MATH 74 BELLENOT

MWF 9:35 – 10:50

TEXT: Apostol's <u>Calculus</u> Vol II (2nd ed.)

TESTS: Midterms (one hour) Wed. Nov. 14
and Wed. Dec. 19

Final (three hours) Fri. Feb. 1

HOMEWORK: Generally speaking, homework will be assigned daily and will be due the next class period. An assignment will consist of five or six problems, of which two or three will be *problems. The *problems are to be written up, turned in, and graded. These *problems are to be <u>your</u> <u>own</u> work, hence the honor system applies to them.

GRADES: As a rule of thumb, 50% on <u>all</u> graded papers is required for a grade of P.

SOLUTION OF A SET OF ORDINARY DIFFERENTIAL EQUATIONS WITH DIGITAL COMPUTER

STATEMENT OF PHYSICAL PROBLEM:

A Biological Epidemic Problem: In a total population of n individuals there are, at any time t, $y(t)$ infectious carriers of a contagious disease, $x(t)$ members of the population who are susceptible to the disease, and $z(t)$ individuals who are recovered and immune. It is clear that $x(t) + y(t) + z(t) = n$ for all $t \geq 0$. The following system of differential equations, known as Kermack-McKendrick equations, characterize the epidemic.

$$
\begin{aligned}
\dot{x}(t) &= -\lambda x(t)y(t) \\
\dot{y}(t) &= \lambda x(t)y(t) - \mu y(t) \qquad (1) \\
\dot{z}(t) &= \mu y(t)
\end{aligned}
$$

If t is the time in days, $\lambda = 0.001$, $\mu = \frac{1}{14}$, $n = 1000$,

$x(0) = 900$, $y(0) = 10$, and $z(0) = 90$.

STATEMENT OF COMPUTER PROBLEM:

Write a ~~NEWTRAN~~ program which will solve the set of differential equation (1) for

a) the number of susceptibles, $x(t)$;

b) the number of infectives, $y(t)$;

c) the number of recovered and immune, $z(t)$; and

d) the epidemic curve $-\dot{x}(t)$, i.e. the rate at which new disease occur.

Your program must have the following features:

?

1. It must use iterative form of improved Euler's Method with number of iterations being 3.
2. It must use interval of computation $h = 0.1$ days but print out the results for every day and up to 15 days.
3. Plot on a single graph paper the variables $x(t)$, $y(t)$, $z(t)$ and $-\dot{x}(t)$ versus time, t.

MODEL: A Biological Epidemic.

In a total population of n individuals there are, at any time t, $y(t)$ infectious carriers of a contagious disease, $x(t)$ members of the population who are susceptible to the disease, and $z(t)$ individuals who are recovered and immune. We have $x(t) + y(t) + z(t) = n$ for all t. The following system of differential equations, know as the Kermack-McKendrick equations, characterize the epidemic.

$$
\begin{aligned}
\dot{x}(t) &= -\lambda\, x(t)\, y(t) \\
\dot{y}(t) &= \lambda\, x(t)\, y(t) - \mu\, y(t) \qquad (*) \\
\dot{z}(t) &= \mu\, y(t)
\end{aligned}
$$

COMPUTER PROBLEM:

Write a program which will solve the set of differential equation (*) for

1) the number of susceptibles, $x(t)$;
2) the number of infectives, $y(t)$;
3) the number of recovered, $z(t)$; and
4) the epidemic curve $-\dot{x}(t)$ (i.e. the rate which new disease occurs);

given that t is the time in days, $\lambda = 0.001$, $\mu = 1/14$, $n = 1000$, $x(o) = 900$, $y(o) = 10$ and $z(o) = 90$.

Your program must have the following features:

1) It must use the iterative form of the improved Euler's Method with the number of iterations being three.
2) It must use the interval of computation $h = 0.1$ days but print out the results for every day up to fifteen days only.

Plot on a single graph paper the variables $x(t)$, $y(t)$, $z(t)$ and $-\dot{x}(t)$ versus time t.

NUMERICAL METHODS FOR SOLVING DIFFERENTIAL EQUATIONS

We will develop two methods for numerically approximating the solution to first order (non-linear) systems of differential equations with initial conditions. Our IVP has the form

$$(*) \qquad \dot{X} = f(X,t)$$
$$X(o) = X_o,$$

where $X(t)$ is the n-tuple $(x_1(t), x_2(t), \ldots x_n(t))$, X_o is the n-tuple $(x_1(o), x_2(o), \ldots x_n(o))$, t is "time" and f is a n-vector valued function of the n+1 variables $x_1(t), x_2(t), \ldots x_n(t), t$. We will solve $(*)$ for $t \geq o$, the solution for $t \leq o$ is similar.

The basic idea is to "divide" the positive t-axis into discrete points $t_o = o$, $t_1, t_2 \ldots$. To facilitate this process, let h be a "small" positive number, called the interval of computation, and define $t_o = o$ and $t_{n+1} = t_n + h$ for $n = 0,1,2, \cdots$. Since we are given $X(t_o)$, our problem is reduced to producing a "good" approximation for $X(t_{n+1})$ given a "good" approximation to $X(t_n)$. We first attempt the case when $(*)$ is an ordinary differential equation (i.e. $n = 1$) and latter extend to the general case. We use the short hand X_n for $X(t_n)$.

EULER'S METHOD:

We are trying to solve the D.E. $dx/dt = f(x,t)$, given $x(o) = x_o = x(t_o)$. Suppose we have obtained x_n and we are trying to make the leap to x_{n+1}. Euler's idea can be expressed as "if h is 'small' enough the tangent line at x_n will be a 'good' approximation to the curve $x(t)$ near t_n." Operationally, this means we assume that the curve and the tangent line are the same between t_n and t_{n+1}. It is simple analytic geometry to obtain $x_{n+1} = x_n + h\, d_n$, where d_n is the slope of the tangent line at x_n, that is $d_n = dx/dt|_{t_n} = f(x_n, t_n)$. Let us do a simple example.

