $\alpha_n < \alpha_m$. Suppose $\alpha_m \to \varepsilon > 0$. Then there must be a sequence $B^m_{j(m)} \supset B^{m+1}_{j(m+1)} \ldots$ with $b^m_{j(m)} - b^m_{j(m+1)} \ge \varepsilon$. Let $B = \bigcap B^m_{j(m)}$ $\mu(B) = \lim B^m_{j(m)} \ge \varepsilon$. Since B is not an atom. $\exists C$ $\mu(B) > \mu(C) > 0$ and again A_j with $\mu(B) > \mu(B \cap A_j) > 0$. But $B \subset B \cap B^j_{j(j)} \subset B \cap A_j$ $\divideontimes$. So we are done.

the above trick applied to Φ^{-1} then we works

Thm: If $\mu(X) < \infty$ ~~then~~ ~~&~~ μ separable, then $L_p(\mu)$ is order isomorphic with a subspace of L_p, all of it if it has no atoms.

pf: Since $\nu(A) = \mu(A)/\mu(X)$ satisfies $\nu(X) = 1$ & $L_p(\mu)$ is order isometric to $L_p(\nu)$ we may assume $\mu(X) = 1$. Let $\Phi : \Sigma/\eta\, \mu \to \mathcal{B}[0,1]/m$ given above. Define $\tilde{\Phi} : L_p(\mu) \to L_p$ by $\tilde{\Phi}(\chi_A) = \chi_{\Phi(A)}$ extend by linear & continuity ($\tilde{\Phi}$ is an isometry & preserves order.) $\tilde{\Phi}(L_p(\mu))$ contains a dense subset if Φ is onto, Done.

Cor If μ is separable & σ-finite then $L_p(\mu)$ is order isometrical with a subspace of L_p (all of it if it has no atoms)

pf: X = pwd union X_n $\quad L_p(\mu) = \ell_p$ sum $L_p(X_n) \cong \ell_p$ sum $L_p \cong L_p$.

<u>Lemma</u>: If $f: X \to \mathbb{K}$ is a function which is Σ_1'-meas. ~~Then~~ $f^{-1}(\mathcal{B})$ is generated by countably many elements.

<u>Lemma</u>: If $\{f_n\} \subset L_p(\mu)$, $p<\infty$ then $[f_n]$ is order isometric to a subspace of L_p.

<u>pf</u>: Let $\Sigma_0 = \sigma$-algebra generated by $\{g^{-1}(B): B \text{ borel}, g \in [f_n]\}$ Since we may assume μ is σ-finite [restricting to $\bigcup$ support f_n if necessary] it suffices to show Σ_0' is separable. It also suffices to use only $g = f_n$ n some n since sums & limits of meas fcns are meas. [conv in norm $\Rightarrow$ conv in meas $\Rightarrow$ subseq conv pt-wise]. It suffices to do this with simple f_n [otherwise let $s_m^n \to f_n$ $m \to \infty$, and note $[f_n] \subset [s_m^n]$]. Hence for each f_n we obtain finitely many sets in Σ_0'. Taking nested unions we obtain countable Σ_∞ which is an algebra, s.t. Σ_∞' is σ-algebra generated by Σ_∞. Using outer measures Σ_∞' is dense in Σ_0.

<u>Remark</u> It will be useful to note that s_m^n have support on a set of finite meas hence Σ_∞ & Σ_0' can be generated with sets of finite meas.

Suppose (X, Σ_1', μ) is not separable. Pick $\{A_\alpha, \alpha \in \Gamma\}$ dense in Σ/η s.t. Card Γ is minimal. Let $(I^\Gamma, \mathcal{B}^\Gamma, m)$ be the product meas. It can be shown that $\Sigma/\eta,\mu$ is isomorphic with a sub algebra of $(\mathcal{B}^\Gamma/\eta, m)$ By basically the same proof using transfinite induction. What is needed is an induction step which can illustrated by Let $\Sigma_0' = $ cl alg generate $\{A_n\}_{n=1}^\infty$ and trying to add one more element, A_ω. Let $\Sigma_0 \cong \mathcal{B}/\Gamma \times I$ we want for $t \in I$ $\{t, s\} \in \Phi(A_\omega)$ if $\mu(\Phi^{-1}(t) \cap A_\omega) = \lambda_t$ & $0 \le s \le \lambda_t$. Needless to say there are alot of details.

<u>Thm</u>: Suppose μ_n are finite measures on Σ_1' and for each $E \in \Sigma_1'$ $\mu(E) = \lim_{n\to\infty} \mu_n(E)$ exists. Then μ is countably additive & ~~$\forall \varepsilon\ \exists \delta$ s.t. $\mu(E) < \delta \Rightarrow \sup \mu_n(E) < \varepsilon$~~ if $\cap E_n = \phi$ $E_n \downarrow \phi$ then $\lim_m \sup_n \mu_n(E_m) = 0$

<u>pf</u>: Let $\lambda_n = |\mu_n| / |\mu_n|(X)$ and $\lambda = \sum_1^\infty \lambda_n 2^{-n}$ then everything is λ-continuous so Vitali Hahn Saks we have the second conclusion, which implies the first ($\mu(X) < \infty$)

Thm: If (X, Σ, μ) has no atoms and $\mu(A) < \infty$, then $\exists$ σ-algebra $\Sigma_0 \subset \Sigma_A$ s.t. $\Sigma_0/\eta\ \mu$ is isomorphic to $\mathcal{B}_{[0,\mu(A)]}/\eta$. m.

pf: Step 1 The (*) statement is false where
(*) $\forall B \in \Sigma_A$ either $\mu(B) < \frac{1}{3}\mu(A)$ or $\mu(B) > \frac{2}{3}\mu(A)$

Let $\Omega = \{B \in \Sigma_A : \mu(B) < \frac{1}{3}\mu(A)\}$. If $B, C \in \Omega$ then $\mu(B \cup C) \le \mu(B) + \mu(C)$ $< \frac{2}{3}\mu(A)$ hence by (*) $B \cup C \in \Omega$.
Let $\lambda = \sup\{\mu(B) : B \in \Omega\}$ The value λ is assumed, if $B_n \in \Omega$ $\mu(B_n) > \lambda - \frac{1}{n}$, then $C_n = \bigcup_1^n B_m \in \Omega$, $C_n \uparrow$, $\lambda - \frac{1}{n} \le \mu(C_n) \le \lambda$ so $C = \bigcup_1^\infty C_n \in \Omega$ and $\mu(C) = \lambda$.

Now we obtain a contradiction since $A \setminus C$ is not an atom $\exists\ B$ $0 < \mu(B) \le \frac{1}{2}\mu(A \setminus C) \le \frac{1}{2}\mu(A) - \frac{1}{2}\mu(C) \le \frac{1}{2}\mu(A)$ so $B \in \Omega$ but then $B \cup C \in \Omega$ and $\lambda \ge \mu(B \cup C) = \mu(B) + \mu(C) > \lambda$ ※

Step 2. If $0 \le \mu(B) \le \frac{1}{2}\mu(A)$, then $\exists\ C \supset B$ $[\frac{1}{2}\mu(A) - \mu(C)] \le \frac{2}{3}[\frac{1}{2}\mu(A) - \mu(B)]$
Let $C_1 = A \setminus B$. Let $D_1 \subset C_1$ with $\frac{1}{3}\mu(C_1) \le \mu(D_1) \le \frac{1}{2}\mu(C_1)$
Inductively define $C_{n+1} = D_n$ & $D_{n+1} \subset C_{n+1}$ with $\frac{1}{3}\mu(C_{n+1}) \le \mu(D_{n+1})$ $\le \frac{1}{2}\mu(C_{n+1})$. We have $(\frac{1}{3})^n \mu(C_1) \le \mu(D_{n+1}) \le (\frac{1}{2})^n \mu(C_1) \to 0$
Since $\mu(B \cup C_1) = \mu(A)$, there is an n with $\mu(B \cup C_n) > \frac{1}{2}\mu(A)$ and $\mu(B \cup C_{n+1}) \le \frac{1}{2}\mu(A)$. If $\frac{1}{2}\mu(A) - \mu(B \cup C_{n+1}) > \frac{2}{3}[\frac{1}{2}\mu(A) - \mu(B)]$ then $\mu(B \cup C_n) \le \frac{1}{2}\mu(A)$ ※.

Step 3 $\exists\ B \in \Sigma_A$ $\mu(B) = \frac{1}{2}\mu(A)$. By step 2 $\exists\ B_n \subset B_{n+1}$ with $\mu(B_n) \le \frac{1}{2}\mu(A)$ & $\frac{1}{2}\mu(A) - \mu(B_{n+1}) \le \frac{2}{3}[\frac{1}{2}\mu(A) - \mu(B_n)]$ hence $B = \bigcup_1^\infty B_n$ works.

Step 4: Write $n = 2^k + \xi$: $0 \le \xi < 2^k$, $k = 0, 1, 2, \ldots$ Let $A_1 = A_0$ if A_n is defined $n = 2^k + \xi$ then $\exists\ B \subset A_n$ $\mu(B) = \frac{1}{2}\mu(A_n)$ let $A_{2^{k+1}+2\xi} = B$, $A_{2^{k+1}+2\xi+1} = A_n \setminus B$. Then $\Sigma_0 = \sigma$-alg gen by $\{A_n\}$ is separable and clearly isomorphic $\mathcal{B}_{[0,\mu(A)]}/\eta$, m.

Thm: If $L_p(\mu)$, $1 \le p < \infty$ is separable, then $L_p(\mu)$ is order isometric to L_p, $L_p \oplus_p \ell_p^n$, $L_p \oplus \ell_p$, ℓ_p^n or ℓ_p.

pf: Separate out the atoms. There are at most countably many of them & what is left is isomorphic to $\{0\}$ or L_p.

§5 Conditional Expectations and Martingales.

At the moment there are two things wrong we the ~~last~~ section 3. The first is there are some proofs we omitted (to be done latter). The second is, except for the results we have not proved, all the theorems required that the limit functional already belongo to $L_p(\mu)$. What is wrong with this? Well, in lots of cases it is generality easier to show that a sequence converges to its limit, then to find a candidate for the limit.

But first (as always a few sidetracks). Let (X, Σ, μ) be a positive measure with $\mu(X) < \infty$ and let $(A_i)_{i=1}^n$ be a partition of X. That is the $\{A_i\}$ are p.w.d. and $X = \cup_{i=1}^n A_i$ and of course each $A_i \in \Sigma$.

Consider the map T which takes the measurable function f onto $\Sigma_{i=1}^n$ (average f on A_i) χ_{A_i}. By the average of f on A_i we mean the freshman calculus type average namely aver f on $A_i = [\mu(A_i)]^{-1}\int_{A_i} f\,d\mu = E(f|A_i)/\mu(A_i)$. To keep from dividing by zero, let us assume $\mu(A_i) \neq 0$ for $i = 1, \dots n$.

Now $\int_{A_i} Tf\,d\mu = \int_{A_i}$ aver f on $A_i\, \chi_{A_i}\,d\mu = \int_{A_i} \frac{1}{\mu(A_i)}\int_{A_i} f\,d\mu\, \chi_{A_i}\,d\mu$

$= \frac{1}{\mu(A_i)}\int_{A_i} f\,d\mu \int_{A_i} \chi_{A_i}\,d\mu = \int_{A_i} f\,d\mu$. Thus for $B = A_i$ $i = 1, \dots n$ we have:

(1) $\int_B Tf\,d\mu = \int_B f\,d\mu$

it is also easy to check that (1) remains true if B is allowed to range over the smallest σ-algebra generated by $\{A_1, \dots A_n\}$.

Other simple facts about T include:

(2) T is a positive operator. That is if $f \geq 0$ then $Tf \geq 0$.

