Bellenot

# LECTURE No. 1
# GEOMETRY 141
# S. BELLENOT

ALL CHARACTERS IN THIS WORK ARE FICTIONAL AND ANY RESEMBLANCE TO ACTUAL OBJECTS, PLACES, OR EVENTS IS PURELY COINCIDENTAL.

## PRELIMINARY MATERIALS

1.1 NAIVE SET THEORY IS NOT CONSISTENT. (Russell Paradox)

LET $C$ BE DEFINED TO BE THE CLASS OF ALL SETS.

$C$ IS NOT A SET. (OTHER WISE $C \in C$)

LET $\mathcal{U} = \{A : A \in C \ \& \ A \notin A\}$

Assume $\mathcal{U} \notin \mathcal{U} \Rightarrow \mathcal{U} \in \mathcal{U}$ by def. of $\mathcal{U}$

or if $\mathcal{U} \in \mathcal{U} \Rightarrow \mathcal{U} \notin \mathcal{U}$ similarly.

1.2 Def: A PARTIALY ORDERED SET $(X, \leq)$ IS A SET $X$ AND A RELATION $\leq \in X \times X$ st

a) $\forall_{x \in X} \ x \leq x$

b) if $x \leq y \ \& \ y \leq x \Rightarrow x = y$

c) if $x \leq y \ \& \ y \leq z \Rightarrow x \leq z$

1.3 Def: If $A \subset (X, \leq)$, $\sup A = x \in X$, if it exists, if $x$ is st $a \leq x \ \forall_{a \in A}$ AND IF $a \leq y \Rightarrow x \leq y$, $y \in X$

AND IF IT EXISTS $\inf A = x \in X$, $x$ st $x \leq a \ \forall_{a \in A}$ AND IF $y \leq a \Rightarrow y \leq x$, $y \in X$.

1.4 REMARK: $\inf \phi = \sup X$ $\sup \phi = \inf X$ if the latter two exist. ALL $x \in X$ SATISFY THE FIRST CONDITION so for $x \leq z$ for $\forall_{x \in X}$, $z = \sup X$ if it exists. If it does not exist neither does $\inf \phi$. $\sup \phi$ CAN be found similarly.

1.5 NOTATION: FOR $x, y \in X$: $x \vee y = \sup\{x, y\}$, $x \wedge y = \inf\{x, y\}$ if they $\exists$.

1.6 **DEF:** A LATTICE IS A POS $(X, \leq)$ st $\forall x,y \in X$ $x \vee y$ & $x \wedge y$ exist.

1.7 **DEF:** A COMPLETE LATTICE IS A POS $(X, \leq)$ st $\forall Y \subset (X, \leq)$ $\inf Y$ AND $\sup Y$ EXIST.

1.8 A COMPLETE LATTICE IS A LATTICE.
$\forall x,y \in X$ $\{x,y\} \subset (X, \leq)$ so $x \vee y$ & $x \wedge y$ EXIST.

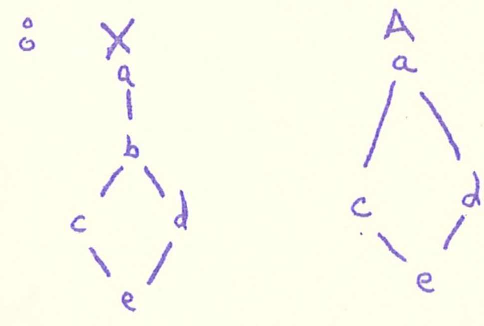
1.9 IF $(X, \leq)$ IS st $\forall B \subset (X, \leq)$ (INCLUDING $\inf \emptyset$) $\inf B$ EXISTS THEN $(X, \leq)$ IS A COMPLETE LATTICE.
LET $C = \{y : y \in X, \forall a \in B \ y \geq a\}$ $B \subset (X, \leq)$ AND $\therefore$ SO IS $C \subset (X, \leq)$. By hypot $\inf C = x$ for some $x \in X$
$\forall a \in B$ $x \geq a$ AND IF $y \geq a \Rightarrow y \in C \Rightarrow y \geq x$. $\therefore x = \sup B$ AND $(X, \leq)$ IS A COMPLETE LATTICE,

1.10 EXAMPLES: $\mathbb{N}$ (NATURAL NUMBERS), $\mathbb{Z}$ (INTEGERS) AND $\mathbb{R}$ (REAL NUMBERS) ARE ALL LATTICES. $\mathbb{N}$ IS A COMPLETE LATTICE. IF $X$ IS A SET $(2^X, \subset)$ IS A COMPLETE LATTICE (THE POWERSET PAIRED WITH INCLUSION).

