Research Assignment

1. Either of these problems (or both) may be handed in and may be used to supplement your in-class quiz score total. Due to the nature of these problems, no specific point assignments will be made, but your work may be worth up to 50 points. You may use any source and may work together.

1. A prime p is called reflexive if the number formed from p by reversing the digits is also a prime. For example, 17 and 71 are both primes, as are 37 and 73. Analyze the set of all reflexive primes. For example, attempt to answer such questions as these:

   i) Are there an infinite number of reflexive primes?

   ii) If $p_1, p_2, \ldots$ denotes the sequence of reflexive primes, does the series $\sum \frac{1}{p_i}$ converge?

   iii) On the average, how many reflexive primes $\leq N$ are there for each integer N?

   iv) Can anything interesting be said about integers whose standard representation consists only of reflexive primes?

2. Any integer N which satisfies the congruence $2^{N-1} \equiv 1 \pmod{N}$ is called an F-number. Investigate the set of all F-numbers.

   Naturally all odd primes are F-numbers, but so are some composite numbers---e.g. 561 and 341 are both F-numbers. Among other properties you may care to explore, attempt to answer questions such as the following:

   i) Are there infinitely many composite F-numbers?

   ii) The integer K is called a Fermat number if $K = 2^{2^m} + 1$ and a Mersenne number if $K = 2^p - 1$ for some prime p. Are any F-numbers also Fermat numbers or Mersenne numbers?

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Supplementary Problems
and
Notes

Problems are due ~~December 18, 1967~~ January 5, 1968. You may work together on the problems, but each person must write up his own notebook.

1. Let $f(x) = \begin{cases} \frac{x^2}{5} & \text{if } 0 < x < 5 \\ 5 & \text{if } x = 5 \\ 10 - \frac{125}{x^2} & \text{if } x > 5 \end{cases}$

   Discuss the continuity of $f(x)$ and sketch its graph.

2. Show that in general three normal lines can be drawn to a parabola from a given point.

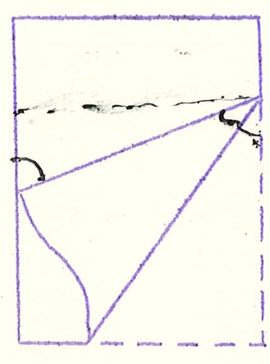
3. A trough 9 ft. long has as its cross section an isosceles right triangle with hypotenuse of length 2 ft. along the top of the trough. If water is pouring into the trough at the rate of 2 ft^3/min, find the depth $h(t)$ of the water t minutes after the water is turned on. Also find the rate at which the depth is increasing when $t = 2$.

4. Discuss the graph of $g(x) = (x-4)^{4/3} + 2(x+4)^{2/3}$.

5. Show that for every real number x, we have

$$\frac{1}{5} \leq \frac{x^2 - 4x + 9}{x^2 + 4x + 9} \leq 5$$

6. The lower right-hand corner of a page is folded over so as to reach the leftmost edge. If the width of the page is six inches, find the minimum length of the crease. You may assume that the page is long enough to prevent the crease from reaching the top of the page.



7. Assume that $|f''(x)| \leq m$ for each x in the interval $[0, a]$, and assume that f takes on its largest value at an interior point of this interval. Use the mean value theorem to prove that $|f'(0)| + |f'(a)| \leq am$.

8. Let $y = f(x)$ be that solution of the differential equation $y' = \dfrac{2y^2 + x}{3y^2 + 5}$ which satisfies the initial condition $f(0) = 0$. (Do not attempt to solve this differential equation.)

i) The D.E. shows that $f'(0) = 0$. Discuss whether f has a relative maximum or minimum or neither at 0.

ii) Notice that $f'(x) \geq 0$ for each $x \geq 0$ and that $f'(x) \geq 2/3$ for each $x \geq 10/3$. Exhibit two positive numbers a and b such that $f(x) > ax - b$ for each $x \geq 10/3$.

9. The integers $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ are called Fibonacci numbers.

i) Prove that the Fibonacci numbers satisfy the inequalities

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} < F_{n+1} < \left(\frac{1+\sqrt{5}}{2}\right)^{n}$$

(cont'd)

9, ii) It can be shown that every positive integer can be written as a sum of distinct terms from the Fibonacci sequence 1, 2, 3, 5, 8, 13, ... Do not prove this theorem, but write a GOTRAN program to verify that it is true for all integers $N \leq 1000$.

iii) Note that $F_2 = \binom{1}{0} = 1$; $F_3 = \binom{2}{0} + \binom{1}{1} = 2$;

$$F_4 = \binom{3}{0} + \binom{2}{1} = 1 + 2 = 3; \quad F_5 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5.$$

Is it true that for all $n \geq 2$,

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-j}{j-1}$$

where the sum of the binomial coefficients on the right terminates with the largest j such that $2j \leq n+1$? Write a computer program to test your hypothesis for all $n \leq 100$.