(3) T maps $L_1(\mu)$ into $L_1(\mu)$ and $\|T\| = 1$

Since if $\|f\|_1 \leq 1$ $\|f\|_1 = \int |f|\,d\mu = \Sigma_{i=1}^n \int_{A_i} |f|\,d\mu$. Now $|f| \geq f \geq -|f| \Rightarrow \int_{A_i} |f|\,d\mu \geq \int_{A_i} f\,d\mu \geq -\int_{A_i} |f|$ hence $\int_{A_i} |f|\,d\mu \geq |\int_{A_i} f\,d\mu|$ so that

$\|f\|_1 \geq \Sigma_1^n |\int_{A_i} f\,d\mu| = \Sigma_1^n |\int_{A_i} f\,d\mu/\mu(A_i)|\,\mu(A_i) = \Sigma_1^n \int_{A_i} |Tf|\,d\mu = \|Tf\|_1$

This shows $\|T\| \leq 1$. To see $\|T\| = 1$ apply it to $f = \chi_{A_1}$ or use:

(4) T is a projection.

It suffices to show $T = \text{iden}$ on range $T = \{\sum_1^n \alpha_i \chi_{A_i}\}$

$$E(\textstyle\sum_1^n \alpha_i \chi_{A_i}, A_j) = \int_{A_j} \sum \alpha_i \chi_{A_i} d\mu = \alpha_j \int_{A_j} \chi_{A_j} d\mu = \alpha_j \mu(A_j)$$

hence $T(\sum \alpha_i \chi_{A_i}) = \sum_i [\alpha_j \mu(A_j)/\mu(A_j)] \chi_{A_i} = \sum \alpha_i \chi_{A_i}$

(5) T maps $L_\infty(\mu)$ into $L_\infty(\mu)$ and $\|T\| = 1$

Note that $\|f\chi_{A_i}\|_\infty \geq \int_{A_i} f\, d\mu \geq -\|f\chi_{A_i}\|_\infty$ if $\mu(A_i) \leq 1$
in fact $\|f\chi_{A_i}\|_\infty \geq |\int_{A_i} f d\mu/\mu(A_i)|$. To see this, $\|f\chi_{A_i}\|_\infty \geq |f\chi_{A_i}|$
so $\int_{A_i} \|f\chi_{A_i}\|_\infty \geq |\int_{A_i} f\chi_{A_i}|$ or $\|f\chi_{A_i}\|_\infty \mu(A_i) \geq |\int_{A_i} f d\mu|$.
And so

$$\|Tf\|_\infty = \sup_{1 \leq i \leq n} \|\textstyle\int_{A_i} f d\mu/\mu(A_i)\|_\infty \leq \sup_{i \leq i \leq n} \|f\chi_{A_i}\|_\infty = \|f\|_\infty .$$

(6) T maps $L_p(\mu)$ into $L_p(\mu)$ and $\|T\| = 1$ for $1 \leq p \leq \infty$

Later we will show that (3) and (5) imply (6). For now note that $\|\sum_1^n \alpha_i \chi_{A_i}\|_p = (\sum_1^n |\alpha_i|^p \mu(A_i))^{1/p}$, and that (since A_i p.w.d. partition) $\|f\|_p = (\sum_1^n \|f\chi_{A_i}\|_p^p)^{1/p}$ so it suffices to show $\|f\chi_{A_i}\|_p^p \geq |\alpha_i|^p \mu(A_i)$ for $\alpha_i = \int_{A_i} f d\mu$. i.e. we want $\int_{A_i} |f|^p \geq |\int_{A_i} f|^p \mu(A_i)$. By Holder's inequality

$$\|g\|_1 \leq \|g\|_p \|\chi_B\|_q \quad \text{where } B = \text{supp}\, g \quad \tfrac{1}{q} + \tfrac{1}{p} = 1$$

Thus $|\int_{A_i} f d\mu \chi_{A_i}| = |\int_{A_i} f d\mu| \mu(A_i) \leq (\int_{A_1} |f\chi_{A_i}|)\mu(A_i) \leq (\int_{A_1} |f|^p)^{1/p} (\mu(A_i))^{1/q}$

since $1 - \frac{1}{q} = \frac{1}{p}$, $|\int_A f d\mu| \mu(A_i)^{1/p} \leq (\int_{A_i} |f|^p)^{1/p}$, taking p-th powers yields the result.

(7) The range of T in $L_p(\mu)$ is order isometric to ℓ_p^n (and L_p^n)

This comes from the introduction §0.

(8) There is no such result for $0 < p < 1$. Suppose $\mu(A) = \mu(B) = 1$
$A \cap B = \emptyset$ $A \cup B = X$ and let $A_1 = X$. Consider $g = 2\chi_A$ then
$Tg = \chi_X$ $\|Tg\|_p = \mu(X)^{1/p} = 2^{1/p} > 2$. But $\|g\|_p = (\int 2^p \chi_A)^{1/p} = (2^p)^{1/p} = 2$.

(9) Note that $\mu(A_i) < \infty$ is important for if $\int_{A_i} f d\mu$ is non-zero finite number there is no constant α so that $\int_{A_i} \alpha d\mu$ is equal to it.

(10) Note that $\mu(A_i) > 0$ is not important. If $\mu(A_i) = 0$ any constant will work and it will not change the norm of anything around.

Let us abstract this construction.

Let (X, Σ_1', μ) be a positive meas and $\Sigma_1'' < \Sigma_1'$ a sub-σ-algebra and let f be Σ_1'-meas. We define $E(f, \Sigma_1'') = g$ if

(i) g is Σ_1''-meas

(ii) $\forall A \in \Sigma_1''$ $\mu(A) < \infty$ $\int_A g \, d\mu = \int_A f \, d\mu$.

We will also consider the stronger condition

(ii') $\forall A \in \Sigma_1''$ ~~$\mu(A) < \infty$~~ $\int_A g \, d\mu = \int_A f \, d\mu$.

The second sense has an advantage over the first in that it is unique if it exists. (Since $\int_A g \, d\mu = 0$ $\forall A \in \Sigma'$ $\Rightarrow g = 0$ a.e.) To see that the first sense is non-unique suppose Σ' has an infinite atom, i.e. $A \in \Sigma_1''$ $\mu(A) = \infty$ and $\forall B \subset A$ $B \in \Sigma_1''$ $\Rightarrow \mu(B) = 0$ or $\mu(B) = \infty$. Then g could be any constant on A and ~~satisfies~~ (ii). Note that ~~this~~ this could happen even if (X, Σ_1', μ) is σ-finite $\{$ i.e. $\Sigma' = \{X, \emptyset\}$ $\}$. We have However

<u>Lemma 1.</u> Let (X, Σ_1', μ) be a pos. meas. and let $\Sigma_1'' \subset \Sigma_1'$ be a sub σ-alg, then the following are equivalent

(1) $\mu | \Sigma'$ is σ-finite

(2) $\exists$ partition $\{A_n\}_{n=1}^{\infty} \subset \Sigma'$ with $\mu(A_n) < \infty$ $X = \cup A_n$

(3) $\mu | \Sigma_1''$ has no infinite atoms and (X, Σ_1', μ) is σ-finite

proof: We know (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). Suppose (3) and let $\{X_n\} \subset \Sigma_1'$ s.t. $\mu(X_n) < \infty$ $X = \cup X_n$ and X_n are pairwise disjoint. Define $\lambda_n = \sup\{\mu(X_n \cap F) : F \in \Sigma_1''$ F is Σ', μ-σ-finite$\}$. There is $F_n \in \Sigma'$, F_n Σ_1'', μ σ-finite so that $\mu(X_n \cap F_n) = \lambda_n$. Indeed if $G_n^i \in \Sigma'$, G_n^i Σ', μ σ-finite so that $\mu(X_n \cap G_n^i) > \lambda_n - \frac{1}{i}$ then $F_n = \cup_i G_n^i$ works. Thus $F = \cup_n F_n$ is Σ', μ σ-finite and $\mu(X_n \cap F) = \lambda_n$ for each n. Let $G = X \setminus F$ and let $H \subset G$ be so that $\mu(H) < \infty$. Now $\mu(H \cap X_n) = 0$, otherwise $F \cup H$ is Σ' μ-σ-finite and $\lambda_n \geq \mu(X_n \cap (F \cup H)) = \mu(X_n \cap F) + \mu(X_n \cap H) > \lambda_n$ a contradiction. Since $\mu | \Sigma_1''$ has no infinite atoms $\mu(G) = 0$ & $X = F \cup G$ is Σ_1' μ-σ-finite.

Remark (X, Σ_1', μ) σ-finite is necessary for (3). Otherwise X uncountable, $\Sigma = \Sigma' = \mathcal{P}(X)$, μ = counting meas is a counterexample.

Lemma 2. $E(f, \Sigma_1')$ is unique [a.e.] if μ/Σ_1' has no infinite atoms.

pf: Suppose g, h satisfy (i) & (ii) and consider $f = g - h$ we have $\forall A \in \Sigma_1'$ $\mu(A) < \infty \Rightarrow \int_A f\,d\mu = 0$. Since f is Σ'-measurable & (in real case) $\exists \varepsilon > 0$ s.t. $A_\varepsilon = \{x : f(x) \geq \varepsilon\}$ has $\mu(A_\varepsilon) > 0$ & $A_\varepsilon \in \Sigma'$ if $f \neq 0$ a.e. (modulo a sign). If $\mu(A_\varepsilon) < \infty$ we have a contradiction. If $\mu(A_\varepsilon) = \infty$, by hyp $\exists B \in \Sigma'$ & $\mu(B) < \infty$ $B \subset A$ and again we have a contradiction.

Lemma 3. Under the conditions of lemma 1 and if $f \in \text{Local-}L_1(\mu)$ then $E(f, \Sigma')$ exists, and in the second sense if $f \in L_1(\mu)$.

pf: Reminder that $f \in \text{Local } L_1(\mu) \iff \int_A |f|\,d\mu < \infty$ for each $A \in \Sigma$ with $\mu(A) < \infty$.

Suppose first $f \in L_1(\mu)$ define ν on Σ_1' by $\nu(E) = \int_E f\,d\mu$ ν is a countable additive (complex or signed) meas on Σ_1' with the property that $\mu(E) = 0 \Rightarrow \nu(E) = 0$ for all $E \in \Sigma'$. Thus $\nu \ll \mu/\Sigma_1'$ so by Radon-Nikodym Thm [Royden p. 238] (Here is where hyp of lemma 1 is used) $\exists g \in L_1(\mu)$ g Σ_1'-meas s.t. $\forall E \in \Sigma'$ $\nu(E) = \int_E g\,d\mu$. That is $g = E(f, \Sigma')$.

Now let $f \in \text{Local-}L_1(\mu)$ and let $A \in \Sigma'$ $\mu(A) < \infty$. Using the above we obtain g_A that works for $f\chi_A$. Note by uniqueness $g_A = g_B$ a.e on $A \cap B$ for $A, B \in \Sigma_1'$ $\mu(A), \mu(B) < \infty$. Thus we can glue all the g_A's together to obtain g. σ-finiteness is again used to show g is Σ_1'-meas. hence $g = E(f, \Sigma')$ in the weaker sense. ~~Clearly~~

Lemma 4. $E(f, \Sigma_1')$ is a positive operator, if the conditions of lemma 3 hold

pf: If $f \geq 0$ but $g = E(f, \Sigma)$ is not then $\{x : g(x) \leq -\varepsilon\}$ $\varepsilon > 0$ has positive measure & is in Σ' a contradiction

Corollary 5 If the conditions of lemma 3 hold, then $\forall A \in \Sigma'$ $\mu(A) < \infty$ we have $\int_A |E(f,\Sigma')| \, d\mu \le \int_A |f|$ & $\|E(f,\Sigma')\|_1 \le \|f\|_1$

pf: $|E(f,\Sigma')| \le E(|f|,\Sigma')$, since it is positive.