Example: Let's take the IVP $\dot{x} = t$, $x(o) = 1$. Here $f(x,t) = t$. Let $h = 1$. The table below gives the values of x_n, $n = 0,1,2,3,4,5$; given by Euler's method and by the exact solution $x(t) = \frac{1}{2} t^2 + 1$.

time	0.	1.	2.	3.	4.	5.
Euler's x(t)	1.0	1.0	2.0	4.0	7.0	11.0
True x(t)	1.0	1.5	3.0	5.5	9.0	13.5

Let's calculate x_4, given x_3; $d_3 = f(x_3,t_3) = f(4,3) = 3$, so $x_4 = x_3 + h\,d_3 = 4 + 3 = 7$. Please note the way the error increases with time. A graph is enlighting.

IMPROVED EULER'S METHOD:

The "improvement" is the observation that it is the secant line through x_n and x_{n+1}, not the tangent line at x_n, that is the "good" approximation for our purposes. Since we do not know what the value of x_{n+1} is, we try instead to find the slope of the secant using x_n and the value $\bar{x}_{n+1}$ obtained from Euler's method (unimproved.)

Our best guesses to the slope of the curve $x(t)$ at t_n and t_{n+1} are $d_n = f(x_n,t_n)$ and $d_{n+1} = f(\bar{x}_{n+1}, t_{n+1})$ respectively. Now if $x(t)$ is a nice curve and h is "small", then the slope of the secant should be some sort of "average" between these two "extremes". With this in mind, we let $S_n^{(1)} = \frac{1}{2}[d_n + d_{n+1}]$. The line through x_n with slope $S_n^{(1)}$ is our approximation to the secant line. Analytic geometry gives the better approximation to $x(t_{n+1})$ as $x_{n+1}^{(1)} = x_n + h\,S_n^{(1)}$. The student may amuse himself (or herself(sorry Tracy)) with the observation that this improvement is enough to give the exact values in the <u>very simple</u> example above. (Of course, in general, this will not happen.)

But why stop now? We have a "better" approximation to x_{n+1} and thus we can obtain a "better" approximation to the slope of the secant. In fact, let $d_n^{(1)} = d_n = f(x_n,t_n)$ and $d_{n+1}^{(1)} = f(x_{n+1}^{(1)},t_{n+1})$, then the slope of the secant should be $S_n^{(2)} = \frac{1}{2}(d_n^{(1)} + d_{n+1}^{(2)})$ and a "better" approximation to $x(t_{n+1})$ would be $x_{n+1}^{(2)} = x_n + h\,S_n^{(2)}$.

This process can be repeated any finite number of times. Let us formulate this by defining $d_n^{(k)} = d_n = f(x_n,t_n)$ and $d_{n+1}^{(k)} = f(x_{n+1}^{(k)}, t_{n+1})$, letting $S_n^{(k+1)} = \frac{1}{2}(d_n^{(k)} + d_{n+1}^{(k+1)})$ and obtaining the value $x_{n+1}^{(k+1)} = x_n + h\,S_n^{(k+1)}$. We have arrived at the iterative form of the improved Euler's method with $k+1$ iterations. It can be shown that if more than two iterations are used that the order of the error reaches and remains at a fixed value. A very "reasonable" approximation.

EXTENSION TO THE GENERAL CASE:

This is quite simple. Since $f(X,t)$ is an n-vector valued function, let its $i^{\underline{th}}$ component be $f_i(X,t)$ (i.e. $f(X,t) = (f_1(X,t), f_2(X,t), \cdots f_n(X,t)$.) Writing this in long hand, we have by equating the $i^{\underline{th}}$ components of $\mathring{X}$ and $f(X,t)$:

$$\mathring{x}_i = f_i(x_1(t),\ x_2(t),\ \cdots x_n(t)),\ t).$$

So to make the leap from $(x_i)_m$ to $(x_i)_{m+1}$ we only need to know $(x_j)_m$, for $j = 1, \cdots n$, if we are using Euler's method. For the improved Euler's to obtain $(x_i)_{m+1}^{(k+1)}$, we need only to know $(x_j)_{m+1}^{(k)}$, for $j = 1, \cdots n$.

This a very small amount of Numerical Solutions to Differential Equations. The subject matter is very useful, challenging and interesting.

BE SURE TO BRING THIS HANDOUT TO MATH CLASS WHEN CLASSES START AGAIN JANUARY 7.

DUE DATE: January 18, 1974, in the Freshman Division office, Parsons #267

Put the standard information on the upper right-hand corner of the first page.

1 Let P_n be the set of polynomials of degree less than or equal to n.

A) Show that P_n is a vector space.

B) Show that $\{1, x, x^2, \cdots x^n\}$ is a basis for P_n.

Let $\xi_1, \xi_2 \cdots \xi_k$ be k-distinct points on the real axis. Define a map $T: P_n \to \mathbb{R}^k$ by the rule: if $p(x)$ is a poly $T(p)$ is the k-vector $(p(\xi_1), p(\xi_2) \cdots p(\xi_k))$

C) Show that T is linear.

D) Show if $k = n+1$, then T is invertible.
[HINT: Show that the null space is trival.]
[SUBHINT: Fundemental Theorem of Algebra.]

E) Show that there always exists a poly $p(x)$ of degree ≤ 3 that has the values
$p(\xi_1) = a \quad p(\xi_2) = b \quad p(\xi_3) = c \quad p(\xi_4) = d$
at the four distinct points ξ_1, ξ_2, ξ_3 and ξ_4.

6. Suppose V is an inner product space.

A) Show: $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

B) If x, y are orthogonal, $\|x\|^2 + \|y\|^2 = \|x+y\|^2$.

C) Show by an example in $\mathbb{R}^2$, that B) need not be true if x and y are not orthogonal.

D) If x, y are orthogonal and non-zero then x, y are independent.

