480
360

MATH 74 TEST SHOW ALL WORK, USE ONE SIDE OF PAPER ONLY, BE NEAT!

RC MINOR FOR

§1: 10 pts ea. : 1. (.975) FIND $\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1/4 & \pi \\ 1 & 2 & 7 \end{pmatrix}$

10-(22); 9; 5

(24,0) 2. Solve $\dot{x} = Ax$, $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; given that A has

(20,4) eigenvalues −1, 2 with eigenvectors (1,−1), (1,1) respectively.

10-(13), 9, 9, 8, 8, 7, 6, 5, 4, 2-(3) (.800)

§2: 15 pts ea. 3. If A is an n×n matrix, show that the

(20,4) rows of A are independent if and only if the columns of A are independent. (.778)

15-(13), 11-(4), 9, 8, 8, 6, 4, 4, 2

4. Let D be the differentiation operator on the vector space C^∞, of all infinitely differentiable real functions, into itself. Show that every real λ is an eigenvalue of D and find the eigenfunctions. (.611)

(17,7)

−2 each of C; −1 for not C^∞; −2 exclude λ=0; −1 (a≠0)

14-(4), 13-(5), 12-(5) 10-(3), 6, 1-(3) 0-(3)

5. Let u be a non-zero solution to the D.E. $y'' + P(x)y' + Q(x)y = 0$.

(19,5) Show that the subsitution $y = uv$ reduces the D.E. $y'' + P(x)y' + Q(x)y = R(x)$ to a 1st order D.E. for v'. (.714)

15-(10), 13, 12, 11, 10, 9-(5), 7, 5, 2, 2, 0

(19,5) 6. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$.

15-(4), 13-(8), 12, 11, 11, 9-(4), 7, 6, 5, 5, 4 (.725)

§3 20 pts.

7. For this problem U is always an upper triangular matrix and $P_A(\lambda) = \det(\lambda I - A)$ is the characteristic poly of the matrix A

(15, 9)

A) Show that the roots of $P_U(\lambda)$ and the entries in the main diagonal of U are exactly the same, counting repetitions.

B.) Suppose $A \sim U$, show that the roots of $P_A(\lambda)$ and the entries in the main diagonal of U are exactly the same, counting repetitions.

(.592)

[HINT: First show $P_A(\lambda) = P_U(\lambda)$.]

20-(2), 19, 17(2) 16 14-(3), 13-(4), 10-(2) 9(3) 8-(3) 6, 4, 0

KEY MATH 74

1. $\det\begin{pmatrix}1 & 2 & 3\\ 0 & 1/4 & \pi\\ 1 & 2 & 7\end{pmatrix} = \det\begin{pmatrix}1 & 2 & 3\\ 0 & 1/4 & \pi\\ 0 & 0 & 4\end{pmatrix} = (1)(1/4)(4) = 1$

Substract row one from row three, we obtain a triangular matrix. minors -2 Row Con 0

2. Let $B = \begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix}$, $B \sim A$ via $C = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. Solving $\dot{Y} = BY$ we get $y_1 = k_1 e^{-t}$ $y_2 = k_2 e^{2t}$. Since $X = CY$,

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}k_1 e^{-t}\\ k_2 e^{2t}\end{pmatrix} = \begin{pmatrix}k_1 e^{-t} + k_2 e^{2t}\\ -k_1 e^{-t} + k_2 e^{2t}\end{pmatrix}.$$ Using $X(0) = \begin{pmatrix}1\\ 0\end{pmatrix}$

$1 = x_1(0) = k_1 + k_2$, $0 = x_2(0) = -k_1 + k_2$; hence $k_1 = k_2 = 1/2$.
Solution is $x_1(t) = \frac{1}{2}[e^{-t} + e^{2t}]$ $x_2(t) = \frac{1}{2}[e^{2t} - e^{-t}]$.

Fredholm alternative; logic

3. The rows of A are independent iff $\det A \neq 0$ iff $\det A^t \neq 0$ iff the rows of A^t are independent iff the columns of A are independent.

4. Let λ be real and suppose $Df = \lambda f$ or $f' = \lambda f$. The general solution to this D.E. is $f(t) = Ce^{\lambda t}$. Since $Ce^{\lambda t}$ is infinitely differentiable, it is an eigenfunction for the eigenvalue λ (if $C \neq 0$.)

5 Let $y = uv$, then $y' = u'v + uv'$ and $y'' = u''v + 2u'v' + uv''$.
Subsitution into the 2nd DE gives:

$$u''v + 2u'v' + uv'' + P(x)[u'v + uv'] + Q(x)(uv) = R(x)$$

collectioning the underlined terms,

$$[u'' + P(x)u' + Q(x)u]v + uv'' + [2u' + P(x)u]v' = R(x)$$

↳ but this $= 0$, since u is a solution to the 1st D.E.
we have: $u(v')' + [2u' + P(x)u](v') = R(x)$ which is 1st order in v'.
[do not divide by u.]