Lemma 6 If (X,Σ,μ) ~~is σ-finite~~ is meas $L_p(\mu) \subset \text{Local-}L_1(\mu)$ for $1 \le p \le \infty$. ~~and~~ Furthermore $\text{Local-}L_1(\mu) \subset L_1(\mu) + L_\infty(\mu)$ if (X,Σ,μ) is σ-finite and separable and purely non-atomic

pf: $p=\infty$ is obvious as is $p=1$, Suppose $1<p<\infty$ & let q be s.t. $\frac{1}{p}+\frac{1}{q}=1$. then $|\int_A f \, d\mu| \le \int |f\chi_A| \, d\mu \le \|f\|_p \|\chi_A\|_q < \infty$.
Now let (X,Σ,μ) be σ-finite & separable. If μ is atomic $\text{Local-}L_1(\mu) =$ set of all real-value measurable functions. which is not contained in $L_1(\mu) + L_\infty(\mu)$. However under the given conditions we may assume $(X,\Sigma,\mu) = (\mathbb{R},\mathcal{B},m)$. Suppose $f \ge 0$ & $f \in \text{Local-}L_1(\mu)$. Let $A_n = \{x : n > f(x) \ge n-1\}$ for $n = 1,2,\ldots$. If $\infty > \int_{\bigcup_{n=m}^{\infty} A_n} f \, d\mu \ge \sum_{n=m}^{\infty} n\,\mu(A_n)$ we are done.

So suppose $\sum_{n=m}^{\infty} n\,\mu(A_n) = \infty$ for each m. (Exercise show $\exists\, b_n$ $0 \le b_n \le \mu(A_n)$ s.t. $\sum b_n < \infty$ but $\sum_1^{\infty} (n-1) b_n = \infty$)
Let $B_n \subset A_n$ $\mu(B_n) = b_n$ then $\mu(\bigcup B_n) < \infty$ but $\int_{\bigcup B_n} f \ge \sum (n-1) b_n = \infty$ which contradicts $f \in \text{Local-}L_1(\mu)$.

Corollary 7. If $f \in L_\infty(\mu)$, then $E(f,\Sigma') \in L_\infty(\mu)$ and $\|E(f,\Sigma')\|_\infty \le \|f\|_\infty$ provided the conditions of lemma 1 hold.

pf: Assume $f \ge 0$ (replace f by $|f|$) & let $g = E(f,\Sigma')$ assume $A = \{x : g(x) > M\} \in \Sigma'$ has positive μ-meas. By passing to a subset we may assume $\mu(A) < \infty$. Hence $\mu(A) M \le \int_A f \, d\mu \le \int f \chi_A \, d\mu \le \|\chi_A\|_1 \|f\|_\infty = \mu(A) \|f\|_\infty$. So that $\|f\|_\infty \ge M$. Hence $\|f\|_\infty \ge \|g\|_\infty$.

Proposition 8. If $1 \le p \le \infty$, and the conditions of lemma 1 hold $E(\cdot,\Sigma')$ is a norm one operator from $L_p(\mu)$ into $L_p(\mu)$.

pf: Let $f \in L_p(\mu)$, then $g = E(f,\Sigma')$ exists and we may assume that $f, g \ge 0$. Suppose $\|g\|_p \ge M$. Then letting $\frac{1}{p}+\frac{1}{q}=1$ $\exists\, h \in L_q(\mu,\Sigma')$ $h \ge 0$ s.t. $\|h\|_q = 1$ and

$\int hg\,d\mu > M$. Since the simple fcn's are dense in $L_q(\mu,\Sigma')$ there is a simple s which is Σ'-meas ≥ 0 $\|s\|_q \leq 1$ and $\int sg\,d\mu > M$. Since $\|s \in L_q(\Sigma,\mu)\|_q = \|s \in L_q(\Sigma',\mu)\|_q$

$$\int sf\,d\mu = \int \Sigma \alpha_i \chi_{A_i} f\,d\mu = \Sigma\alpha_i \int_{A_i} f\,d\mu = \Sigma\alpha_i \int_{A_i} g\,d\mu = \int sg > M$$

Thus $\|f\|_p \geq M$. Hence $\|f\|_p \geq \|g\|_p$.

<u>Corollary 9</u>. L_p is a ~~com~~ norm one complemented subspace of each finite not purely atomic $L_p(\mu)$. ℓ_p is a norm one complemented subspace of each infinite dimensional $L_p(\mu)$

<u>pf:</u> Take a separable non-atomic $\Sigma' < \Sigma_1'$ and $E(\cdot,\Sigma_1')$ for the first. For the second let A_n be s.t. $0 < \mu(A_n) < \infty$. and A_n p.w.d. First restricted to $\Sigma_{\cup A_n}$ then do $E(\cdot\,\Sigma')$ where $\Sigma' = \sigma$-alg generated by $\{A_n\}$.

<u>Lemma $8\frac{1}{2}$</u> $E(\cdot\,\Sigma')$ is a projection on $L_1(\mu)$ if the conditions of lemma 1 holds

pf. $E(E(f,\Sigma'),\Sigma') = E(f,\Sigma')$ by uniqueness.

<u>Definition</u> Let $\mathcal{B}_1 < \mathcal{B}_2 \subset \mathcal{B}_3 \subset \ldots$ be a finite or infinite sequence of sub-σ-algebra's ~~with~~ of Σ_1' and let (X,Σ,μ) be a Probability space

A sequence of functions $\{f_n\}$ is said to be a martingale (with respect to $\{\mathcal{B}_n\}$) if
$$E(f_{n+1},\mathcal{B}_n) = f_n.$$

<u>Lemma</u> $E(f_{n+k},\mathcal{B}_n) = f_n$. $\quad k = 0,1,2,\ldots$

<u>pf:</u> by uniqueness.

<u>Corollary</u> If $\{f_n\}$ is martingale w/resp $\{\mathcal{B}_n\}$ then $\{f_{n_i}\}$ is martingale w/resp $\{\mathcal{B}_{n_i}\}$.

We will restrict most of our attention to $[0,1]$ although proofs are more general.

Let us go back to section 3. And consider $f_n, f \in L_p$ $1<p<\infty$ we had if $f_n \to f$ wkly & $\|f_n\|_p \to \|f\|_p$ then $f_n \to f$ in L_p-norm. What is the trouble with this? Well it assumes we already have a candidate f for the limit and that this $f \in L_p$. Normally, we would just have the sequence $\{f_n\} \subset L_p$ & we would be happy to just know that it has a limit, mostly not caring what it is (the limit)

Well let us translate the conditions about into statements just about the f_n's.

$\{f_n\} \subset L_p(\mu)$ $1 \le p < \infty$ is said to be wkly Cauchy if $\forall g \in L_q(\mu)$ $\frac{1}{p}+\frac{1}{q}=1$ we have $\lim_{n\to\infty} \int g f_n \, d\mu$ exists

this is equivalent to $\forall A \in \Sigma'$ $\mu(A)<\infty$ $\lim_{n\to\infty} \int_A f_n \, d\mu$ exists for $1<p<\infty$ and for $p=1$ if $\mu(X)<\infty$. [see page 3.15 Lemma] provided $\sup_n \|f_n\|_p < \infty$.

So let's assume $\lim_n \|f_n\|_p$ exists & $\forall A \in \Sigma'$ $\mu(A)<\infty$ $\lim_{n\to\infty} \int_A f_n \, d\mu$ exists. Need $\exists f \in L_p(\mu)$ s.t. $f_n \to f$ in L_p-norm?

No. Let $f_n = 2^{n/p} \chi_{[2^{-n}, 2^{-n+1}]}$ then $f_n \to 0$ wkly & $\|f_n\|_p = 1$

Well what can we do? Let $X = [0,1]$ and let $\mathcal{B}_k$ be the σ-algebra generated by $\{[0, 2^{-k}], \dots [i/2^{-k}, (i+1)/2^{-k}), \dots [2^k-1/2^k, 1]\}$ for $k = 0,1,2,\dots$ and suppose f is the limit. Then $g_k = E(f, \mathcal{B}_k)$ is a martingale. Suppose $g_k \to f$ in norm. & suppose $\|g_k\| \nearrow \lim_n \|f_n\|$. Then $f_n \to f$ in norm. The important fact is that g_k can be defined without knowing f. namely

(3.17) $$g_k(t) = \left\{ \lim_{n\to\infty} \frac{1}{2^k} \int_{[i/2^k, (i+1)/2^k)} f_n \, dm \quad \text{if } t \in [i/2^k, (i+1)/2^k) \right]$$

<u>Thm</u>: If $\{f_n\} \subset L_p$ $1<p<\infty$ $\cancel{\|f_n\|_p \to \|f\|}$, $\lim_{n\to\infty} \|f_n\|_p$ exists and $\{f_n\}$ is wkly Cauchy. Then g_k is a martingale (defined) by (3.17) and if $\{g_k\}$ converges in L_p-norm & $\lim \|g_k\|_p = \lim_{n\to\infty} \|f_n\|_p$ then f_n converges in L_p-norm to the same limit.

proof: Suppose g_k is a martingale & $g_k \to f$ in L_p-norm & $\lim_k \|g_k\| = \lim_n \|f_n\|$. Then $\|f\| = \lim_n \|f_n\|$ & if Borel $= \mathcal{B}_\infty \overset{k}{=}$ σ-algebra generated by $\cup \mathcal{B}_k$. If $A \in \mathcal{B}_k$ then

$$\int_A f\,d\mu = \lim_{m\to\infty} \int_A g_m\,d\mu = \int_A g_k\,d\mu = \lim_{n\to\infty} \int_A f_n\,d\mu.$$

Since the simple fcns which are $\cup\mathcal{B}_k$-measurable are dense in $L_p(\mu)$, $f_n \to f$ wky hence $f_n \to f$ in L_p-norm.

To see that g_k is a martingale note that

$$\int_{i/2^k}^{(i+1)/2^k} f_n = \int_{2i/2^{k+1}}^{(2i+1)/2^{k+1}} f_n + \int_{(2i+1)/2^{k+1}}^{(2i+2)/2^{k+1}} f_n$$

hence g_k on $[i/2^k, (i+1)/2^k]$ is the average of g_{k+1} on the same interval.

Remark. Since $E(g_{k+1}, \mathcal{B}_k) = g_k$ we have $\|g_k\|_p \le \|g_{k+1}\|_p$ hence $\lim_{k\to\infty} \|g_k\|_p = \sup_k \|g_k\|_p$

Thus our problem is reduced to that of finding when martingales convergence in L_p-norm. This requires some work.