10. A certain utility company charges a \$10 flat fee for the first 50 (or less) hours of service each month, and then charges 10¢/hr for each hour in excess of 50 hours. Furthermore, a 5% penalty charge is added to the balance of any customer who has not paid his previous month's bill by the first of the month. Thus, if customer A still owes \$100 from past bills and has used 75 hours of service, his new bill will be \$10 + 25(.10) + \$100 + (.05)(\$100) = \$117.50.

Design a GOTRAN program (beginning with a READ statement) which prints out a customer's code number, his previous balance, the number of hours of service he has used, and his new bill. Use your program to compute the bills of the following customers:

Code #	old balance	# hours used
7332	\$0.00	85
810	\$10.75	27
4	\$0.00	25
1000	\$4.00	90

(140)

S-155
Part III

Difficult problems. Name STEVE BELLENOT

1. (20 points) If p is an odd prime, show that any prime which divides $a^p + 1$ but does not divide $a + 1$ is of the form $2pK + 1$.

(20)

q a prime $\quad q \mid (a^p+1) \Rightarrow q \mid (a+1)$ or $q \mid (a^{p-1} \mp a^{p-2} + \mp \ldots \pm - 1)$

$a^p + 1 \equiv 0 \bmod q$

$a^p \equiv -1 \bmod q$ but $a \not\equiv -1 \bmod q$

$\forall_a \quad a^{2p} \equiv 1 \bmod q \qquad 2p \mid \phi(q)$

what does this have to do with the problem?

$2pk = q - 1$

$q = 2pk + 1$

2. (20 points) Show that if $n > 2$ and $x^n + y^n = z^n$, then x,y,z cannot be consecutive positive integers.

(15)

$x^n + (x+1)^n = (x+2)^n$

$2x^n + nx^{n-1} + \frac{n(n-1)x^{n-2}}{2} + \ldots \frac{n(n-1)\cdots(n-r+1)x^{n-r}}{r!} + 1 =$

$x^n + nx^{n-1}2 + \frac{n(n-1)x^{n-2}2^2}{2} \ldots\ldots + 2^n$

x can't be even for then left side would be odd + rt even.

if x is odd $(x+2)^n - x^n = (x+1)^n = nx^{n-1}2 + \frac{n(n-1)x^{n-2}2^2}{2} + \ldots + 2^n$

since $2 \mid x+1 \quad 2^n \mid (x+1)^n \Rightarrow 2 \mid nx^{n-1} \Rightarrow 2 \mid n$ so

n is even thus $(x^a)^2 + (y^a)^2 \neq (z^a)^2$

why is this impossible?

3. (20 points) Show that the ~~exponential~~ factorial equation $x! + y! = z!$ has exactly one positive integral solution, and find it.

20

$x! + y! = z!$ let $x < y$ since $x! \mid y!$, & $z!$

$\left(1 + \frac{y!}{x!}\right) = \frac{z!}{x!}$

$y(y-1)\ldots(x+1) - z(z-1) \ldots (x+1) = -1$

$(x+1) \mid -1 \Rightarrow x = 0$ or $x = y$ in which case

$x+1 = 2 \quad x = 1 \quad y = 1 \quad z = 2 \qquad 1! + 1! = 2 = 2!$

✓

4. (20 points) Use the fact that any number Not of the form $n = 4^{\ell}(8k + 7)$ with $k \geqslant 0$, $\ell \geqslant 0$ may be represented as the sum of three squares to show that there exist infinitely many primes of the form $a^2 + b^2 + c^2 + 1$.

20

Since all primes p ~~are~~ $\ni$ $p-1 \not\equiv 7 \bmod 8$

$\forall_p \quad p-1 = a^2+b^2+c^2$ for some integers a, b, c

so $\forall_p \quad p = a^2+b^2+c^2+1$ Since no. of primes is infinite thus the no. of primes of the form $a^2+b^2+c^2+1$ is infinite.

✓

7

5. (20 points) If $p \equiv 1 \pmod 6$ and p is a prime, show that there exist integers x,y such that $p = 3x^2 + y^2$.

if $p \mid k^3+1$ where $k \equiv 1 \bmod 3$ then from 3rd test

$\exists$ integers x, y $\ni p = 3x^2+y^2$ ✓

∴ show this

6. (20 points) Prove that if n is a positive integer which has exactly 10 divisors, then either $n = p^9$ or $n = pq^4$, where p and q are primes.