7. Show that $\{\pi, \tan x, \text{[illegible]}, e^x, |x|\}$ are independent.

MATH 74 TEST SHOW ALL WORK, USE ONE SIDE OF PAPER ONLY, BE NEAT!

§1: 10 pts ea. : 1. FIND $\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1/4 & \pi \\ 1 & 2 & 7 \end{pmatrix}$

2. Solve $\dot{x} = Ax$, $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; given that A has eigenvalues $-1, 2$ with eigenvectors $(1,-1), (1,1)$ respectively

§2: 15 pts ea. 3. If A is an $n \times n$ matrix, show that the rows of A are independent if and only if the columns of A are independent.

4. Let D be the differentiation opeator on the vector space C^∞, of all infinitely differentiabe real functions, into itself. Show that every real λ is an eigenvalue of D and find the eigenfunctions.

5. Let u be a non-zero solution to the D.E. $y'' + P(x)y' + Q(x)y = 0$. Show that the subsitution $y = uv$ reduces the D.E. $y'' + P(x)y' + Q(x)y = R(x)$ to a 1^{st} order D.E. for v!

6. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$.

§3 20 pts.

7. For this problem U is always an upper triangular matrix and $P_A(\lambda) = \det(\lambda I - A)$ is the characteristic poly of the matrix A

A) Show that the roots of $P_U(\lambda)$ and the entries in the main diagonal of U are exactly the same, counting repetitions.

B.) Suppose $A \sim U$, Show that the roots of $P_A(\lambda)$ and the entries in the main diagonal of U are exactly the same, counting repetitions.

[HINT: First show $P_A(\lambda) = P_U(\lambda)$.]

1. $c_1e^x + c_2xe^x + c_3x^2e^x + c_4e^{2x} + c_5$ [illegible]

2. Dependent, since $\cos^2 x + \sin^2 x = 1$.

3. $T(x+y) = T((x_1+y_1, \ldots, x_n+y_n)) = (\lambda_1 x_1 + \lambda_1 y_1, \ldots, \lambda_n x_n + \lambda_n y_n)$

$$= (\lambda_1 x_1, \ldots \lambda_n x_n) + (\lambda_1 y_1, \ldots \lambda_n y_n) = T(x) + T(y)$$

$$T(\alpha x) = T(\alpha x_1, \ldots \alpha x_n) = (\lambda_1 \alpha x_1, \ldots \lambda_n \alpha x_n)$$
$$= \alpha(\lambda_1 x_1, \ldots \lambda_n x_n) = \alpha T(x)$$

Since $T(e_i) = \lambda_i e_i$ (e_i the usual basis of $\mathbb{R}^n$),

$$m(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix}$$

4. <u>T invertible $\Rightarrow \lambda_i \neq 0$ $i = 1, 2, \ldots n$.</u> Suppose not then some $\lambda_i = 0$; Thus the matrix of T has a row zero's and hence is singular. ※ <u>Proof 2</u>: Suppose not, then some $\lambda_i = 0$, Suppose $\lambda_j = 0$, $T(e_j) = \lambda_j e_j = 0$, so that $N(T) \neq \{0\}$, T is not 1-1 and so could not be invertible. ※

<u>$\lambda_i \neq 0$ $i = 1, 2, \ldots n \Rightarrow T$ invertible</u>: Let S be the matrix

$$\begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \lambda_2^{-1} & \\ & & \ddots \\ 0 & & & \lambda_n^{-1} \end{pmatrix}.$$

It is clear that $ST = TS = I$ and so $S = T^{-1}$ and T is invertible.

5. See me.

6. Integrating factor e^{e^x}, so that $(e^{e^x} y)'$

$= e^{e^x} y' + e^{e^x} e^x y = e^{e^x} e^{-e^x} = 1$. Integrating both sides

$$e^{e^x} y = \int_0^x 1\, dt \quad \text{(no constant term since } y(0) = 0\text{.)}$$

so $e^{e^x} y = x$ or $y = xe^{-e^x}$.

7. (Actually one must add the assumption that all $\xi_i \neq 0$, otherwise the set $\{0\}$ is orthogonal, but not independent.)

Suppose $\xi_1, \xi_2, \dots \xi_n$ are non-zero orthogonal vectors and suppose $c_1, c_2, \dots c_n$ are scalars such that $\sum_1^n c_i \xi_i = 0$. Taking the inner product of both sides with ξ_j gives

$$0 = \langle \xi_j, 0 \rangle = \langle \xi_j, \sum_{i=1}^n c_i \xi_i \rangle = \sum_{i=1}^n c_i \langle \xi_j, \xi_i \rangle = c_j \langle \xi_j, \xi_j \rangle,$$

but $\langle \xi_j, \xi_j \rangle > 0$, so $c_j = 0$. Therefore $\xi_1, \xi_2, \dots \xi_n$ are independent.

In $\mathbb{R}^2$ note that $(1,0)$ and $(1,1)$ are independent but not orthogonal.

8. Suppose $f(x), g(x) \in C_{\infty}$ and α a scalar, then

Thus C_{∞} is a vector space.

$$\begin{cases} \lim_{x\to\infty} (f(x)+g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = 0+0 = 0 \\ \lim_{x\to-\infty} (f(x)+g(x)) = \lim_{x\to-\infty} f(x) + \lim_{x\to-\infty} g(x) = 0+0=0 \\ \lim_{x\to\infty} \alpha f(x) = \alpha \lim_{x\to\infty} f(x) = \alpha\cdot 0 = 0 = \alpha \cdot 0 = \alpha \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \alpha f(x) \end{cases}$$

That e^{-x^2} is in C_{∞} is trival.

9. Suppose $x, y \in N$, Then $S(x+y) = S(x) + S(y) = 0+0 = 0$ and $S(\alpha x) = \alpha S(x) = \alpha \cdot 0 = 0$, and so $x+y, \alpha x \in N$. Thus N is a subspace of V.

$\underline{S \text{ is } 1\text{-}1 \Rightarrow N = \{0\}}$: This is easy since if $S(x) = 0 = S(0)$, then by 1-1ness $x = 0$.