Proposition A. Let $\{f_n\}_{n=1}^k$ be a finite martingale w/resp $\{\mathcal{B}_n\}$ on (X, Σ, μ) with $\mu(X) = 1$ then for $t > 0$ we have

$$t\,\mu(\sigma_t) \le \int_{\sigma_t} |f_k|\,d\mu$$

where $\sigma_t = \{x \in X : \max_{1\le n\le k} |f_n(x)| > t\}$.

proof define $T: X \to \{1, 2, \ldots\ldots k+1\}$ by

$$T(x) = \begin{cases} \min\{j : |f_j(x)| > t\} & \text{if } x \in \sigma_t \\ k+1 & \text{if } x \notin \sigma_t \end{cases}$$

(This is an example of a "stopping time"

Let $1 \le j \le k$ and $X_j = \{x \in X : T(x) = j\} = \{x \in X : |f_j(x)| > t \text{ \& } \max_{1\le k<j} |f_k(x)| \le t\} \in \mathcal{B}_j$

Since cond exp is pos $|f_j| \le E(|f_k|, \mathcal{B}_j)$ $1 \le j \le k$ thus we have.

$$t\mu(\sigma_t) = t\mu\{x : T(x) \le k\} = t\sum_{j=1}^{k}\mu(X_j)$$

$$\le \sum_{j=1}^{k}\int_{X_j}|f_j|\,d\mu \qquad (\text{since } f_j \ge t \text{ on } X_j)$$

$$\le \sum_{j=1}^{k}\int_{X_j}E(|f_k|\,d\mu) = \sum_{j=1}^{k}\int_{X_j}|f_k|\,d\mu = \int_{\sigma_t}|f_k|\,d\mu.\ \text{DONE.}$$

(Doob) Lemma B: $\{f_n\}_1^k$ is martingale w/resp $\{B_n\}_1^k$ on (X,Σ,μ) with $\mu(X)=1$ & $1<p<\infty$ then for $\frac{1}{p}+\frac{1}{q}=1$

$$\|\sup_n |f_n|\|_p \le q\,\|f_k\|_p$$

proof: Let $f = \max_{1\le n\le k}|f_n|$ and $\sigma_t = \{x\in X : f(x) > t\}$

opps we need

(*) If $1\le p<\infty$, $g\in L_p(X,\Sigma,\mu)$ $\mu(X)=1$ then

$$\int_X |g|^p\,d\mu = \int_0^\infty p t^{p-1}\mu(\{x\in X : |g(t)|>t\})\,dt$$

(see below)

Now

$$\|f\|_p^p = \int_0^\infty p t^{p-1}\mu(\sigma_t)\,dt \le \int_0^\infty p t^{p-2}\left(\int_{\sigma_t}|f_k|\,d\mu\right)dt$$

$$= \int_X |f_k|\int_0^\infty p t^{p-2}\chi_{\sigma_t}(x)\,dt\,d\mu(x)$$

$$= \int_X |f_k|\int_0^{f(x)} p t^{p-2}\,dt\,d\mu(x) \qquad (\text{since } \chi_{\sigma_t}(x)=0 \quad t>f(x))$$

$$= \int_X |f_k|\frac{p}{p-1}f(x)^{p-1}\,d\mu(x) = q\int_X |f_k| f^{p-1}\,d\mu \qquad \left(\tfrac{1}{p}+\tfrac{1}{q}=1 \quad q=\tfrac{1}{1-1/p}=\tfrac{p}{p-1}\right)$$

Hölder's

$$\le q\,\|f_k\|_p\,\|f^{p-1}\|_q = q\|f_k\|_p\left(\int |f|^{(p-1)q}\right)^{1/q} = q\,\|f_k\|_p\,\|f\|_p^{p/q}$$

Hence $\|f\|_p^{p-p/q} = \|f\|_p \le q\|f_k\|_p$

(Doob) Corollary C If $\{f_n\}_1^\infty$ is a martingale w/ resp $\{B_n\}_1^\infty$ on (X,Σ,μ) $\mu(X)=1$ then $1<p<\infty$ $\frac{1}{p}+\frac{1}{q}=1$

$$\|\sup |f_n|\|_p \le q\sup\|f_n\|_p.$$

proof: since $\sup_n|f_n| = \lim_k \max_{1\le n\le k}|f_n|$ & the second seq is increasing

$$\|f\|_p^p = \int|\sup_n f_n|^p \le \lim_n\int|\max_{1\le h\le k}|f_n||^p \le \lim_n q\,\|f_n\|_p^p = q(\sup\|f_n\|_p)^p$$

Now to prove $(*)$ Let $g = \sum_1^k \alpha_i \chi_{A_i}$ w/ A_i p.w.d $\mu(A_i) < \infty$ & Assume (as we may) all $\alpha_i > 0$ and $\alpha_1 > \alpha_2 > \dots > \alpha_n$

$$\int |g|^p = \sum_1^k |\alpha_i|^p \mu(A_i) \quad \text{and}$$

$$\int_0^\infty p t^{p-1} \mu(\{x : |g(t)| > t\})\, dt = \int_0^{\alpha_k} \_\_ + \int_{\alpha_k}^{\alpha_{k-1}} \_\_ + \dots + \int_{\alpha_1}^\infty \_\_$$

and

$$\int_{\alpha_1}^\infty p t^{p-1} \mu((x : |g(t)| > t))\, dt = 0$$

$$\int_0^{\alpha_k} p t^{p-1} \mu((x : |g(t)| > t))\, dt = (\alpha_k^p - 0^p)(\mu(A_1) + \dots \mu(A_k))$$

$$\int_{\alpha_i}^{\alpha_{i-1}} p t^{p-1} \mu(x : |g(t)| > t)\, dt = (|\alpha_i|^p - |\alpha_{i-1}|^p)(\mu(A_1) + \dots \mu(A_{i-1}))$$

hence this telescoping sum is $\sum_1^k |\alpha_i|^p \mu(A_i) = \int |g|^p$.

Now let $g \in L_p$ assume $g \geq 0$ (as we may) and let $0 \leq g_n \uparrow g$ pt-wise hence $\int |g|^p = \lim_n \int |g_n|^p$

Consider $p t^{p-1} \mu(\{x : g_n(x) > t\}) \uparrow \leq p t^{p-1} \mu(\{x : g(x) > t\})$
claim left hand $\to$ pt-wise to rt hand. Since $\{x : g(x) > t\} = \bigcup_n \{x : g_n(x) > t\}$
. Monotone conv. Thm kills it.

<u>Lemma D</u> Let $\{f_n\}_1^k$ martingale w/resp $\{B_n\}$ on (X, Σ, μ) $\mu(X) = 1$
Let $S = (|f_1|^2 + \sum_{j=2}^k |f_j - f_{j-1}|^2)^{1/2}$
then if $p = 2^m$ $\exists\, C_p$

$$C_p^{-1} \|f_k\|_p \leq \|S\|_p \leq C_p \|f_k\|_p$$

<u>proof</u>: Let $f_0 \equiv 0$, $\Delta f_j = f_j - f_{j-1}$ so $S = (\sum_1^k |\Delta f_j|^2)^{1/2}$

Suppose $p = 2$, then $C_2 = 1$ since Δf_j Δf_n $j < n$ are orthogonal. Since

$$\int_X \Delta f_j \Delta f_n\, d\mu = \int_X E(\Delta f_j \Delta f_n)\, d\mu = \int_X \Delta f_j (E \Delta f_n)\, d\mu$$

$= \int \Delta f_j \cdot 0\, d\mu = 0$. Thus

$$\|S\|_2^2 = \int \sum_1^k |\Delta f_j|^2 = \int (\sum_1^k \Delta f_j)(\overline{\sum_1^k \Delta f_j}) = \sum_1^k \|\Delta f_j\|_2^2 = \|\sum_1^k \Delta f_j\|_2^2$$
$$= \|f_k\|_2^2$$

For general $p = 2^m$, the proof is done by induction.
Suppose it is true for p with C_p for every finite martingale
we have

$$|f_k|^2 = \Big|\sum_{j=1}^{k} \Delta f_j\Big|^2 = \sum_{j=1}^{k} |\Delta f_j|^2 + 2 \sum_{\substack{m,n=1\\ m<n}}^{k} \Delta f_m \overline{\Delta f_n}$$

$$= S^2 + 2\sum_{n=2}^{k} \overline{\Delta f_n} \sum_{m=1}^{n-1} \Delta f_m = S^2 + 2\sum_{n=2}^{k} f_{n-1}\overline{\Delta f_n}$$

Hence

$$\|S^2\|_p - 2\Big\|\sum_{n=2}^{k} f_{n-1}\overline{\Delta f_n}\Big\| \le \|f\|_p^2 \le \|S^2\|_p + 2\Big\|\sum_{n=2}^{k} f_{n-1}\overline{\Delta f_n}\Big\|$$

Now let $g_1 \equiv 0$ & $g_j = \sum_{n=2}^{j} f_{n-1}\overline{\Delta f_n}$ $\quad 2 \le j \le k$
then $\{g_j\}_1^k$ is martingale w/resp $\{\mathcal{B}_j\}_1^k$.
for $E(g_2, \mathcal{B}_1) = E(f_1 \overline{\Delta f_2}, \mathcal{B}_1) = 0$ since $E(\overline{\Delta f_2}, \mathcal{B}_1) = 0$
and $E(g_{j+1}, \mathcal{B}_j) = E\Big(\sum_{n=2}^{j+1} f_{n-1}\overline{\Delta f_n}, \mathcal{B}_j\Big) = \sum_{n=2}^{j+1} E(f_{n-1}\overline{\Delta f_n}, \mathcal{B}_j)$

$$= \sum_{n=2}^{j} f_{n-1}\overline{\Delta f_n} + E(f_j \overline{\Delta f_{j+1}}, \mathcal{B}_j) = g_j$$

Applying the inductive hypothesis

$$\Big\|\sum_{n=2}^{k} f_{n-1}\overline{\Delta f_n}\Big\|_p \le C_p \Big\|\Big(\sum_{n=2}^{k} |f_{n-1}\overline{\Delta f_n}|^2\Big)^{1/2}\Big\|_p$$

Put $f = \max_{1\le n\le k} |f_n|$ $\qquad Sf = \Big(\sum_{j=1}^{k} |\Delta f_j|^2\Big)^{1/2} f \ge \Big(\sum_{j=1}^{k} f^2 |\Delta f_j|^2\Big)^{1/2} \ge \Big(\sum_{n=2}^{k} |f_{n-1}\overline{\Delta f_n}|^2\Big)^{1/2}$

$$\Big\|\sum_{n=2}^{k} f_{n-1}\overline{\Delta f_n}\Big\|_p \le C_p \|Sf\|_p \le C_p \|S\|_{2p}\|f\|_{2p} \quad \text{since } \tfrac{1}{2p} + \tfrac{1}{2p} = \tfrac{1}{p} \text{ Gen Hölders}$$

$$\le C_p\Big(\frac{2p}{2p-1}\Big)\|S\|_{2p}\|f_k\|_{2p} \quad \text{(Doob's)} \quad \Big[q = \frac{1}{1-\frac{1}{p}}\Big]$$

Combining: since $\|S_k\|_{2p}^2 = \|S^2\|_p$
we have

$$\|S\|_{2p}^2 - 2C_p\frac{2p}{2p-1}\|S\|_{2p}\|f_k\|_{2p} \le \|f_k\|_{2p}^2 \le \|S\|_{2p}^2 + 2C_p\frac{2p}{2p-1}\|S\|_{2p}\|f_k\|_{2p}$$

Let $\alpha = 2C_p\frac{2p}{2p-1}$ $\quad \Lambda = \|S\|_{2p}/\|f_k\|_{2p}$ divide by $\|f_k\|_{2p}^2$ we get

$$\Lambda^2 - \alpha\Lambda \le 1 \le \Lambda^2 + \alpha\Lambda \quad \text{which yields}$$

$$\frac{1}{\alpha+1} \le \frac{-\alpha + \sqrt{\alpha^2+4}}{2} \le \Lambda \le \frac{\alpha + \sqrt{\alpha^2+4}}{2} \le \alpha + 1$$

so $C_{2p} \le 1 + 2C_p\Big(\frac{2p}{2p-1}\Big)$ $\qquad$ done.