TEST KEY (con't)

6. $\det\begin{pmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)(2-\lambda)$

To get a non-zero term of the det. we must pick the 2^{nd} entry in the second row; this forces us to pick the 1^{st} entry in the 1^{st} row and thus the last entry in the last row. Thus the eigenvalues are $\lambda = 1, 2, 3$.

$\lambda = 1$,

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

hence $y = 0$ and $-x + z = 0$

So the eigenvectors are $k(1,0,1)$ $k \neq 0$

$\lambda = 2$

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

hence $y = 0$, $x = 0$

So the eigenvectors are $k(0,0,1)$ $k \neq 0$

$\lambda = 3$

$$\begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

hence $z = 0$ and $x = y$

So the eigenvectors are $k(1,1,0)$ $k \neq 0$.

upper triangular

7. Let $U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} \cdots & u_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 \cdots & 0 & u_{nn} \end{pmatrix}$ we have that

$$P_U(\lambda) = \det\begin{pmatrix} \lambda-u_{11} & -u_{12} & \cdots & -u_{1n} \\ 0 & \lambda-u_{22} & \cdots & -u_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 \cdots & 0 & \lambda-u_{nn} \end{pmatrix} = (\lambda-u_{11})(\lambda-u_{22})\cdots(\lambda-u_{nn})$$

Since the matrix is in triangular form.

eigenvectors or values no help

A) follows from the above.

B) Let $A = C^{-1}UC$ $\quad P_A(\lambda) = \det(\lambda I - A)$

$= \det(\lambda I - C^{-1}UC) = \det(C^{-1}\lambda I C - C^{-1}UC)$

$= \det(C^{-1}(\lambda I - U)C) = \det C^{-1} \det(\lambda I - U) \det C$

$= \det(\lambda I - U) = P_U(\lambda)$. B) now follows from A).

Computation Problem #1. Due October 1, 1973

Evaluation of a Polynomial.

(a) The algorithm to be applied to the evaluation of a polynomial by the "brute force" method is more or less obvious. Nonetheless, in order to gain experience in charting of algorithms, prepare a block diagram of all of the steps which you must program the computer to do in order to obtain the value of y for a given value of x.

The polynomial to be evaluated is the following.

$$y = a_4 + a_3x + a_2x^2 + a_1x^3 + a_0x^4.$$

The program should be general enough to utilize any given set of five real coefficients, and, given the desired value of x, should provide a listing of the value of x and then a corresponding value of y.

For this problem we shall use the following set of coefficients.

$$a_o = -5,\ a_1 = 2,\ a_2 = -1,\ a_3 = 2,\text{ and } a_4 = 3.$$

(b) Translate the block diagram which you have prepared in part (a) into a computer program using BASIC language. The input value of x will be 2.03118.

Turn in to your instructor the block flow diagram and computer program.

(c) (Optional -- involves looping.)
Horner's method allows us to evaluate the polynomial,

$$p_n(x) = a_n + a_{n-1}x + \cdots + a_1\ x^{n-1} + a_o\ x^n.$$

using only n multiplications and n additions instead of the 2 n-1 multiplications and n additions of the brute force method. This is done by writing $p_n(x)$ in a nested fashion:

$$p_n(x) = (((\cdot\cdot(a_o\ x + a_1)x + a_2)x + \cdots + a_{n-1})\ x + a_n.$$

Let n = 4 and chart the appropriate algorithm. Prepare a BASIC program which will accept values of the five coefficients: $a_o = -5$, $a_1 = 2$, $a_2 = -1$, $a_3 = 2$, $a_4 = 3$. The program must print the values of p_4 corresponding to the printed values of x: x = 0,1,2,..., 9.

Can you find a general procedure that would be equally easy to program for n = 20? (This would involve the use of a DIMension statement. See Chapter 3 in the BASIC Conversational Language Manual.)

Note: On this problem and all future computations problems which are assigned in the computations sequence this semester include in a block at the upper-right-hand corner of the front page the following information.

Name (yours)/(name of instructor who assigned the problem)
Date due/Problem number
Total time (minutes) spent on problem
Time (minutes) on computer terminals

Example: T. Student/Beeman
Sept. 23, 1971/Prob. 1-A
55 min./ 0 min.