$\underline{N = \{0\} \Rightarrow S \text{ is } 1\text{-}1}$: Suppose $Sx = Sy$, Then $S(x-y) = Sx - Sy = 0$. Hence by hypothesis $x - y = 0$ or $x = y$. $\therefore$ S is 1-1.

10. $e_1 = \frac{1}{2} f_1 + \frac{1}{2} f_2$, $e_2 = \frac{1}{2} f_1 - \frac{1}{2} f_2$

$\{f_1, f_2\}$ is independent because they span a two-dimensional space.

10, not passing, no attempt

58 / ~~57~~ / 29

MATH 74 TEST 14 NOV. 19

SHOW ALL WORK; USE ONE SIDE OF PAPER ONLY, BE CLEAR!

1. WRITE THE GENERAL SOLUTION TO THE D.E. 15/2/0

$$(D-1)^3(D-2)(D^2+1)\,y = 0.$$

10 (15), 9, 8, 7, 7, 5, 5, 5, 4, 2 no attemps 0

2. LET $\mathfrak{X} = \{1, e^x, \sin x, \cos^2 x, x+3, e^{e^x}, \sin 2x, \sin^2 x, 2^x\}$.

IS $\mathfrak{X}$ INDEPENDENT? WHY? 11/12/0

10 (11), 9, 2-(8), 1-(4) NA 0

3. SUPPOSE $\lambda_1, \lambda_2, \dots, \lambda_n$ ARE REAL NUMBERS. DEFINE $T: \mathbb{R}^n \to \mathbb{R}^n$

mul 3
add 3 +1
Matrix 3

BY $T((x_1, x_2, \dots, x_n)) = (\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n)$. SHOW THAT T IS LINEAR AND FIND THE MATRIX OF T WITH RESPECT TO THE USUAL BASIS OF $\mathbb{R}^n$.

10-(7), 9-(3), 8-(2), 7-(4), 6, 5, 5, 3, 2, 2, 0 NA 1 7/4/1

3
3
3

4. SHOW THAT T IN #3 IS INVERTIBLE, IF AND ONLY IFF, ALL $\lambda_i \neq 0$ $i = 1, 2, \dots n$, IN THIS CASE FIND THE MATRIX OF T^{-1}.

10, 9, 8-(3), 7-(6), 6, 6, 5, 4-(3), 3, 2, 1, 0 NA 3 1/7/3

5. ORTHONORMALIZE THE SET $\{1, x, x^2\}$ WITH RESPECT TO THE INNER PRODUCT $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

8-(3), 7-(6), 6-(3), 5-(2), 4, 4, 3, 2, 1-(4) NA 2 0/8/2

6. SOLVE THE IVP: $y' + e^x y = e^{-e^x}$ $y(0) = 0$.

10-(7), 9, 8, 7-(6) 5-(6), 4 ~~4, 4, 2, 1, 0~~ NA 2 7/1/2

7. IF $\xi_1, \xi_2, \dots \xi_n$ ARE ORTHOGONAL, SHOW THAT $\xi_1, \xi_2, \dots \xi_n$ ARE INDEPENDENT. SHOW THAT THE CONVERSE IS FALSE FOR $n=2$ IN $\mathbb{R}^2$.

10-(3), 7, 6-(3), 5, 5, 4, 4, 2, 1, 1, 0-(4) NA 6 3/9/6

add 4
mul 4
e^{-x^2} 2

8. LET $C_{00} = \{f : \mathbb{R} \to \mathbb{R} \mid \lim_{x\to\infty} f(x) = 0 = \lim_{x\to -\infty} f(x)\}$. SHOW THAT C_{00} IS A VECTOR SPACE. SHOW THAT e^{-x^2} IS IN C_{00}.

10-(11), 9, 6-(5), 4, 3, 3 NA 4 11/3/4

9. SUPPOSE $S: V \to W$ IS A LINEAR MAP; LET N BE THE NULL SET OF S. SHOW THAT N IS ~~THE~~ A SUBSPACE OF V. SHOW THAT S IS ONE-ONE, IF AND ONLY IF, $N = \{0\}$.

10, 9, 8, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 2, 2, 2, 1, 1 NA 7 1/8/7

10. LET e_1, e_2 BE THE USUAL BASIS OF $\mathbb{R}^2$. LET $f_1 = (1,1)$ AND $f_2 = (1,-1)$. EXPRESS e_1 AND e_2 AS LINEAR COMBINATIONS OF f_1 and f_2. THIS IMPLIES THAT f_1 AND f_2 ARE INDEPENDENT. WHY?

10, 10, 9-(4), 8, 6-(6), 5-(4), 3, 3, 3 NA 4 2/3/4

MATH 74

TESTING THE INDEPENDENCE OF m VECTORS IN AN n-DIMENSIONAL VECTOR SPACE.

WE CAN ASSUME $m \leq n$ (why?)
WRITE EACH VECTOR AS A ROW IN A MATRIX.
THE FOLLOWING OPERATIONS ARE ALLOWABLE:

(i) INTERCHANGING ANY TWO ROWS

(ii) MULTIPLYING ANY ROW BY ANY NON-ZERO SCALAR.

(iii) ADDING ANY ROW TO ANOTHER ROW.