The Haar basis $\{h_n\}_{n=1}^{\infty}$ are functions on $[0,1]$ as follows:

$h_1 = \chi_{[0,1]}$

$h_2 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$

$h_3 = \chi_{[0,1/4)} - \chi_{[1/4,1/2)}$

$h_4 = \chi_{[1/2,3/4)} - \chi_{[3/4,1)}$

etc.

if $n = 2^k + i \qquad 1 \le i \le 2^k \qquad k = 0, 1, \dots \qquad (\text{so } n \ge 2)$

$$h_n(t) = \begin{cases} 1 & \text{if } t \in [(2i-2)/2^{k+1}, (2i-1)/2^{k+1}) \\ -1 & \text{if } t \in [(2i-1)/2^{k+1}, 2i/2^{k+1}) \\ 0 & \text{otherwise} \end{cases}$$

Lemma $[h_n]_{n=1}^{2^k}$ is the collection of functions measurable with resp to the σ-algebra $\mathcal{B}_k = \{[i/2^k, (i+1)/2^k) : i = 1, \dots 2^k\}$ for $k = 0, 1, 2, \dots$.

pf: Clearly each h_n $n = 1, \dots, 2^k$ is measurable w/resp to $\mathcal{B}_k$, hence ~~their~~ closed linear span = linear span is measurable w/resp $\mathcal{B}_k$. Conversely, it suffices to show $\chi_{[i/2^k, (i+1)/2^k)} \in [h_n]_{i=1}^{2^k}$. ~~Note~~ This is done by induction. For $k=1$, we have $\chi_{[0,1/2)} = h_1 + h_2$ $\chi_{[1/2,1)} = h_1 - h_2$. If $f = \chi_{[i/2^{k-1}, (i+1)/2^{k-1})} \in \mathcal{B}_{k-1}$ then $\chi_{[2i/2^k, (2i+1)/2^k)} = f + h_{2^{k-1}+i+1}$ & $\chi_{[(2i+1)/2^k, (2i+1)/2^k)} = f - h_{2^{k-1}+i-1}$

Corollary $[h_n]_{n=1}^{\infty} = L_p \quad 1 \le p < \infty$ (Here closure in L_p-norm)

pf: The simple functions which are measurable w/resp $\cup \mathcal{B}_k$ are dense in L_p (This is false for the non-separable space L_∞ of course)

A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is said to be a Schauder basis if $\forall x \in X$ there is a unique scalar sequence $\{\alpha_n\}$ so that $\quad x = \lim \sum_1^N \alpha_n x_n \quad$ in norm.

Note That an orthogonal basis in Hilbert space are Schauder basis but there are Schauder basis in Hilbert space which are not orthogonal. A Schauder basis is always linearly independent, but it is a Hamel basis exactly if X is finite dimensional. If X is infinite dimensional, a Hamel basis is never a Schauder basis.

<u>Thm</u> The Haar basis is a Schauder basis in L_p and $1 \le p < \infty$

We will actually show more

<u>Lemma</u> For all N the map $P: [h_n]_{n=1}^{N+1} \to [h_n]_{n=1}^{N}$ given by $\Sigma_1^{N+1} \alpha_n h_n \to \Sigma_1^N \alpha_n h_n$ has norm one in L_p $1 \le p \le \infty$

<u>pf:</u> $h_{N+1} = \chi_{[2i/2^k, (2i+1)/2^k)} - \chi_{[(2i+1)/2^k, (2i+2)/2^k)}$ for some i and k.

Now P is well-defined since $\{h_n\}_1^\infty$ is linearly independent. Let $x = \Sigma_1^N \alpha_n h_n$. Note $x + \alpha_{N+1} h_{N+1}$ & $x - \alpha_{N+1} h_{N+1}$ have the the same norm in L_p since x is constant on $[2i/2^k, (2i+2)/2^k)$

Thus $\|P(\Sigma_1^{N+1} \alpha_n h_n)\| = \|\Sigma_1^N \alpha_n h_n\| = \|x\| = \|\frac{1}{2}(x + \alpha_{N+1} h_{N+1} + x - \alpha_{N+1} h_{N+1})\|$

$\le \frac{1}{2}(\|x + \alpha_{N+1} h_{N+1}\| + \|x - \alpha_{N+1} h_{N+1}\|) = \|x + \alpha_{N+1} h_{N+1}\| = \|\Sigma_1^{N+1} \alpha_n h_n\|$.

& we are done.

<u>Cor</u>: $P_N : \text{Span}\{h_n\}_{n=1}^\infty \to [h_n]_{n=1}^N$ $P_N(\Sigma \alpha_n h_n) = \Sigma_1^N \alpha_n h_n$ has norm one in L_p $1 \le p \le \infty$

<u>pf.</u> If $x \in \text{Span}\{h_n\}_{n=1}^\infty$, $x = \Sigma_1^M \alpha_n h_n$, some $M < \infty$

$\|Px\| = \|\Sigma_1^N \alpha_n h_n\| \le \|\Sigma_1^{N+1} \alpha_n h_n\| \le \|\Sigma_1^{N+2} \alpha_n h_n\| \le \dots \le \|\Sigma_1^M \alpha_n h_n\| = \|x\|$

using the lemma finitely many times.

<u>Proof of Thm</u>: Since $\text{Span}\{h_n\}$ is dense in L_p, $1 \le p < \infty$ the operators P_N are norm one on L_p. Clearly $P_N(L_p) = [h_n]_{n=1}^N$ since the latter space is finite dimensional & complete. Further $P_N = id$ on $[h_n]_1^N$ hence it is a projection.

Let $f \in L_p$. Since $P_N P_M = P_{\min[m,N]}$ we easly obtain $\{\alpha_n\}$ s.t. $P_N f = \Sigma_1^N \alpha_n h_n$. Next we show that $f = \text{norm}\lim \Sigma_1^N \alpha_n h_n$

Since $\exists f^n \in \text{span}\{h_n\}$ s.t. $\|f - f^n\| < 1/n$ thus

$\|\Sigma_1^N \alpha_i h_n - f^n\| = \|P_N(f - f^n)\| \le \|f - f^n\| < \frac{1}{n}$ for large enuff N

so $\|f - \Sigma_1^N \alpha_n h_n\| \le \|f - f^n\| + \|f^n - \Sigma_1^N \alpha_n h_n\| < \frac{2}{n}$ for large enuff N.

Finally we show uniqueness, suppose $f = \text{norm}\lim \Sigma_1^N \beta_n h_n$

$\Sigma_1^N \alpha_n h_n = P_N f = \lim P_N(\Sigma_1^m \beta_n h_n) = \lim \Sigma_1^N \beta_n h_n = \Sigma_1^N \beta_n h_n$

which implies $\alpha_n = \beta_n$ $1 \le n \le N$ since $\{h_n\}$ are linearly independent

<u>Remark</u>: The fact $P_N(\Sigma a_n x_n) \to \Sigma_1^N a_n x_n$ have uniformly bdd norms is equivalent to the linear indep seq with dense span being a Schauder basis.

In this case $\|P_N\| = 1$ such basis are called monotone

Thm Let $1<p<\infty$, $\mathcal{B}_k = \sigma$-alg gen $\{[i/2^k,(i+1)/2^k) : i=0,\dots 2^k-1\}$ and $\{g_k\}_1^\infty$ a martingale w/resp to $\{\mathcal{B}_k\}_1^\infty$. $\{g_k\}_1^\infty$ converges in L_p-norm to L_p function if and only if $\sup_k \|g_k\|_p < \infty$.

proof: If $g_k \to f$ in L_p-norm, then $E(f, \mathcal{B}_j) = \lim_k E(g_k, \mathcal{B}_j)$ $= E(g_j, \mathcal{B}_j) = g_j$ hence $\|g_j\|_p \le \|f\|_p$ and the $\sup_h \|g_j\|_p \le \|f\|_p < \infty$.

Now suppose $\sup_k \|g_k\|_p < \infty$. Since g_k is $\mathcal{B}_k$-meas $g_k = \sum_1^{2^k} \alpha_n h_n$. Note $E^k(g_k, \mathcal{B}_{k-1}) = \sum_1^{2^{k-1}} \alpha_n h_n = g_{k-1}$, since $E(h_{2^{k-1}+i}, \mathcal{B}_{k-1}) \equiv 0$. Since $\|\sum_1^N \alpha_n h_n\| \le \|\sum_1^{N+1} \alpha_n h_n\|$, the condition $\sup_k \|g_k\|_p < \infty$ is equivalent to $\sup_N \|\sum_1^N \alpha_n h_n\|_p < \infty$. If $\sum_1^N \alpha_n h_n$ converges in norm so does any subsequence including g_k. Thus to complete the proof it suffices to show

(*) If $\{\alpha_n\}$ is a sequence s.t. $\sup_N \|\sum_1^N \alpha_n h_n\| < \infty$ implies $\sum a_n h_n$ converges./ Thus the theorem follows from the Prop. below since L_p $1<p<\infty$ is reflexive.

Definition A basis $\{h_n\}$ that satisfies (*) is called boundedly complete.

Proposition. A basis of a reflexive space is boundly complete.

pf: Let X, x_n be a reflexive space, X^* its dual & identify X with X^{**}, let $\{\alpha_n\}$ be given so that $\sup_N \|\sum_1^N \alpha_n x_n\| < \infty$ and let $y_N = \|\sum_1^N \alpha_n x_n\|$. Since y_N is contained in a multiple of the unit ball it has a weak-cluster point y_∞. Claim $P_N(y_\infty) = \sum_1^N \alpha_n x_n$ Consider $x_n' \in X^*$, defined by $x_n'(\sum a_m x_m) = \alpha_n$ is cont (Actually this takes some work.) & $x_n'(y_\infty) = \lim x_n'(y_m) = \alpha_n$. Since $\{x_n\}$ is a basis something converges to y_∞ hence it must be $\sum \alpha_n x_n$. done.

The Theorem is false for L_1 Let $g_k = 2^k \chi_{[0,1/2^k)}$

Actually, the Haar basis has stronger properties for $1<p<\infty$ A basis $\{x_n\}$ is uncondition if $\sum \alpha_n x_n$ converges unconditionally to x for each x.

<u>Example</u> The Haar basis is not unconditional in L_1

$$\| h_1 + h_2 + 2h_3 + 4h_5 + 8h_9 + \cdots 2^k h_{2^k+1} \|_1 = 1$$

$$\| h_1 + 2h_3 + 8h_9 + \cdots \quad 2^{2k+1} h_{2^{2k+1}+1} + \cdots \|_1 = \infty$$

<u>Remark</u> $\Sigma_1^\infty \alpha_n x_n$ converges unconditionally $\iff$ $\Sigma \pm \alpha_n x_n$ conv all choices of signs $\iff$ $\Sigma_i \alpha_{n_i} x_{n_i}$ conv all subseq (n_i) of integers $\iff$ $\Sigma_n \alpha_{\pi(n)} x_{\pi(n)}$ all permutations: $\pi : \mathbb{N} \to \mathbb{N}$.

<u>Thm</u>: The Haar basis is an unconditional basis for L_p, for $1 < p < \infty$.

<u>Proof</u>: For $p = 2^m$ $m = 1, 2, \ldots$. For $p = 2$, Haar basis is orthogonal & orthogonal basis are unconditional.

Let $p = 2^m$, & let $\{\alpha_n\}_{n=1}^k$ be given let $\theta_n = \pm 1$

Let $f_j = \Sigma_1^j \alpha_n h_n$, $g_j = \Sigma_1^j \alpha_n \theta_n h_n$ these are martingales with respect to $\mathcal{A}_j$ = smallest σ-algebra ~~for~~ which $\{h_n\}_{n=1}^j$ is meas. [same proof as above]

Since S given by Thm D? is the same for f_j & g_j

$$C_p^{-2} \| f_k \|_p \le \| g_k \|_p \le C_p^2 \| f_k \|_p$$

Thus the map $T : \Sigma_1^\infty \alpha_n h_n \mapsto \Sigma \alpha_n \theta_n h_n$ has norm at most C_p^2 on span $\{h_n\}_1^\infty$ hence on all L_p. If $\Sigma_1^\infty \alpha_n h_n$ converges we have $T(\Sigma_1^\infty \alpha_n h_n) = \lim T(\Sigma \alpha_n h_n) = \lim \Sigma_1^N \alpha_n \theta_n h_n$ so $\Sigma \alpha_n \theta_n h_n$ conv.

The proof is completed by using interpolation for $2 \le p < \infty$ and duality for $1 < p \le 2$ [i.e. define $x_n'(\Sigma \alpha_m x_m) = \alpha_n$, if (x_n) is unconditional so is (x_n') unconditional. & for h_n, h_n' is h_n in $L_{p'}$.].