1.11 IF $(X, \leq)$ POS AND $A \subset X$ THEN $A$ IS CANONICALLY A POS SET VIA $\leq / A \times A = \leq_A$ OR $\forall x,y \in A$ $x \leq_A y$ iff $x \leq y$ in $X$.
IT IS EASY TO SEE $\leq_A$ IS REFLEXIVE, ANTI-SYMMETRIC, AND TRANSITIVE, AND THUS A P.O.

1.12 **DEF:** $(A, \leq_A)$ IS SUBLATTICE OF $(X, \leq)$ IF $A \subset X$, $\leq_A = \leq / A \times A$ AND $\forall x,y \in A$ $x \wedge y$ & $x \vee y$ ARE THE SAME AS THEY WERE IN $(X, \leq)$

1.13 EXAMPLE: AN $(A, \leq_A)$ THAT IS LATTICE, BUT NOT A SUBLATTICE ON $(X, \leq)$:




$c \vee d = b$ in $X$ but $c \vee d = a$ in $A$

1.14 **DEF** $x \in X$ is a minimal element if $\forall y \in X$ $y \leq x \Rightarrow y = x$
$x \in X$ is a maximal element if $\forall y \in X$ $x \leq y \Rightarrow x = y$

1.15 **DEF** $(X, \leq)$ has minimality condition if $\forall A \subset X$ st $A \neq \emptyset$ AND $A$ HAS A minimal element. $\mathbb{N}$ satisfies this condition but $\mathbb{Z}$ and $\mathbb{R}$ do not.

1.16 DEF: $(X,\leq)$ SATISFIES THE INDUCTION CONDITION, IF $(X,\leq)$ POS AND IF $A \subset X$ st

1) every minimal element of $X$ is in $A$

2) whenever $a \in X$ AND $\{x : x < a\} \subset A \Rightarrow a \in A$

THEN $A = X$.

1.17 Th IF $(X \leq)$ pos Then Fsae

1) $(X,\leq)$ SATISFIES THE MINIMALITY CONDITION

2) $(X,\leq)$ SATISFIES THE INDUCTION CONDITION

PROOF

1) $\Longrightarrow$ 2)

LET $A \subset X$ AND EVERY MINIMAL ELEMENT OF $X$ IS IN $A$ AND WHENEVER $a \in X$ AND $\{x : x < a\} \subset A \Rightarrow a \in A$

LOOK AT THE SET $X - A$. ASSUME THAT $X - A \neq \phi$, THEN BY THE MINIMALITY CONDITION $\exists x$ A MINIMAL ELEMENT $\in X - A$.

$\{y : y < x\} \subset A$ BECAUSE IF $y < x$ $y \notin X - A \Rightarrow y \in A$ AND BY THE 2nd PART OF THE HYPOT $x \in A$ AND SO $x \notin X - A$ AND

$\therefore X = A$

2) $\Rightarrow$ 1) WILL NOT BE PROVIDED

1.18 DEF: A POS $(X, \leq)$ IS LINEARLY ORDERED IF $\forall_{x,y} \in X$ EITHER $x \leq y$ OR $y \leq x$

1.19 DEF: $X$ IS WELL ORDERED IF $X$ IS LINEARLY ORDERED AND SATISFIES THE MINIMALITY CONDITION.

1.20 DEF: A CARDINAL NUMBER IS AN ISOMORPHIC CLASS OF SETS.

1.21 PRINCIPAL OF TRANSFINITE INDUCTION

LET $X, Y$ be sets AND $\leq$ A P.O. SATISFING THE MINIMALITY CONDITION. A SCHEME TO DEFINE A FUNCTION FROM $X$ TO $Y$:

1) DEFINE $f$ ON THE MINAMAL ELEMENTS OF $X$
2) ASSUME $f$ IS DEFINED FOR $y < x$ AND THEN DEFINE $xf$ IN TERMS OF $yf$, $y < x$.