AT ANY POINT IN THIS PROCESS, THE ROWS REPRESENT m-VECTORS EACH IN THE LINEAR SPAN OF THE ORIGINAL m-VECTORS. BUT, IF THERE APPEARS A ROW OF ALL ZEROS, THEN THE VECTORS (THE ORIGINAL m-VECTORS) ARE DEPENDENT. MORE GENERALLY, THIS PROCESS IS REVERSIBLE (i.e. IF YOU CAN GET FROM MATRIX A TO MATRIX B VIA A FINITE NUMBER OF APPLICATIONS OF OPERATIONS (i), (ii) OR (iii), THEN THERE IS A FINITE NUMBER OF APPLICATIONS OF OPERATIONS OF (i), (ii) OR (iii) THAT WILL TURN MATRIX B INTO MATRIX A.) TO SEE THIS NOTE THAT TO UNDO (i) JUST DO (i) AGAIN WITH THE SAME ROWS, TO UNDO (ii) JUST DO (ii) WITH THE RECIPROCAL SCALAR. TO UNDO ADDITION OF ROW A TO ROW B (iii), ONE CAN DO THE

FOLLOWING: MULTIPLY ROW A BY -1 (ii) THEN ADD THE RESULTING ROW A TO ROW B (iii) AND LASTLY, MULTIPLY ROW A AGAIN BY -1 (ii).

LET'S SUM UP WHAT WE HAVE JUST SHOWN WITH THE FOLLOWING:

<u>THEOREM</u> A: SUPPOSE WE HAVE WRITTEN m VECTORS IN AN n-DIMENSIONAL VECTOR SPACE AS THE ROWS OF A MATRIX M, AND FURTHER SUPPOSE THAT THE MATRIX N IS OBTAINABLE FROM M BY A FINITE NUMBER OF OPERATORS OF (i), (ii) OR (iii); THEN THE ROWS OF N CONSIDERED AS "n-DIMENSIONAL" VECTORS ARE IN THE LINEAR SPAN OF THE ROWS OF M, AND <u>CONVERSELY</u>.

<u>COROLLARY</u>: THE ROWS OF M ARE INDEPENDENT IF, AND ONLY IF, THE ROWS OF N ARE INDEPENDENT.

<u>proof</u>: SUPPOSE THE ROWS OF N ARE DEPENDENT, THEN THE ROWS OF N ARE IN THE LINEAR SPAN OF K VECTORS, WHERE $K < m$. HENCE, BY THEOREM A, THE ROWS OF M ARE IN THE LINEAR SPAN OF K VECTORS. AND THUS, BY THEOREM 1.5 OF APOSTOL, THE ROWS OF M ARE DEPENDENT. THE CONVERSE FOLLOWS BY INTERCHANGING THE ROLE OF M AND N

WE HAVE CHANGED THE PROBLEM OF TESTING INDEPENCE OF THE ROWS OF M, TO THE PROBLEM OF TESTING INDEPENCE OF THE ROWS OF N. BUT IT MAY BE JUST AS HARD TO DUE THE LATTER, THE REST OF THIS HANDOUT WILL SHOW THAT WE CAN PICK N TO BE IN <u>ROW-CANONICAL</u> FORM, AND FOR SUCH A MATRIX, THE TEST OF INDEPENCE IS TRIVAL.

A MATRIX IS SAID TO BE IN <u>ROW-CANONICAL</u> FORM IF THE FOLLOWING ARE TRUE:

(a) THE FIRST NON-ZERO ELEMENT IN EACH ROW IS A ONE. (THERE MAYBE NO NON-ZERO'S AT ALL.)

(b) IN A COLUMN WHICH A FIRST ONE OCCURS, ALL OTHER ENTRIES ARE ZERO

(c) IF IN ROW α, THE FIRST ONE IS IN THE β^{th} COLUMN. AND IN ROW γ, THE FIRST ONE IS THE δ^{th} COLUMN, THEN $\beta < \delta$ IF, AND ONLY IF $\alpha < \gamma$.

CAN ANY MATRIX M, THOURGH A FINITE NUMBER OF APPLICATIONS OF (i), (ii) OR (iii), BE EQUIVALENT TO A MATRIX N IN ROW-CANONICAL FORM? THE ANSWER IS YES, AND WE SKETCH A PROOF BELOW.

TAKE THE FIRST ROW WITH A NON-ZERO ELEMENT, AND LET THIS BE THE ELEMENT IN ROW R AND COLUMN C. MAKE ALL THE OTHER ROWS ZERO IN COLUMN C BY FIRST (ii) MAKING ~~OUR~~ ELEMENT IN ROW R AND COLUMN C THE ADDITIVE INVERSE, THEN (iii) TO ANNIHILATE THE ELEMENT IN ANOTHER ROW, BUT THE SAME COLUMN. WITH THIS COMPLETE USE (ii) TO MAKE THE ELEMENT IN ROW R AND COLUMN C ONE.

REPEAT THIS PROCESS UNTIL ALL ROWS EITHER HAVE A FIRST ONE, OR ALL ZEROS. NOTE THAT ONCE A COLUMN HAS BEEN CHANGED SO TO INCLUDE ZERO'S WITH AT MOST 1 (ONE) ONE, OUR PROCESS HAS NO EFFECT ON THIS COLUMN.

FINALLY, WE CAN ORDER THE ROWS SO TO SATISFY (c), DONE

WE NEED AN EASY TEST FOR INDEPENCE OF THE ROWS, IN A ROW-CANONICAL MATRIX. WE FORMULATE THIS IN:

THEOREM B: IF N IS A ROW-CANONICAL MATRIX WITH m-ROWS (i.e. m-VECTORS), THEN THE ROWS OF N ARE INDEPENT EXACTLY WHEN EACH OF THE m ROWS HAS A FIRST ONE.

Proof: IF OUR MATRIX N HAS FEWER THEN m FIRST ONES, THEN IT HAS A ROW ENTIRELY OF ZERO'S AND THUS IS DEPENDENT BY THE COROLLARY ABOVE.

CONVERSELY, IF N HAS m ROWS EACH WITH A FIRST ONE, WE NEED TO SHOW THAT THESE ROWS OF N ARE INDEPENDENT. SUPPOSE NOT, THEN THERE EXIST SCALARS $c_1, c_2, \ldots, c_m$, NO ALL ZERO, SUCH THAT

$$\sum_{i=1}^{m} c_i R_i = \text{ZERO VECTOR}$$

(WHERE R_i IS THE i^{th} ROW.) LET i BE BETWEEN 1 AND m. IN THE i^{th} ROW, THERE IS A FIRST ONE IN SAY COLUMN j. IN ALL OTHER ROWS THE j^{th} COLUMN HAS A ZERO. SO, IN ORDER FOR THE SUM ABOVE TO BE THE ZERO VECTOR, THE SUM OF THE ELEMENTS IN THE j^{th} COLUMN MUST BE ZERO; BUT THIS SUM IS JUST c_i. HENCE ALL $c_i = 0$, $i = 1, 2, \ldots m$. AND THE CONTRADICTION ESTABLISHED THE THEOREM.