20

$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$

the no. of divisors $= (\alpha_1+1)(\alpha_2+1) \dots (\alpha_k+1)$

since this equals $= 10 = 2 \cdot 5$

either $\alpha_1 = 9$ ~~&~~ $\alpha_i = 0$ $i > 1$

or $\alpha_1 = 1$ $\alpha_2 = 4$ $\alpha_i = 0$ $i > 2$

$n = p^9$ or pq^4 ✓

7. (20 points) Prove that if $p \equiv 3 \pmod 4$ and if $q = 2p + 1$ is a prime greater than 7, then $2^p - 1$ is composite. In fact, $q \mid 2^p - 1$. Thus, show that $2^{431} - 1$ is not a prime. p is a prime

3

$$2^{4k+3} - 1 = (2-1)(2^{4k+2} + 2^{4k+1} + \dots 1)$$

~~$2^{10} - 1 = 1023 = 3 \cdot 11 \cdot 31 \quad (2^5-1)(2^5+1)$~~

$2^{15} - 1 =$ q prime $2^{\frac{q-1}{2}} - 1 = 2^{4k+3} - 1$

q is of the form $8k+7$ ✓

1024
32
2048
3072
32768
32767

8. (20 points) If p is a prime each of whose digits in the decimal expansion is 1, prove that the number of digits must be prime. (e.g. 11 is prime, as is 2) Is the converse true?

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If the no. of digits was composite = ab the number with a 1's would divide it (ie 111,111 is divisible by 111 & 11) so to be a prime the no. of digits must be prime. prove this

$111 = 3 \cdot 37$ so the converse is false ✓

9. (25 points) Six professors began courses of lectures on Monday, Tuesday, Wednesday, Thursday, Friday, and Saturday of the first week of 1968, respectively. At the same time, the same six men announced their intentions of lecturing at intervals of two, three, four, one, six, and five days, respectively. If the university forbids Sunday lectures, what will be the date of the first day when all six professors have to omit a scheduled lecture on the same Sunday?

JAN 1

Sundays $\equiv 0$ (7)

10

S F W M Sa T T S

setup?

$M \quad 2k+1 \equiv 0 \bmod 7 \qquad k \equiv 3 \ (7)$

$T \quad 3m+2 \equiv 0 \bmod 7 \qquad m \equiv 4 \ (7)$

$W \quad 4z+3 \equiv 0 \bmod 7 \qquad z \equiv 1 \ (7)$

$F \quad 6x+5 \equiv 0 \bmod 7 \qquad x \equiv 5 \ (7)$

$S \quad 5y+6 \equiv 0 \bmod 7 \qquad y \equiv 3 \ (7)$

alternate sundays 7, 21

14 + 21

7 + 28

5(6) = 35

21

119
31 Jan
88
29 Feb
57
31 Mar
26

1 +
119
119
119
119
119

$M = 7 + 14k$
$T = 14 + 21\ell$
$W = 7 + 28z$
T all
$F = 35 + 42x$
$S = 21 + 49y$

$2k = 4z$
$\ell = 5$

$4 + 6x = 4z$

$1 + 3\ell = 4z$

$2 + 6x = 7y$

$2 + 3x = 2z$

Apr 26 the 119th day of the year

10. (30 points) Prove that $(x+1)^3 - x^3 = y^2$ is equivalent to $(2y)^2 - 3(2x+1)^2 = 1$. Conclude that in order to solve $(x+1)^3 - x^3 = y^2$ in integers, it suffices to find integral solutions of the Pell equation $w^2 - 3z^2 = 1$ such that w is even and z is odd.

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ii) Prove that all positive integral solutions of $(x+1)^3 - x^3 = y^2$ are contained in the sequence (x_n, y_n) defined recursively by $x_0 = 0$, $y_0 = 1$; $x_k = x_{k-1} + 4y_{k-1} + 3$ and $y_k = 12x_{k-1} + 7y_{k-1} + 6$, $k = 1, 2, \ldots$.

iii) Characterize the positive integral solutions of the equation $(u-v)^5 = u^3 - v^3$.

$(x+1)^3 - x^3 = y^2 \Leftrightarrow 3x^2 + 3x + 1 = y^2$

$(2y)^2 - 3(2x+1)^2 = 1 \Leftrightarrow 4y^2 - 3(4x^2+4x) - 3 = 1 \Leftrightarrow$ ✓

$4(y^2 - 3x^2 - 3x - 1 = 0)$ which is $\Leftrightarrow$ with

so to solve 1st in integers it suffices $w^2 - 3z^2 = 1 >$ w is even, z is odd

$\sqrt{3} = \langle 1, \overline{1, 2} \rangle$ ✓

if	k -2	-1	0	1	2	3		
a_k			1	1	2	1	2	
P_k	0	1	1	2	5			
q_k	1	0	1	1	3			

(P_i, q_i) is solution if $i = 2k-1$ or if i is odd

$P_1 = 2$ $\quad$ $P_k = a_k P_{k-1} + P_{k-2}$ (odd) $= P_{k-1} + P_{k-2}$ $\quad$ general soln.?

$q_1 = 1$

$2x_0 + 1 = 1$

$x_0 = 0$

$2y_0 = 2$

$y_0 = 1$