Note that the Rademachers are as far from being a martingale as possible w/resp to $\{\mathcal{B}_k\}$ above. Also $r_n = \sum_{i=2^n+1}^{2^{n+1}} h_i$, a "block-basic" sequence of the Haar basis.

This completes section 5

Note: if $X \subseteq Y$, then $X \cap Y = X$ and $X + Y = Y$. Furthermore $\|f\|_{X\cap Y} = \|f\|_X$ and $\|f\|_{X+Y} = \|f\|_Y$ if we have $\|\cdot\|_Y \le \|\cdot\|_X$

The next step is to consider when $X = L_p(\mu)$ $Y = L_q(\mu)$
Suppose $1 \le p < r \le q \le \infty$. Then (see page 2.4) $\exists\,\theta$ $0 \le \theta \le 1$
$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$ and

$$\|f\|_r \le \|f\|_p^{\theta}\|f\|_q^{1-\theta} \le \max(\|f\|_p, \|f\|_q)$$

so $L_p(\mu) \cap L_q(\mu) \subset L_r(\mu)$ and the inclusion map has norm one.

The space $L_p(\mu) + L_q(\mu)$ is harder to deal with. consider for $\lambda > 0$ and measurable f the functions f^λ, f_λ

$f^\lambda = f\chi_{\{t:\,|f(t)| \ge \lambda\}}$ $\quad f_\lambda = f\chi_{\{t:\,|f(t)| < \lambda\}}$

note that f^λ, f_λ have disjoint supports and $f = f^\lambda + f_\lambda$.

Now suppose $1 \le p \le r \le q \le \infty$, since $|f_\lambda| \le \lambda$, $|f_\lambda| \le |f|$ and $|f^\lambda| \le |f|$ we have

$|f_\lambda|^r \le \lambda^{r-p}|f_\lambda|^p$ $\quad$ since $0<t<\lambda \Rightarrow t^r \le t^p\lambda^{r-p}$

$|f^\lambda|^r \le \lambda^{r-q}|f^\lambda|^q$ $\quad$ since $\lambda<t<\infty \Rightarrow \lambda^{q-r}t^r \le t^q$

take $\lambda = 1$ we have

$\|f\|_r^r \le \|f_\lambda\|_p^p + \|f^\lambda\|_q^q$ $\quad$ which goes the wrong way

try $\|f^\lambda\|_p^p + \|f_\lambda\|_q^q \le \|f^\lambda\|_r^r + \|f_\lambda\|_r^r = \|f\|_r^r$

so $L_r(\mu) \subset L_p(\mu) + L_q(\mu)$. But what is the norm of the inclusion map?

Consider $|||f||| = \sup\{\|f\chi_A\|_p : \mu(A) \le 1\}$. If
$f = g + h$ $\quad \|f\chi_A\|_p \le \|g\|_p + \|h\chi_A\|_p$. $\quad$ Let $p \le q$ $\quad r = \frac{q}{p}$ r'
and $\quad \|h\chi_A\|_p \le \|h\|_q$ $\quad$ s.t. $\frac{1}{r} + \frac{1}{r'} = 1$

$$\int |h|^p\chi_A \le \||h|^p\|_r\,\|\chi_A\|_{r'} = \left(\int |h|^q\right)^{p/q}\cdot 1$$

hence $|||f||| \le \|f\|_{L_p+L_q}$ $\quad p \le q$

~~Lets us look at $L_1(\mu) + L_\infty(\mu)$~~

<u>Riesz – Thorin</u> <u>interpolation thm</u>:

Let (X_i, Σ'_i, μ_i) $i=1,2$ be measure spaces Let $p_1 \neq q_1$. $p_2 \neq q_2$ be numbers in $[1,\infty]$. Let T be a linear map: $L_{p_1}(\mu_1) + L_{q_1}(\mu_1) \longrightarrow L_{p_2}(\mu_2) + L_{q_2}(\mu_2)$ which is strong type (p_1, p_2) & (q_1, q_2)

Then for every $0<\theta<1$, T is of strong type (r_1, r_2) where $\frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}$ $i=1,2$ Moreover

$$\|T\|_{r_1, r_2} \leq 1^* \|T\|^{\theta}_{p_1 p_2} \|T\|^{1-\theta}_{q_1 q_2}$$

[*if these are real spaces the one must be replaced by 2].
proved via complex method

~~Let~~

<u>Marcinkiewicz</u> <u>interpolation thm</u>. Let (X_i, Σ_i, μ_i) p_i, q_i as above. Let T be a quasilinear mapping $L_{p_1}(\mu_1) + L_{q_1}(\mu_1) \longrightarrow$ the space of measurable functions on X_2 which is weak type (p_1, p_2) and (q_1, q_2).

Then for every $0<\theta<1$, T is strong type (r_1, r_2) where $\frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}$ provided $r_1 \leq r_2$
proved via real method.

<u>Definitions</u>

1 $T: X \to Y$ is quasi-linear if $|T(\alpha x)| = |\alpha| |Tx|$ scalars α, & $x \in X$ & $\exists$ constant C so that $|T(x+y)| \leq C(|Tx| + |Ty|)$
Linear $\Rightarrow$ quasi-linear. $S(\Sigma' \alpha_i x_i) = (\Sigma' |\alpha_i x_i|^2)^{1/2}$ is quasi-linear but not linear.

2. T is strong type (p_1, p_2) (as above) if T is a bounded operator $L_{p_1}(\mu_1) \longrightarrow L_{p_2}(\mu_2)$

3. T is weak type (p_1, p_2) $p_1 \neq \infty$ $1 \leq p_2 \leq \infty$ if T is a bounded operator $L_{p_1,1}(\mu) \longrightarrow L_{p_2,\infty}(\mu_2)$
T is weak type (∞, p_2) $1 \leq p_2 \leq \infty$ if T is a bounded operator $L_\infty(\mu_1) \longrightarrow L_{p_2,\infty}(\mu_2)$

4. For $1 \leq p < \infty$, $1 \leq q < \infty$ $L_{p,q}(\mu)$ is the space of all locally integrable real valued functions f on X s.t.

$$\|f\|_{p,q} = \frac{q}{p} \int_0^\infty [t^{1/p} f^*(t)]^q \, t^{-1} \, dt < \infty$$

For $1 \leq p \leq \infty$ $L_{p,\infty}$ all functions so that $\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) < \infty$

~~$L_{\infty,\infty} = L_\infty$~~ $L_{\infty,q} = L_\infty$ for $q < \infty$ $\| \ \|_{\infty,\infty} = \sup_{t>0} f^*(t) = \| \ \|_\infty$

5 <u>Definition of $f^*(t)$</u>: Assume (X, Σ, μ) is σ-finite and let $f: X$ be locally integrable.

$$d_f(t) = \mu(\{x \in X : |f(x)| > t\}) \qquad 0 \le t < \infty$$

Note that $d_f(t)$ is a decreasing function

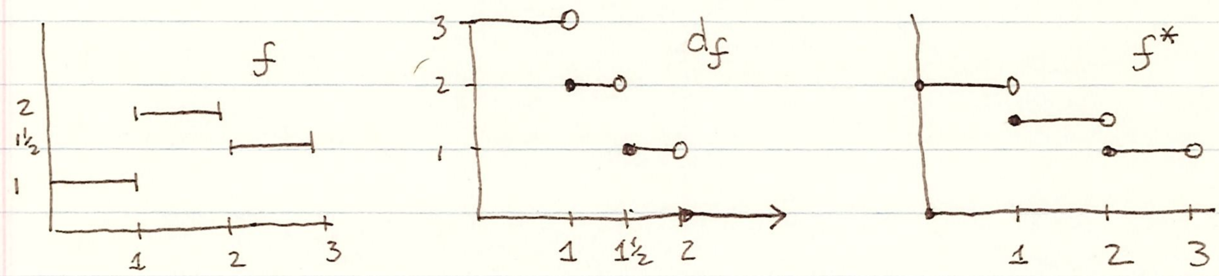
f^* is defined to be the right continuous inverse of d_f i.e.

$$f^*(s) = \begin{cases} \inf\{t > 0 : d_f(t) \le s\} & 0 \le s < \mu(X) \\ 0 & \mu(X) \le s. \end{cases}$$

Note that f^* is a decreasing function

On page 5.9 we proved $\int_X |f|^p d\mu = \int_0^\infty p t^{p-1} d_f(t)\, dt$ ($)

Now $d_{f^*}(t) = m(\{s : |f^*(s)| > t\})$



~~but~~ $f^*(s) = |f^*(s)| > t \Rightarrow \inf\{r > 0 : d_f(r) \le s\} > t$

$\Rightarrow d_f(t) > s \Rightarrow$ ~~sin~~ $\inf\{r > 0 : d_f(r) \le s\} > t \Rightarrow f^*(s) > t$

since d_f dec

so $\{s : f^*(s) > t\} = \{s : d_f(t) > s\} = [0, L)$ so $d_f(t) = L$
$= [0, L)$

not used

Let $d_f(t) = A = \mu\{x \in X : |f(x)| > t\} < \infty$

then $f^*(A) = \inf\{t > 0 \ \ d_f(t) \le A\} \le t$ if $f^*(A) < t$

$\exists\, s < t$ s.t. $d_f(s) \le A$ since d_f is decreasing $d_f(s) = A$

hence $\mu\{x \in X : t \ge f(x) > s\} = 0$

Now the function $d_f(t)$ is decreasing ≥ 0 so it can have at most countably many jumps. between t & ∞

$$f^*(s) > t \Leftrightarrow d_f(t) > s \Leftrightarrow \mu\{x : |f(x)| > t\} > s$$

$m\{s : f^*(s) > t\} = m(\{s : \mu(\{x : f(x) > t\}) > s\}) = \mu(\{x : f(x) > t\})$

$\therefore \ d_{f^*}(t) = d_f(t)$ hence

$\|f\|_p^p = \|f^*\|_p^p$ by ($) $\qquad p < \infty \qquad$ If $p = \infty$

$\|f\|_\infty = \inf\{s > 0 : \mu\{x \in X : |f(x)| > s\} = 0\} = \inf\{s > 0 : d_f(s) = 0\}$

$= \|f^*\|_\infty$

6.5

Some niceness for f is require so that f^* is not rediculess if $f(t) = t$ on $[0,\infty)$ then $f^*(t) = \infty$ on $[0,\infty)$.

not quite true → If ~~f is local L_1 then~~ $d_f(t)$ ~~must~~ is be finite for all $t>0$ (~~since σ-finite $\Rightarrow$ no infinite atoms~~) then Also $\lim_{t\to\infty} d_f(t) = 0$

otherwise $\{x : |f(x)| = \infty\} = \bigcap_n \{x : |f(x)| > n\}$ so

$\mu\{x : |f(x)| = \infty\} = \lim_{n\to\infty} d_f(n) \neq 0$ & $f \notin$ Local L_1 ※.