EXAMPLES:

(I) $\begin{pmatrix} 0&0&1&0&7&0&-1&0&0&0 \\ 0&0&0&0&0&1&-2&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$ (II) $\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&4 \\ 0&0&0&0 \end{pmatrix}$

(III) $\begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$ IV $\begin{pmatrix} 2&0 \\ 0&0 \end{pmatrix}$ (V) $\begin{pmatrix} 1&0&7 \\ 0&1&5 \\ 0&0&1 \end{pmatrix}$

(I) AND (II) ARE IN ROW-CANONICAL FORM WITH (I) BEING INDEPENDENT AND (II) BEING DEPENDENT. EACH OF (III), (IV) AND (V) FAIL TO BE IN ROW-CANONICAL FORM.

PROBLEMS: PUT EACH IN ROW-CANONICAL FORM BY USE OF (i), (ii) OR (iii), INDEPENDENT OR NOT?

(B.) $\begin{pmatrix} 1&2&-1&-2&0&4 \\ 2&0&-1&1&1&1 \\ -1&-1&0&2&1&3 \\ 1&14&-1&-22&-8&4 \end{pmatrix}$ (C.) $\begin{pmatrix} 3&1&-1 \\ 2&-1&0 \\ 1&2&1 \end{pmatrix}$

MATH 74 FINAL SHOW ALL WORK; USE ONE SIDE OF PAPER ONLY; BE NEAT; ALL "PARTS" WORTH 10 pts, 200 TOTAL.

0. A) Show $\{e^x, \sin x, 1\}$ is an independent set.

B) If $T: V \to W$ is a linear map between vector spaces Show N = null space of T is a vector subspace of V.

C) Solve the IVP $y' - xy = 0$ $y(0) = 1$.

I. Let $(*)$ $(D-1)(D-2)y = f(t)$. Find the general solution to $(*)$ when:

A) $f(t) \equiv 0$. B) $f(t) = t^2$. C) $f(t) = \sqrt[3]{t}$.

II. Let $C[0,1]$ be the continuous real-valued functions on $[0,1]$ and let $W = \{f \in C[0,1] \mid \int_0^1 f(t)\,dt = 1\}$.

A) Show that W is not a subspace.

B) Show that W is a flat.

III. A) Show that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ have the same characteristic polynomial but are not similar.

B) Find e^{tA}, both for $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

C) SOLVE $\dot{x} = Ax$ $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in both cases.

IV. Let V be a vector space and $A: V \to V$ a linear map. Suppose $A^2 = A$.

A) Suppose $x \in V$ and $y = Ax \neq 0$. Show y is an eigenvector for A. What is the eigenvalue?

B) Suppose $x \in V$ and $z = x - Ax \neq 0$. Show z is an eigenvector for A. what is the eigenvalue?

C) Show that any $x \in V$ can be written $x = z + y$ where z belongs to the null space of A and y belongs to the range of A.

D) Show that the only eigenvalues that A can have are 0 or 1.

sin t

cos t

2nd PAGE

V. Solve $\dot{\mathfrak{X}}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathfrak{X}(t)$, $\mathfrak{X}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, any way you can.

VI. Let $u_1(t)$ and $u_2(t)$ be independent solutions to the second order homogenuous D.E.

$$y''(t) + a(t)y'(t) + b(t)y(t) = 0.$$

Let $w(t) = \det \begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix}$.

A.) Show that: u_1, u_2 are never zero together, u_1 and u_1' are never zero together, u_2 and u_2' are never zero together and u_1' and u_2' are never zero together.

B.) Suppose a and b are points where $u_1(a) = u_1(b) = 0$ and for all ξ between a and b, $u_1(\xi) \neq 0$.
Show $u_1'(a)\,u_1'(b) < 0$.

C.) Suppose a and b are points as in B). Show that there is a point ξ between a and b where $u_2(\xi) = 0$. [HINT: $w(x)$ cannot change sign. why?]

D.) Use part C) to show that between any two zero's of $\sin x$ there is a zero of $\cos x$.

MODEL: A Biological Epidemic.

In a total population of n individuals there are, at any time t, $y(t)$ infectious carriers of a contagious disease, $x(t)$ members of the population who are susceptible to the disease, and $z(t)$ individuals who are recovered and immune. We have $x(t) + y(t) + z(t) = n$ for all t. The following system of differential equations, know as the Kermack-McKendrick equations, characterize the epidemic.

$$(*) \quad \begin{aligned} \dot{x}(t) &= -\lambda\, x(t)\, y(t) \\ \dot{y}(t) &= \lambda\, x(t)\, y(t) - \mu\, y(t) \\ \dot{z}(t) &= \mu\, y(t) \end{aligned}$$

COMPUTER PROBLEM:

Write a program which will solve the set of differential equation (*) for

1) the number of susceptibles, $x(t)$;

2) the number of infectives, $y(t)$;

3) the number of recovered, $z(t)$; and

4) the epidemic curve $-\dot{x}(t)$ (i.e. the rate which new disease occurs);

given that t is the time in days, $\lambda = 0.001$, $\mu = 1/14$, $n = 1000$, $x(o) = 900$, $y(o) = 10$ and $z(o) = 90$.

Your program must have the following features:

1) It must use the iterative form of the improved Euler's Method with the number of iterations being three.

2) It must use the interval of computation $h = 0.1$ days but print out the results for every day up to fifteen days only.