THIS METHOD MAY SEEM A LITTLE VAGUE. IT IS NOT A PLUG IN FORMULA WHICH INSTANDLY GIVES A WELL-DEFINE FUNCTI[ON] IT IS JUST A WAY A FUNCTION CAN' BE BASE[D] FROM ANY TWO SETS.

1.22 Axiom OF CHOICE

IF $(A_\alpha : \alpha \in I)$ IS A FAMILY OF SETS $\neq \emptyset$ $\exists$ A CHOICE FUNCTION $f : I \to \bigcup_{\alpha \in I} A_\alpha$ st $\forall_{\alpha \in I}$ $\alpha f \in A_i$.

1.23 Th The Axiom of choice is equivalent to every onto function $f : X \to Y$ has a left inverse

__Proof:__ If $f$ is onto, define $Y \xrightarrow{g} X$ by $\forall_{y \in Y} \exists\, yg \in X$ st $ygf = y$. By axiom of choice, such function $g$ exists.

Conversely, suppose $(A_\alpha : \alpha \in I)$ is a family of non-$\emptyset$ sets. $A =_{df} \{(\alpha, x) : \alpha \in I,\ x \in A_\alpha\}$. $A \xrightarrow{f} I$, $(\alpha, x) \mapsto \alpha$ is onto, so $\exists\ I \xrightarrow{g} A$ st $gf = 1_I$. Define $A \xrightarrow{h} \bigcup A_\alpha$, $(\alpha, x) \mapsto x$. $I \xrightarrow{gh} \bigcup A_\alpha$ is the desired choice function.

# Geometry

10-2-67

Lecture 2

notation

fsae $=_{dn}$ following statements are equivalent

fsav $=_{dn}$ " " " valid

## Some properties of Cardinal Numbers

$\alpha, \beta$ cardinals $\Rightarrow$ $\alpha \le \beta =_{dF}$ there exist sets $X$ and $Y$ with card $(X) = \alpha$ and card $(Y) = \beta$ and there is a 1-1 function mapping from $X$ into $Y$.

If $\alpha$ is a fixed cardinal, then $\{\beta : \beta$ is a cardinal, $\beta \le \alpha\}$ is a set whose cardinal is $\alpha$.

Moreover, $(\{\beta : \beta \le \alpha\}, \le)$ is a well-ordering

$\le$ is reflexive by the identy function [if card $X = \beta$, $X \to X \Rightarrow \beta \le \beta$]

$\le$ is antisymmetric (Schroeder - Bernstein Thm). $X, Y$ sets such that $X \rightarrowtail Y \rightarrowtail X$ then $X \cong Y$ (isomorphic) i.e, $X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$

Hence $\beta \le \alpha$ and $\alpha \le \beta \Rightarrow \alpha = \beta$

$\le$ is transitive since $X \xrightarrow[1\text{-}1]{f} Y \xrightarrow[1\text{-}1]{g} Z \Rightarrow X \xrightarrow[1\text{-}1]{fg} Z$, so $\alpha \le \beta \le \gamma \Rightarrow \alpha \le \gamma$.

Finally, $\emptyset \ne A \subset [\beta : \beta \le \alpha] \Rightarrow A$ has a minimal (least) element.

**Definition:** If $(X, \le)$ is a partially ordered set, a chain in $(X, \le) =_{dF}$ a non-$\emptyset$ subset $\mathbb{C}$ of $X$ such that for all $x, y \in \mathbb{C}$, either $x \le y$ or $y \le x$; i.e, $\mathbb{C}$ in the induced partial ordering is a linear ordering.

**Zorn's Lemma:** If $(X, \le)$ is a partially ordered set with $X$ non-empty and such that each chain in $X$ has an upper bound in $X$, then $X$ has a maximal element.

**Well-Ordering Theorem:** If $X$ is a set, there exists a well-ordering for $X$ (i.e, there exists a relation $\le \subset X \times X$ such that $(X, \le)$ is a well-ordered set).

Thm: fsae
(a) Axiom of Choice
(b) Zorn's Lemma
(c) Well-ordering Theorem

Proof of (a) $\Rightarrow$ (b):

Let $(\mathbb{X}, \leq)$ be a partially ordered set with $\mathbb{X} \neq \emptyset$ such that for each chain $\mathbb{C} \subset \mathbb{X}$ there exists $x \in \mathbb{X}$ with $x \geq \mathbb{C}$. Assume $\mathbb{X}$ has no maximal element (and look for a contradiction).