Hense $f^*(s)$ is finite for $s > 0$

Second formula

$$\int_0^s f^*(t)\,dt = \sup\left\{ \int_A |f(x)|\,d\mu : \mu(A) \le s \right\}$$

$$\cancel{\int_0^s f^*(t)\,dt = \int_0^\infty f^*(t)\chi_{[0,s)}\,dt = \int_0^\infty d_{f^*\chi_{[0,s)}}(t) = \int}$$

$f\chi_A = g \qquad d_g(t) = \mu\{x\in A : |f(x)| > t\} \le d_f(t)$

& $d_g(t) \le \mu(A)$ for all t

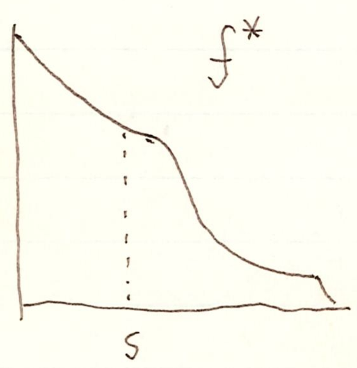
$\int_A |f|\,d\mu = \int_X |f|\chi_A\,d\mu = \int_0^\infty d_g(t)$

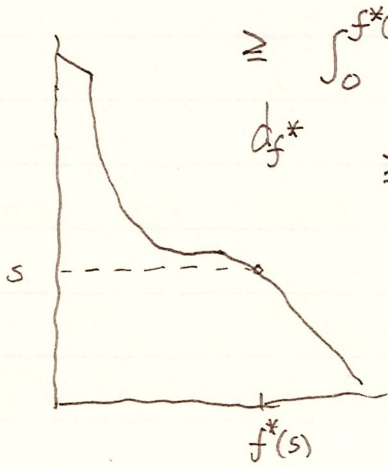
$f^*\chi_{[0,s)} = h \qquad d_h(t) = m\{r : f(r) > t \text{ \& } r < s\}$

$$= \begin{cases} d_{f^*}(t) & \text{if } t > f^*(s) \\ d_{f^*}(f^*(s)) & \text{if } 0 < t \le f^*(s) \end{cases}$$

$$\int_0^s f^*\,dt = \int_0^\infty d_h(t) = \int_0^{f^*(s)} d_{f^*}(f^*(s))\,dt + \int_{f^*(s)}^\infty d_{f^*}(t)\,dt$$

if $\mu(A) \le s = d_{f^*}(f^*(s))$
$$\ge \int_0^{f^*(s)} \mu(A) + \int_{f^*(s)}^\infty d_{f^*}(t)\,dt$$
$$\ge \int_0^{f^*(s)} \mu(A) + \int_{f^*(s)}^\infty d_g(t)\,dt$$
$$\ge \int_0^\infty d_g(t)\,dt$$
$$= \int_A |f|\,dt.$$





~~Cor If $f \in$ Local $L_1[0,\infty) \Rightarrow f \in L_1 + L_\infty$~~

~~$f \in$ Local $L_1[0,\infty)$~~, ~~then~~ If $\sup_{\mu(A)\le 1} \int_A |f(x)|\,dx < \infty \Rightarrow f \in L_1 + L_\infty$

Since $f^* = f^*\chi_{[0,1)} + f^*\chi_{[1,\infty)}$ first in L_1 second bdd by $f^*(1)$.

Note that $(f+g)^*(t)$ need not be $\leq f^*(t) + g^*(t)$

example: $f(t) = t$ $g(t) = 1-t$ on $[0,1]$ then $(f+g)^*(t) \equiv 1$ on $[0,1]$ but $f^*(t) = g^*(t) = g(t) = 1-t$ on $[0,1]$ & $g^*(1) + f^*(1) = 0$. However we have

Lemma: $(f+g)^*(t+s) \leq f^*(s) + g^*(t)$

pf: $\{x : |f(x)+g(x)| > f^*(s)+g^*(t)\}$
$\subset \{x : |f(x)| > f^*(s)\} \cup \{x : |g(x)| > g^*(t)\}$

since $|a+b| > w+v \Rightarrow$ either $|a| > w$ or $|b| > v$ since if both $|a| \leq w$ $|b| \leq v$ then $|a+b| \leq |a|+|b| \leq w+v$.

Thus $d_{f+g}(f^*(s) + g^*(t)) \leq d_f(f^*(s)) + d_g(g^*(t)) = s+t$

$(f+g)^*(t+s) = \inf\{r>0 : d_{f+g}(r) \leq t+s\} \leq f^*(s) + g^*(t)$

Sublemma $d_f(f^*(s)) = s$

$d_f(f^*(s)) = d_{f^*}(f^*(s)) = m(\{r : f^*(r) > f^*(s)\}) = m([0,s)) = s$

In particular: $(f+g)^*(2t) \leq f^*(t) + g^*(t)$

Thm: Let $I = [0,1]$ or $[0,\infty)$ $1 \leq p < q \leq \infty$ let T be linear: $L_p(I) + L_q(I) \to$ measurable fcns on I which is weak type (p,p) & (q,q) then T is strong type (r,r) for $p<r<q$.

Main Lemma: Under same assumptions $\exists M < \infty$ so that

$$(Tf)^*(2t) \leq M\left(\int_0^1 f^*(tu)u^{(1-p)/p}\,du + \int_1^\infty f^*(tu)u^{(1-q)/q}\,du\right.$$

for $f \in L_{p,1} + L_{q,1}$ $0<t<\infty$ if $I = [0,\infty)$ $0<t\leq \frac{1}{2}$ $I = [0,1]$

pf: Let $\|T\|_{p,p} = M_p$, $\|T\|_{q,q} = M_q$

Let $f \in L_{p,1}(I) + L_{q,1}(I)$ $u, t \in I$ write

$$g_t(u) = \begin{cases} f(u) - f^*(t) & \text{if } f(u) > f^*(t) \\ f(u) + f^*(t) & \text{if } f(u) < -f^*(t) \\ 0 & \text{if } |f(u)| \le f^*(t) \end{cases}$$

$$h_t(u) = f(u) - g_t(u)$$

(1) $g_t^*(u)$ is 0 if $u \geq t$

(2) $g_t^*(u) \le f^*(u)$

(3) $|h_t(u)| = \min(|f(u)|, f^*(t))$

Apply T is weak type (p,p) to g_t

$\left\{ \begin{array}{l} T: L_{p,1} \to L_{p,\infty} \\ \|f\|_{p,1} = \frac{1}{p}\int_0^\infty \frac{t^{1/p}}{t} f^*(t)\,dt \le 1 \\ \Rightarrow \|Tf\|_{p,\infty} = \\ \sup_{t>0} t^{1/p}(Tf)^*(t) \le M_p \end{array} \right.$

$$t^{1/p}(Tg_t)^*(t) \le M_p \frac{1}{p}\int_0^\infty s^{1/p-1} g_t^*(s)\,ds \le \frac{M_p}{p}\int_0^t f^*(s)\, s^{(1-p)/p}\,ds \quad \text{by (1)\&(2)}$$

Let $s = tu$ $\quad ds = t\,du \quad s=0\ u=0$ & $s=t\ u=1$

$$= \frac{M_p}{p}\int_0^1 f^*(tu)\,(tu)^{(1-p)/p}\, t\,du = \frac{M_p}{p} t^{1/p}\int_0^1 f^*(tu)\, u^{(1-p)/p}\,du$$

Apply T is weak type (q,q) to h_t $\quad$ if $q < \infty$

$$t^{1/q}(Th_t)^*(t) \le M_q \frac{1}{q}\int_0^\infty s^{1/q-1} h_t^*(s)\,ds \le M_q\frac{1}{q}\left(\int_0^t f^*(t)\, s^{(1-q)/q}\,ds + \int_t^\infty h_t^*(s)\, s^{(1-q)/q}\right)$$

$$= M_q \frac{1}{q}\left(f^*(t)\, q\, t^{1/q} + t^{1/q}\int_1^\infty h_t^*(tu)\, u^{(1-q)/q}\,du \right)$$

$$\le \frac{M_q t^{1/q}}{q}\left(\frac{q}{p}\int_0^1 f^*(tu)\, u^{(1-p)/p}\,du + \int_1^\infty f^*(tu)\, u^{(1-q)/q}\,du \right)$$

$\geq \frac{q}{p}\int_0^1 f^*(t)\, u^{\frac{1}{p}-1}\,du = \frac{q}{p} f^*(t)\, p$

because $|h_t(s)| \le |f(s)|$ so $f^* \geq h_t^*$

Now $|Tf| \le |Tg_t| + |Th_t|$ $\quad$ [note quasi-linear works here]

so

$$(Tf)^*(2t) \le (Tg_t)^*(t) + (Th_t)^*(t)$$
$$\le \left(\frac{M_p}{p} + \frac{M_q}{p}\right)\int_0^1 f^*(tu)\, u^{(1-p)/p}\,du + \frac{M_q}{q}\int_1^\infty f^*(tu)\, u^{(1-q)/q}\,du$$

so $M = \frac{1}{p}(M_p + M_q)$ works. (If $q = \infty$ $\quad (Th_t)^*(t) \le M_\infty \sup_{s>0} h_t^*(s) \le M_\infty \sup_s h_t^*(s)$ $\quad 0 \cdot \int_1^\infty f^*(tu)\, u^{-1}\,du = 0$) ?

$p < r < q$ choose p_0, q_0 so that $p < p_0 < r < q_0 < q$

$\cancel{\int_I |Tf|^r = \int_0^\infty (Tf^*)^r(t)\,dt}$

some $g \in L_{r'}(I)$ $\|g\|_{r'} = 1$ s.t. $g(t) \geq 0$

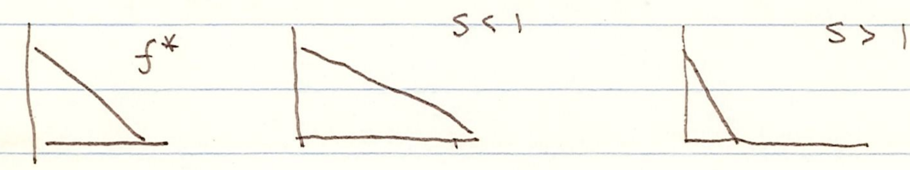
$\|Tf\|_r = \int_I Tf^*(t)\, g(t)\, dt$ if $I = [0,1]$ take $g(t) = 0$ $t > 1$

$$\leq \int_I g(t) \left\{ M \int_0^1 f^*\left(\tfrac{tu}{2}\right) u^{(1-p)/p}\, du + \int_1^\infty f^*\left(\tfrac{tu}{2}\right) u^{(1-q)/q}\, du \right\} dt$$

$$M \int_0^\infty \int_0^1 f^*\left(\tfrac{tu}{2}\right) g(t)\, u^{(1-p)/p}\, du\, dt = M \int_0^1 u^{(1-p)/p} \int_0^\infty f^*\left(\tfrac{tu}{2}\right) g(t)\, dt$$

$$\int_0^\infty f^*\left(\tfrac{tu}{2}\right) g(t)\, dt \leq \|f^*\left(\tfrac{tu}{2}\right)\|_r \|g(t)\|_{r'} = \left(\tfrac{2}{u}\right)^{1/r}$$

Lemma: $\cancel{\|f^*\left(\tfrac{tu}{2}\right)\|_r}$ $\|f^*(st)\|_r = s^{-1/r} \|f\|_r$



$$\int_0^\infty |f^*(st)|^r\, dt = \int_0^\infty |f^*(u)|^r \tfrac{1}{s}\, du = \tfrac{1}{s} \|f\|_r$$

$u = st$ $du = s\,dt$

$$\to M 2^{1/r} \int_0^1 u^{1/p - 1} u^{-1/r}\, du = 2^{1/r} M \frac{1}{\frac{1}{p} - \frac{1}{r}} \|f\|_r$$

$$M \int_0^\infty \int_1^\infty g(t) f^*\left(\tfrac{tu}{2}\right) u^{1/q - 1}\, du\, dt = M \int_1^\infty u^{1/q-1} \int_0^\infty f^*(tu/2)\, g(t)\, dt$$

$$\leq M \int_1^\infty u^{1/q - 1}\, 2^{1/r} u^{-1/r}\, du = 2^{1/r} M \int_1^\infty u^{1/q - 1/r - 1}\, du$$

$\frac{1}{q} < \frac{1}{r}$
$$= 2^{1/r} M \frac{1}{-\frac{1}{q} + \frac{1}{r}} \|f\|_r$$

Thus $\|Tf\|_r \leq 2^{1/r} M \left[\frac{1}{-\frac{1}{q} + \frac{1}{r}} + \frac{1}{\frac{1}{p} - \frac{1}{r}} \right] \|f\|_r$

done.