Plot on a single graph paper the variables $x(t)$, $y(t)$, $z(t)$ and $-\dot{x}(t)$ versus time t.

NUMERICAL METHODS FOR SOLVING DIFFERENTIAL EQUATIONS

We will develop two methods for numerically approximating the solution to first order (non-linear) systems of differential equations with initial conditions. Our IVP has the form

$$(*) \qquad \dot{X} = f(X,t)$$
$$X(o) = X_o,$$

where $X(t)$ is the n-tuple $(x_1(t), x_2(t), \ldots x_n(t))$, X_o is the n-tuple $(x_1(o), x_2(o), \ldots x_n(o))$, t is "time" and f is a n-vector valued function of the n+1 variables $x_1(t)$, $x_2(t), \ldots x_n(t)$, t. We will solve $(*)$ for $t \geq o$, the solution for $t \leq o$ is similar.

The basic idea is to "divide" the positive t-axis into discrete points $t_o = o$, t_1, t_2 $\ldots$. To facilitate this process, let h be a "small" positive number, called the interval of computation, and define $t_o = o$ and $t_{n+1} = t_n + h$ for $n = 0,1,2, \cdots$. Since we are given $X(t_o)$, our problem is reduced to producing a "good" approximation for $X(t_{n+1})$ given a "good" approximation to $X(t_n)$. We first attempt the case when $(*)$ is an ordinary differential equation (i.e. $n = 1$) and latter extend to the general case. We use the short hand X_n for $X(t_n)$.

EULER'S METHOD:

We are trying to solve the D.E. $dx/dt = f(x,t)$, given $x(o) = x_o = x(t_o)$. Suppose we have obtained x_n and we are trying to make the leap to x_{n+1}. Euler's idea can be expressed as "if h is 'small' enough the tangent line at x_n will be a 'good' approximation to the curve $x(t)$ near t_n." Operationally, this means we assume that the curve and the tangent line are the same between t_n and t_{n+1}. It is simple analytic geometry to obtain $x_{n+1} = x_n + h\, d_n$, where d_n is the slope of the tangent line at x_n, that is $d_n = dx/dt|_{t_n} = f(x_n, t_n)$. Let us do a simple example.

Example: Let's take the IVP $\dot{x} = t$, $x(o) = 1$. Here $f(x,t) = t$. Let $h = 1$. The table below gives the values of x_n, $n = 0,1,2,3,4,5$; given by Euler's method and by the exact solution $x(t) = \frac{1}{2}t^2 + 1$.

time	0.	1.	2.	3.	4.	5.
Euler's x(t)	1.0	1.0	2.0	4.0	7.0	11.0
True x(t)	1.0	1.5	3.0	5.5	9.0	13.5

Let's calculate x_4, given x_3; $d_3 = f(x_3,t_3) = f(4,3) = 3$, so $x_4 = x_3 + h\, d_3 = 4 + 3 = 7$. Please note the way the error increases with time. A graph is enlighting.

IMPROVED EULER'S METHOD:

The "improvement" is the observation that it is the secant line through x_n and x_{n+1}, not the tangent line at x_n, that is the "good" approximation for our purposes. Since we do not know what the value of x_{n+1} is, we try instead to find the slope of the secant using x_n and the value $\bar{x}_{n+1}$ obtained from Euler's method (unimproved.)

Our best guesses to the slope of the curve $x(t)$ at t_n and t_{n+1} are $d_n = f(x_n,t_n)$ and $d_{n+1} = f(\bar{x}_{n+1}, t_{n+1})$ respectively. Now if $x(t)$ is a nice curve and h is "small", then the slope of the secant should be some sort of "average" between these two "extremes". With this in mind, we let $S_n^{(1)} = \frac{1}{2}[d_n + d_{n+1}]$. The line through x_n with slope $S_n^{(1)}$ is our approximation to the secant line. Analytic geometry gives the better approximation to $x(t_{n+1})$ as $x_{n+1}^{(1)} = x_n + h\, S_n^{(1)}$. The student may amuse himself (or herself(sorry Tracy)) with the observation that this improvement is enough to give the exact values in the very simple example above. (Of course, in general, this will not happen.)

But why stop now? We have a "better" approximation to x_{n+1} and thus we can obtain a "better" approximation to the slope of the secant. In fact, let $d_n^{(1)} = d_n = f(x_n,t_n)$ and $d_{n+1}^{(1)} = f(x_{n+1}^{(1)},t_{n+1})$, then the slope of the secant should be $S_n^{(2)} = \frac{1}{2}(d_n^{(1)} + d_{n+1}^{(2)})$ and a "better" approximation to $x(t_{n+1})$ would be $x_{n+1}^{(2)} = x_n + h\, S_n^{(2)}$.

This process can be repeated any finite number of times. Let us formulate this by defining $d_n^{(k)} = d_n = f(x_n,t_n)$ and $d_{n+1}^{(k)} = f(x_{n+1}^{(k)}, t_{n+1})$, letting $S_n^{(k+1)} = \frac{1}{2}(d_n^{(k)} + d_{n+1}^{(k+1)})$ and obtaining the value $x_{n+1}^{(k+1)} = x_n + h\, S_n^{(k+1)}$. We have arrived at the iterative form of the improved Euler's method with $k+1$ iterations. It can be shown that if more than two iterations are used that the order of the error reaches and remains at a fixed value. A very "reasonable" approximation.

EXTENSION TO THE GENERAL CASE:

This is quite simple. Since $f(X,t)$ is an n-vector valued function, let its $i^{\underline{th}}$ component be $f_i(X,t)$ (i.e. $f(X,t) = (f_1(X,t),\ f_2(X,t), \cdots\ f_n(X,t).)$ Writing this in long hand, we have by equating the $i^{\underline{th}}$ components of $\mathring{X}$ and $f(X,t)$:

$$\mathring{x}_i = f_i(x_1(t),\ x_2(t), \cdots x_n(t)),\ t).$$

So to make the leap from $(x_i)_m$ to $(x_i)_{m+1}$ we only need to know $(x_j)_m$, for $j = 1, \cdots n$, if we are using Euler's method. For the improved Euler's to obtain $(x_i)_{m+1}^{(k+1)}$, we need only to know $(x_j)_{m+1}^{(k)}$, for $j = 1, \cdots n$.