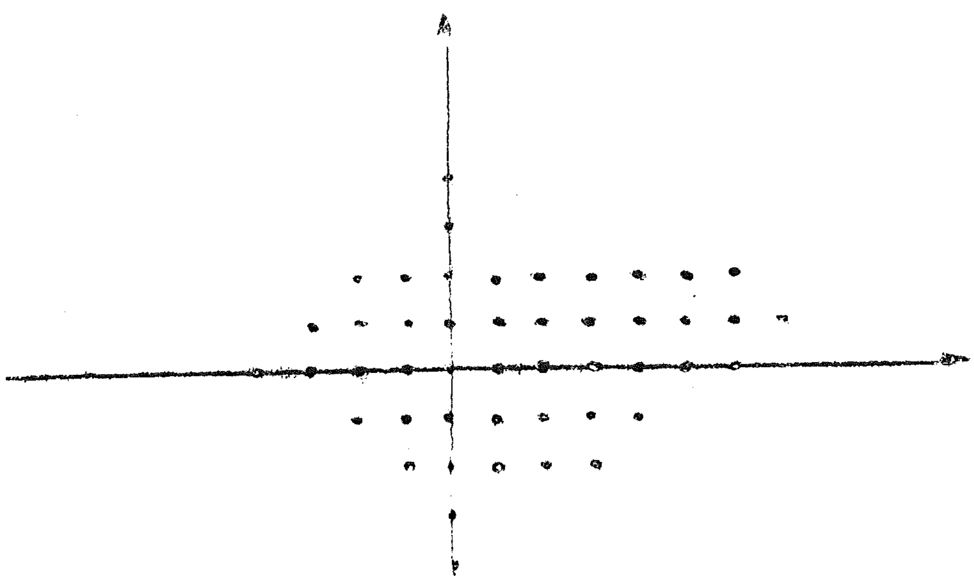
Freshman Computation
Problem 2
Due Date: October 5, 1973
In computations recitation sections.

Some Preliminary Comments

In this second problem, you are asked to re-examine a question first studied in detail by the great mathematician, K. F. Gauss. It is basically one of counting, and the program required for your work will not be extensive. Still, your result will have intriguing aspects to it - hopefully, some indication of the sheer computational speed of the modern digital computer will also become evident to you.

The Gauss Lattice Point Problem

We define, for the purposes of this problem, the infinite unit lattice to be the collection of all the points in the xy-cartesian plane with integer co-ordinates including the origin, (0,0). Thus, (7,-2) is a lattice point, while (π,7) is not. The sketch shows how this lattice looks near the origin.



Suppose now that we draw a circle with radius r, centered at the origin. We define the function f(r) as follows:

$$f(f) \triangleq \text{number of lattice points inside or on the boundary of the circle.}$$

With this definition, we have $f(o) = 1$ and $f(1) = 5$.

The computation of f(r), for any given r, is just a problem in counting. Of course, as r gets large, this can become a tedious task. Here is where we can use the digital computer to help us.

(a) Draw a flow diagram of an algorithm that computes f(r). There is more than one way to approach this problem, and some are "better" than others.

(b) Using your flow diagram, write a BASIC program and use it to find f(r) for
r = 10, 50, 100, 200, and 300.

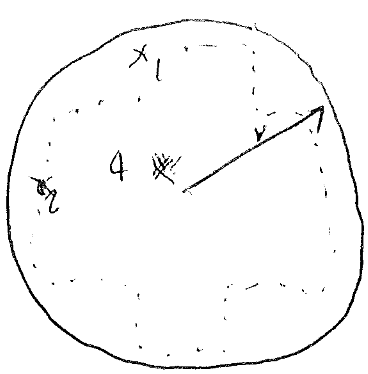
(c) For the above values of r, compute the value of

$\frac{f(r)}{r^2}$. Do your results suggests anything to you? Can you now see why Gauss was interested in this problem?

(d) The value of f(r) is always odd. See if you can develop a simple proof of this statement.
actually $f(r) = 4k+1$

Note: As with problem #1, include the following information at the top of the first page of your solution:

Name (yours)/(name of instructor who assigned the problem)
Date due/Problem number
Total time (minutes) spent on problem
Time (minutes) on computer terminals



COMPUTATIONS PROBLEM 3

Due: Monday, October 15
no later than noon
Mrs. Graham's office, Parsons 267

a) Write a BASIC program to find the roots of an arbitrary quadratic $Ax^2 + Bx + C$. Your program should handle the ten triples of coefficients given below, in each case printing A,B,C and the roots. The program must be able to identify complex roots and to handle the case A = 0.

	A	B	C
1	1	1	-2
2	1	2	1
3	1	2	3
4	0	5	6
5	-7	3	2
6	2	-3	5
7	4	11	12
8	13	15	-212
9	0	0	0
10	0	0	5

b) Submit a flow chart, a list, and a printout of your results.

c) In the upper-right-hand corner of the first page, include the following information:

Name/Math Instructor Name
Math Section letter
Date/Problem #
Total time/time on terminal, minutes

the problem will be returned to you in your math class.