Lemma: $L_{p,1} \subset L_{p,p} \subset L_{p,\infty}$

if $\frac{1}{p} \int_0^\infty \frac{t^{1/p}}{t} f^*(t)\, dt < \infty \Rightarrow \frac{p}{p} \int_0^\infty [t^{1/p} f^*(t)]^p t^{-1}\, dt < \infty \Leftrightarrow \int |f^*(t)|^p dt$

$\Rightarrow \sup_{t>0} t^{1/p} f^*(t) < \infty$ [prove it for simple decreasing functions]

$f = \sum_{k=1}^{n} a_k \chi_{[t_{k-1}, t_k]}$ $\qquad 0 = t_0 < t_1 < \dots < t_n$

$\qquad a_1 > a_2 > \dots \quad a_n > 0$

$\gamma = \frac{1}{p} < 1$ [let $b_k = a_k^p$ $\quad s_k = t_k$]

want

$$\sum_{k=1}^{n} b_k (s_k - s_{k-1}) \le \left(\sum_{k=1}^{n} b_k^\gamma (s_k^\gamma - s_{k-1}^\gamma) \right)^{1/\gamma}$$

prove by induction showing $\varphi''(x) \ge 0$ $\quad \varphi(0) \le 0$

$\varphi(b_{n-1}) \le 0$

$$\varphi(x) = \sum_{k=1}^{n-1} b_k (s_k - s_{k-1}) + x(s_n - s_{n-1}) - \left\{ \sum_{k=1}^{n-1} b_k^\gamma (s_k^\gamma - s_{k-1}^\gamma) + x^\gamma (s_n^\gamma - s_{n-1}^\gamma) \right\}^{1/\gamma}$$

since $0 \le b_{n-1} \le b_n$

hence $\varphi(b_n) \le 0$.

<u>Example</u>: $S_\Theta(\sum' \alpha_n h_n) = \sum \alpha_n \Theta_n h_n \qquad \Theta_n = \pm 1$ is weak type $(1,1)$. If $\| \sum' \alpha_n h_n \|_1 \le 1$ we need to show $\sup_{t>0} t\, (\sum' \alpha_n \Theta_n h_n)^*(t) \le M < \infty$. Assuming this is true we get the Haar basis is unconditional in L_p $1 < p \le 2$ since S_Θ are strong type $(2,2)$

Also using $(2,2)$ & $(2^m, 2^m)$ with $m = 2, 3, \dots$ we get the Haar basis is unconditional in L_p $2 \le p < \infty$. So much for the real method.

Now for the complex method

<u>Three lines lemma</u>: Let $f(x)$ be analytic for $\{x + iy : s \le x \le t\}$

Let $M(r) = \sup_y |f(r+iy)|$ then

$$\log M(\theta s + (1-\theta)t) \le \theta \log M(s) + (1-\theta) \log M(t)$$

$$M(\theta s + (1-\theta)t) \le (M(s))^\theta (M(t))^{1-\theta}$$

pf:

Ignore the statement of 3-lines lemma on 6.9.

Let $S = \{ x+iy \in \mathbb{C} : a \le x \le c \}$.

<u>Lemma 1</u> If $f(z)$ is analytic & $\lim_{M \to \infty} \sup \{ |f(x+iy)| : \underset{x+iy \in S}{,} |y| = M \} = 0$

(*) then $\sup_{z \in S} |f(z)| \le \sup_{\substack{z \in S \\ x = a \text{ or } c}} |f(z)|$

<u>proof</u>: Let $\varepsilon < \sup \{ |f(z)| : z \in S \}$ choose M s.t. $|y| \ge M$ implies $|f(x+iy)| < \varepsilon$ for $x+iy \in S$.
The usual max mod principle applied to the box $x=a$, $x=c$ $|y| = M$ yields that $\max |f(z)|$ in the box occurs when $x=a$, $x=c$ or $|y| = M$. Since $|f|$ is small on $|y| = M$ it occurs when $x=a$ or $x=c$. This must be $\sup_{z \in S} |f(z)|$ since $|f|$ is small outside the box.

<u>Lemma 2</u> If $f(z)$ is bounded & analytic in S then (*) is true

<u>Proof</u>: Assume (as we may) that $a \ge 1$. Then $f_n(z) = f(z)/z^{1/n}$ satisfies the hypothesis of lemma 1 and $|f_n(z)| \uparrow |f(z)|$ pointwise.
Hence (*) is true.

<u>Three lines lemma</u>: If $f(z)$ is bounded & analytic in S and if $a \le b \le c$ define $M(b) = \sup \{ |f(x+iy)|, x = b \}$ then if $0 \le \theta \le 1$

$$M(\theta a + (1-\theta)c) \le [M(a)]^{\theta} [M(c)]^{1-\theta}$$

<u>proof</u>. Let $b = \theta a + (1-\theta)c$ so that $a \le b \le c$. Choose r real by $ra + \log M(a) = rc + \log M(c)$ ie so that $e^{ra} M(a) = e^{rc} M(c)$. & $r = \frac{1}{a-c} [\log M(c) - \log M(a)]$

Now $e^{rz} f(z)$ is bounded and analytic in S hence by Lemma 2 $e^{rb} M(b) \le \max \{ e^{ra} M(a), e^{rc} M(c) \} = e^{rc} M(c)$

Thus $rb + \log M(b) \le rc + \log M(c)$.

~~bsc~~. $c - b = -\theta(a-c)$ so that

$$\log M(b) \le r(c-b) + \log M(c) = -\theta(a-c)r + \log(M(c))$$
$$= -\theta [\log M(c) - \log M(a)] + \log M(c)$$
$$= \theta \log M(a) + (1-\theta) \log M(c).$$

done

Let ~~$\mathcal{B}$~~ L = the simple functions with support on a set with finite measure on (X, Σ, μ) and define $G: L \times L \to \mathbb{C}$ by

$G(f,g) = \int (Tf)(x)\, g(x)\, d\mu(x)$ where $T: L \to M$ is linear and M = measurable functions on (X, Σ). Assume that G is defined that is the integral exists

Let $1 \le p, q < \infty$ [the case where p or $q = \infty$ is left to reader]

Lemma 1 The norm of $T: L_p \to L_q$ is

$$\sup \{ |G(f,g)| : \|f\|_p \le 1, \|g\|_{q'} \le 1 \} \qquad 1/q' + 1/q = 1$$

proof: $|G(f,g)| \le \|Tf\|_q \|g\|_{q'} \le \|T\| \|f\|_p \le \|T\|$

Conversely, since the simple fcns are dense in L_p, L_q

$\|T\| = \|T|_{\text{simple fcns in } L_p}\|$ & $\|Tf\|_q = \|Tf|_{\text{simple fcns finite meas in } L_{q'}}\|$

Hence $\sup\{ |G(f,g)| : \|f\|_p \le 1, \|g\|_{q'} \le 1 \} = \|T\|$

Lemma 2: $G: L \times L \to \mathbb{C}$ is analytic in the sense if $z_1 \ldots z_n \in L \times L$ then $f(\lambda_1, \ldots \lambda_n) = G(\lambda_1 z_1 + \ldots \lambda_n z_n)$

$f: \mathbb{C}^n \to \mathbb{C}$ is analytic.

$$f(\lambda_1, \ldots \lambda_n) = \int T(\lambda_1 x_1 + \ldots \lambda_n x_n)(\lambda_1 y_1 + \ldots \lambda_n y_n)\, d\mu \quad \text{where } z_i = (x_i, y_i)$$

$$= \sum_{1 \le i,j \le n}' \lambda_i \lambda_j \int T x_i\, y_j\, d\mu = \sum \alpha_{ij} \lambda_i \lambda_j \leftarrow \text{quadratic.}$$

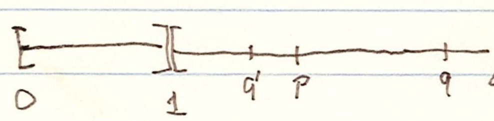
Riesz Thorin Thm: $\log M(\frac{1}{p}, \frac{1}{q})$ is convex function of $\frac{1}{p}, \frac{1}{q}$ $1 \le p, q \le \infty$

where

$$M(\tfrac{1}{p}, \tfrac{1}{q}) = \sup\{ |G(f,g)| : \|f\|_p \le 1, \|g\|_{q'} \le 1$$

pf: It suffices to show if $A(\frac{1}{p}, \frac{1}{q}) = \{(f,g) : \int |f|^p, \int |g|^q \le 1$ then and

$$M(\tfrac{1}{p}, \tfrac{1}{q}) = \sup_{f \in A(\frac{1}{p}, \frac{1}{q})} |G(f)| \quad \text{then } \log M(\tfrac{1}{p}, \tfrac{1}{q}) \text{ is convex}$$



Since $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$ $\quad \frac{1}{r'} = 1 - \frac{1}{r} = \theta + 1 - \theta - \frac{1}{r'}$

$$= \theta(1 - \tfrac{1}{p}) + (1-\theta)(1 - \tfrac{1}{q}) = \theta/p' + \tfrac{1-\theta}{q'}$$

Thus it suffices to prove

Lemma: G analytic $L \times L \to \mathbb{C}$ then $\log M(a,b)$ is convex for $0 < a, b \le 1$.

~~Let~~

pf Consider $\{ (f_1^a g_1, f_2^b g_2) : \|f_i\|_1 \le 1 \wedge \|g_i\|_\infty \le 1 \wedge f_i \ge 0 \} = B(a,b)$

Claim $A(a,b) = B(a,b)$

Since $\int |f^a g|^{1/a} = \int |f||g| \le \|f\|_1 \|g\|_\infty \le 1$

we have $A(a,b) \subset B(a,b)$

if $(h_1, h_2) \in A(a,b)$ then $f_1 = |h_1|^{1/a}$ & $g_1 = \operatorname{sgn} h_1$

similarly for h_2 & $(h_1, h_2) \in B(a,b)$

Thus $M(a,b) = \sup\{|G(f^\alpha g)| : f^\alpha g = (f_1^a g_1, f_2^b g_2) : \|f_i\|_1 \le 1 \ \|g_i\|_\infty \le$

$= \sup_t \sup_\alpha |G(f^{\alpha+it} g)| \quad f, g$ as above)

Since $f^{\alpha+it} g \in A(a,b)$ because $|f^{it}| = |e^{rti}| = 1$

$= \sup_{f,g} \sup_t \{|G(f^{\alpha+it} g)| \quad f, g$ as above$\}$

$p_n(i)$ positive $z_n(i)$ – characteristic function

Now write $\lambda_1 = a + it_1 \quad \lambda_2 = b + it_2$ Assume $f_1, f_2 \; g_1, g_2$ are simple. $f_i^{\lambda_i} g_i = \sum_{n=1}^{N(i)} p_n^{\lambda_i}(i) z_n(i)$. Since G is analytic

$G(f^\lambda g)$ is analytic as a function of (λ_1, λ_2) and bounded for each strip $0 \le \operatorname{Re} \lambda_i \le c_i < \infty$ [Since $p_n \ge 0$]

To show

$$\log M(a,b) = \sup_{\substack{f \in A(1,1)\, f \ge 0 \\ g \in A(0,0)}} \log \{\sup_t |G(f^{\alpha+it})|\}$$

is convex it suffices to show two lemmas

<u>Lemma</u> A: $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is analytic & bounded on a set of the form $S = \{(x_1 + iy_1, x_2 + iy_2 : x_1, x_2 \in C\}$ where C is convex then $\log M(x)$ is convex function of $x = (x_1, x_2)$ where

$$M(x) = \sup_{y_1, y_2} (f(x_1 + iy_1, x_2 + iy_2)$$

<u>Pf</u>: Connect points with straight line

<u>Lemma</u> B: If $\{M_\gamma \ \gamma \in \Gamma\}$ are convex, so is $\sup_\alpha M_\alpha$