This a very small amount of Numerical Solutions to Differential Equations. The subject matter is very useful, challenging and interesting.

BE SURE TO BRING THIS HANDOUT TO MATH CLASS WHEN CLASSES START AGAIN JANUARY 7.

DUE DATE: January 18, 1974, in the Freshman Division office, Parsons #267

Put the standard information on the upper right-hand corner of the first page.

This is an open book, note and Library exercise, but work is to be done independently. Remember, results about vector spaces are fair game.

To prove $A = B$ as sets, show <u>both</u> $A \subset B$ and $B \subset A$.

To prove $A \subset B$ show <u>either</u> (a) everything in A is in B (in symbols, $x \in A \Rightarrow x \in B$) or (b) everything not in B is not in A (in symbols, $x \notin B \Rightarrow x \notin A$).

<u>Definition.</u> A linear combination $\sum_1^k c_i x_i$, of a finite set of vectors $\{x_1, \cdots, x_k\}$, is an <u>affine</u> combination if $\sum_1^k c_i = 1$.

<u>Definition.</u> A subset P of a vector space is a <u>flat</u> if P contains all the affine combinations of the non-empty finite subsets of P.

<u>Theorem 1.</u> The following are equivalent:

(1) P is a flat.

(2) For each $p \in P$, the set $\{x-p: x \in P\}$ is a vector subspace.

(3) P is the translate of a unique vector subspace. (i.e., there is a vector w and a <u>unique</u> subspace V, such that $P = w + V = \{w+x: x \in V\}$.)

<u>Theorem 2.</u> The flats of $\mathbb{R}^3$ are either:

(1) a single point,

(2) a straight line,

(3) a plane,

or

(4) the whole space.

Theorem 3. A flat P is a subspace, if and only if, the origin belongs to P.

Theorem 4. The intersection of a set of flats (possible infinite) is either empty or a flat.

Theorem 5. Suppose we have an linear system of equations $AX = B$, where A is an $n \times n$ matrix, B is an n column vector and X is the unknown n column vector. Show that the solution space of $AX = B$ (i.e., $\{X: AX = B\}$) is a flat or empty. Furthermore show that this flat is a subspace if and only if B is the origin.

Open Book To Anything in Chapters 1 – 3 or 6

1 A map $T: V_1 \times \cdots \times V_n \to W$ (where $V_1, \cdots V_n, W$ are vector spaces) is said to <u>multilinear</u> if it is linear in each variable. (T here is considered as a function of n-variables, one from each V_i). A multilinear map T is said to be <u>symmetric</u> [<u>alternating</u>] if $T(x_1 \cdots \underline{x_i} \cdots \underline{x_j} \cdots x_n) = T(x_1 \cdots \underline{x_j} \cdots \underline{x_i} \cdots x_n)$
$[\, T(x_1 \cdots \underline{x_i} \cdots \underline{x_j} \cdots x_n) = -T(x_1 \cdots \underline{x_j} \cdots \underline{x_i} \cdots x_n).]$
(<u>Note that</u> in these definitions it is assumed that $V_1 = V_2 = \cdots = V_n$)

Let $M = \{ f: V_1 \times \cdots \times V_n \to W \mid f \text{ is multilinear} \}$
$S = \{ f \in M \mid f \text{ is symmetric} \}$ $(V_1 = V_2 \cdots = V_n)$
$A = \{ f \in M \mid f \text{ is alternating} \}$. $(V_1 = V_2 \cdots = V_n)$

<u>Examples:</u> a real inner product is symmetric bilinear and det is alternating n-linear.

<u>Show</u>: A) M is a vector space and both S and A are subspaces of M.

B) If $T \in S$, and $T \in A$ then T is the zero map.

C) Suppose $T: V \times V \to W$ is bilinear show that $T_A(x,y) = \frac{1}{2}(T(x,y) - T(y,x)) \in A$ and $T_S(x,y) = \frac{1}{2}(T(x,y) + T(y,x)) \in S$ and $T = T_A + T_S$.

4. FIND THE GENERAL SOLUTION TO THE D.E.

$$D^3(D-1)(D^2+1)y = \cos 2x$$

FIND THE SOLUTION OF THE ABOVE D.E WITH THE INTIAL CONDS $y(0) = y'(0) = y''(0) = y'''(0) = y^{(iv)}(0) = y^{(v)}(0) = 0$.

5. Suppose $T: V \to V$ is linear and let

N_1 = null space of T $\quad$ R_1 = range of T (ie, image)

N_2 = null space of T^2 $\quad$ R_2 = range of T^2 (ie $T^2(V)$)

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

N_m = null space of T^m $\quad$ R_m = range of T^m (etc)

A) Show that $N_1 \subset N_2 \subset N_3 \cdots \subset N_m$ and $R_1 \supset R_2 \supset R_3 \cdots \supset R_m$.

B) Suppose V is finite dimensional, then show that there is an M (an integer) such that $N_M = N_{M+1} = \ldots N_{M+j}$ and $R_M = R_{M+1} = \cdots R_{M+j}$ for all integers j.

6. Suppose $T: V \to V$ is a linear map on the inner product space V and suppose $\lambda_1, \cdots \lambda_n$ are <u>distinct</u> real numbers, different from zero, such that there exist vectors $x_1 \cdots x_n$ in V with $Tx_i = \lambda_i x_i$ $\quad i = 1, \cdots n$.

A) Show that $\{x_1 \cdots x_n\}$ is indepedent

B) Suppose for all $x, y \in V$ that $\langle Tx, y\rangle = \langle x, Ty\rangle$ then show that $\{x_1 \cdots x_n\}$ is orthogonal.