Open Book To Anything in Chapters 1–3 or 6

1 A map $T: V_1 \times \cdots \times V_n \longrightarrow W$ (where $V_1, \cdots V_n, W$ are vector spaces) is said to <u>multilinear</u> if it is linear in each variable. (T here is considered as a function of n-variables, one from each V_i). A multilinear map T is said to be <u>symmetric</u> [<u>alternating</u>] if $T(x_1 \cdots \underline{x_i} \cdots \underline{x_j} \cdots x_n) = T(x_1 \cdots \underline{x_j} \cdots \underline{x_i} \cdots x_n)$ [$T(x_1 \cdots \underline{x_i} \cdots \underline{x_j} \cdots x_n) = -T(x_1 \cdots \underline{x_j} \cdots \underline{x_i} \cdots x_n)$.]
(<u>Note that</u> in these definitions it is assumed that $V_1 = V_2 = \cdots = V_n$)

Let $M = \{ f: V_1 \times \cdots \times V_n \to W \mid f \text{ is multilinear} \}$

$S = \{ f \in M \mid f \text{ is symmetric} \}$ $(V_1 = V_2 \cdots = V_n)$

$A = \{ f \in M \mid f \text{ is alternating} \}$. $(V_1 = V_2 \cdots = V_n)$

<u>Examples:</u> a real inner product is symmetric bilinear and det is alternating n-linear.

<u>Show</u>: A) M is a vector space and both S and A are subspaces of M.

B) If $T \in S$, and $T \in A$ then T is the zero map.

C) Suppose $T: V \times V \to W$ is bilinear show that ~~$T_A = \frac{1}{2}$~~ $T_A(x,y) = \frac{1}{2}(T(x,y) - T(y,x)) \in A$ and $T_S(x,y) = \frac{1}{2}(T(x,y) + T(y,x)) \in S$ and $T = T_A + T_S$.

2. Let $T: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be bilinear. Let $e_1 \cdots e_n$ be the usual basis for $\mathbb{R}^n$ and $f_1 \cdots f_m$ be the usual basis for $\mathbb{R}^m$.

A) Show T is uniquely determined by its values on (e_i, f_j) $i = 1, \cdots n$, $j = 1, \cdots m$.

B) Show that conversely given c_{ij} $i = 1, \cdots n$ $j = 1, \cdots m$. there exists a unique bilinear $T: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with $T(e_i, f_j) = c_{ij}$ $i = 1 \cdots n$, $j = 1 \cdots m$.

Let V be a vector space of $\dim = n \times m$, and let $e_i \otimes e_j$ be a basis for V. $(i = 1 \cdots n, j = 1, \cdots m)$

C) Show that for every bilinear $T: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ there exist a unique $S: V \to \mathbb{R}$ with $S(e_i \otimes e_j) = T(e_i, e_j)$ $i = 1, \cdots n$ $j = 1, \cdots m$, and conversely (i.e. given S there exists a unique T)

Thus we have shown that Bilinear maps on $\mathbb{R}^n \times \mathbb{R}^m$ are the same thing as linear maps on V.

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ with $x = \sum c_i e_i$ and $y = \sum d_i f_i$ define: $x \otimes y = \sum_{i,j} c_i d_j \, e_i \otimes f_j$ and define $\varphi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ by $\varphi(x, y) = x \otimes y$.

D) show φ is bilinear and if T & S are as in C) then $T(x, y) = S(x \otimes y)$. (Note: $V = \mathbb{R}^n \otimes \mathbb{R}^m$).

3. Let $x, y \in \mathbb{R}^4$ define $[\cdot, \cdot] : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ by
$$[x, y] = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$$

A) Show $[\cdot, \cdot]$ satisfies all the axioms of a real inner product but one.

B) Show $\exists\, x \neq 0$ with $[x, x] = 0$ and $\exists\, y \neq 0$ with $[y, y] < 0$.

Define $\|[x]\| = [x, x]^{\frac{1}{2}}$ (the "norm" of x)

C) use B) to show that the norm of a non-zero element could be zero or even imaginary.

D) Show that $[\cdot, \cdot]$ has the following property: For each non-zero vector $x \in \mathbb{R}^4$ there is a vector y in $\mathbb{R}^4$ with $[x, y] > 0$.

A vector with $\|[x]\|$ ~~$\neq 0$ is~~ real is said to be "space-like" and one with $\|[x]\|$ imaginary is said to be "time-like."

E) Show that neither the set of space-like vectors or the set of time like vector form a vector space.

F) Show $S = \{(x, y, z, t) \in \mathbb{R}^4 : t = 0\}$ consists of space-like vectors and $T = \{(x, y, z, t) \in \mathbb{R}^4 : x = y = z = 0\}$ consists of time-like vectors.

4. FIND THE GENERAL SOLUTION TO THE D.E.

$$D^3(D-1)(D^2+1)y = \cos 2x$$

FIND THE SOLUTION OF THE ABOVE D.E WITH THE INTIAL CONDS $y(0) = y'(0) = y''(0) = y'''(0) = y^{(iv)}(0) = y^{(v)}(0) = 0$.

5. Suppose $T: V \to V$ is linear and let

N_1 = null space of T $\quad$ R_1 = range of T (ie. image)

N_2 = null space of T^2 $\quad$ R_2 = range of T^2 (ie $T^2(V)$)

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

N_m = null space of T^m $\quad$ R_m = range of T^m (etc)

A) Show that $N_1 \subset N_2 \subset N_3 \cdots \subset N_m$ and $R_1 \supset R_2 \supset R_3 \cdots \supset R_m$.

B) Suppose V is finite dimensional, then show that there is an M (an integer) such that $N_M = N_{M+1} = \dots N_{M+j}$ and $R_M = R_{M+1} = \cdots R_{M+j}$ for all integers j.

6. Suppose $T: V \to V$ is a linear map on the inner product space V and suppose $\lambda_1, \cdots \lambda_n$ are <u>distinct</u> real numbers, different from zero, such that there exist vectors $x_1, \cdots x_n$ in V with $Tx_i = \lambda_i x_i$ $\quad i = 1, \cdots n$.

A) Show that $\{x_1, \cdots x_n\}$ is indepedent

B) Suppose for all $x, y \in V$ that $\langle Tx, y\rangle = \langle x, Ty\rangle$ then show that $\{x_1, \cdots x_n\}$ is orthogonal.