Computation Problem #5
Due November 7
For Math 74 Students

<u>Gauss Elimination</u> is a very general computational technique with useful applications in all branches of science and engineering. The technique yields solutions of sets of simultaneous linear algebraic equations with constant coefficients. It has been applied to the analysis of the network equations that describe an electric circuit, to solve the difference equations used to approximate more complicated sets of differential equations, and in the computation of the inverse of a matrix.

<u>Some Basic Theory</u>

Consider the set of equations:

$$\sum_{k=1}^{n} a_{1,k} x_k = b_1$$
$$\sum_{k=1}^{n} a_{2,k} x_k = b_2$$
$$\vdots$$
$$\sum_{k=1}^{n} a_{n,k} x_k = b_n$$

The a's and b's are all known constants. We define the augmented coefficient matrix $\tilde{A}$ as

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & & & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & b_n \end{bmatrix}$$

The Gauss Elimination Method now reduces $\tilde{A}$, through a series of row operations to

$$\tilde{A}' = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ 0 & a'_{2,2} & \cdots & a_{2,n}' & b_2' \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & a_{n,n}' & b_n' \end{bmatrix}$$

where all the elements below the diagonal of the original coefficient matrix have been reduced to zeroes. This reduction technique is really just the way you would probably solve the equation set by hand, eliminating variables successively. Thus, the first step in the reduction is to replace the first column of $\mathfrak{A}$ by zeroes, except $a_{1,1}$. This is done by subtracting the product of the first row and the number $a_{k,1}/a_{1,1}$ from the k'th row, where $2 \leq k \leq n$. The result is that $a_{k,1}$ is replaced by 0 for $2 \leq k \leq n$, while $a_{k,i}$ and b_k are replaced by

$$a'_{k,i} = a_{k,i} - \frac{a_{k,1}}{a_{1,1}} a_{1,i}, \quad b'_k = b_k - \frac{a_{k,1}}{a_{1,1}} b_1,$$

where $2 \leq k \leq n$ and $1 \leq i \leq n$. This first step reduces $\mathfrak{A}$ to

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ 0 & a'_{2,2} & \cdots & a'_{2,n} & b'_2 \\ \vdots & & & & \\ 0 & a'_{n,2} & \cdots & a'_{n,n} & b'_n \end{bmatrix}$$

The reduction process is now repeated on the sub-matrix formed by ignoring the first row and column of this new matrix. The whole procedure repeats until A' is generated. The set of equations for which A' is the augmented coefficient matrix is

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a'_{2,2} & \cdots & a'_{2,n} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a'_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{bmatrix}.$$

From this, we immediately write the solution for x_n, i.e.

$$x_n = \frac{b_n'}{a'_{n,n}}$$

Using this result, we can then write

$$x_{n-1} = \frac{b'_{n-1} - a_{n-1,n}\, x_n}{a'_{n-1,n-1}}$$

In general,

$$x_k = \frac{b'_k - a'_{k,n}\, x_n - a_{k,n-1}\, x_n \cdots - a_{k,k+1}\, x_{k+1}}{a'_{k,k}}$$

where $1 \leq k \leq n$.

First Task: Write a BASIC program that implements the Gauss Elimination Method for up to 100 variables. Write it so that the number of variables is an input parameter (called for by the program when it begins execution), and the a's and b's are specified in BASIC statements. Your program should include the feature of substituting the answers back into the original equations to determine "how well" the answers work.

Second Task: Using your program solve the following set of equations:

$$0.357\, x_1 + 0.203\, x_2 + 0.714\, x_3 = 0.017$$

$$-0.206\, x_1 + 0.295\, x_2 + 0.371\, x_3 = 0.111$$

$$0.412\, x_1 + 0.315\, x_2 + 0.604\, x_3 = 0.149$$

also state what the left sides of the above equations equal when the program results are substituted back into them.

Third Task: By the further use of Gauss elimination (as discussed in class or in Apostol), a matrix is obtained, if all goes well, in the form:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} b''_1 \\ b''_2 \\ \cdot \\ \cdot \\ \cdot \\ b''_n \end{bmatrix} .$$

Write a program that will reduce the augmented matrix A to the row canonical form above. Your program must recognize values that come within error limits of zero as zero (but have it print out exactly its calculated value).

Repeat the equation of task II using the new program and check results as before. Compare.

Instructions

1. Run everything as one program on the computer.
2. Submit a flowchart program listing, and printout of results.
3. Turn in the problem to Mrs. Graham.
4. The problem will be returned to you in math class.
5. In the upper-right-hand corner of the first page, include the following information.

Your Name/Name of your instructor
Math course number
Date/Problem number
Total time/Terminal time, minutes