Let $\mathcal{C} = \{\mathbb{C} : \mathbb{C}$ is a chain in $\mathbb{X}\}$

For each $\mathbb{C} \in \mathcal{C}$, there is $x_{\mathbb{C}} \in \mathbb{X}$ with $x_{\mathbb{C}} \geq \mathbb{C}$.

Since $x_{\mathbb{C}}$ is not maximal, there exists $y_{\mathbb{C}} \in \mathbb{X}$ with $y_{\mathbb{C}} \gneqq x_{\mathbb{C}} \geq \mathbb{C}$

Noting that $\mathbb{C} \cup \{y_{\mathbb{C}}\}$ is a chain in $\mathbb{X}$, we may use the Axiom of Choice to arrive at the existence of a function $\mathcal{C} \xrightarrow{f} \mathcal{C}$ with the property that $\forall \mathbb{C} \in \mathcal{C}$, $\mathbb{C}f \supsetneqq \mathbb{C}$.

Let $\alpha$ be a cardinal greater than $\operatorname{card}(\mathcal{C})$. We can define $[\beta : \beta \leq \alpha] \xrightarrow{g} \mathcal{C}$ by induction.

For $\beta = 0$, let $0g =_{df} \mathbb{C}_0$, where $\mathbb{C}_0$ is any chain in $\mathcal{C}$. [Since $\mathbb{X} \neq \emptyset$ and for $x \in \mathbb{X}$, $\{x\}$ is a chain, such a $\mathbb{C}_0$ does exist]. Now suppose that for $\gamma < \beta$ $(\beta \leq \alpha)$, $\gamma g$ has been defined in such a manner that $\gamma_1 \leq \gamma_2 < \beta \Rightarrow \gamma_1 g \subset \gamma_2 g$. Then define $\beta g =_{df} \left(\bigcup_{\gamma < \beta} \gamma g\right) f \in \mathcal{C}$

Observe that $\gamma < \beta \Rightarrow \gamma g \subsetneqq \beta g$

$g$ is 1-to-1 since for $\beta_1 \neq \beta_2$, either $\beta_1 < \beta_2$ or $\beta_2 < \beta_1$.

We know from the properties of cardinals (above), $\alpha = \operatorname{card}[\beta : \beta \leq \alpha]$.

Since we have constructed a 1-to-1 function $[\beta : \beta \leq \alpha] \xrightarrow{g} \mathcal{C}$, we know by the definition of $\leq$ (above) that $\alpha \leq \operatorname{card}(\mathcal{C})$, which contradicts the above definition of $\alpha$.

We have proven that the Axiom of Choice $\Rightarrow$ Zorn's Lemma.

S.G.

S.G.

# Errata

## Lecture 1

Delete 1.4. Replace with:

**1.4 Remark.** If either of $\inf \emptyset$, $\sup X$ exist they both do and $\inf \emptyset = \sup X$. Suppose $\inf \emptyset$ exists. $\forall y \in X$ the statement "$\forall a \in \emptyset .\ y \leq a$" is (vacuously) true, and hence $y \leq \inf \emptyset$, which says $\inf \emptyset = \sup X$. Now suppose $\sup X$ exists. If $y \in X$ such that $y \leq a\ \forall a \in \emptyset$ then, using only that $y \in X$, $y \leq \sup X$. Also, "$\sup X \leq a\ \forall a \in \emptyset$" is (vacuously) true. $\therefore$ $\sup X = \inf \emptyset$.

Similarly $\sup \emptyset = \inf X$. To prove it, define $x \leq' y =_{df} x \geq y$. Then $(X, \leq')$ is a partially ordered set and $\sup \emptyset$ in $(X, \leq) = \inf \emptyset$ in $(X, \leq') = \sup X$ in $(X, \leq') = \inf X$ in $(X, \leq)$.

1.9. delete proof and replace with:

Let $B \subset X$. $\sup B =_{df} \mathrm{INF}\{y \in X : y \geq B\}$. Let $b \in B$. Because $b \leq y\ \forall y \geq B$, $b \leq \sup B$. Suppose $x \in X$ such that $x \geq B$. Clearly $\sup B \leq x$. $\therefore$ $\sup B$ really is the supremum of $B$ in $(X, \leq)$.

1.10. $\mathbb{N}$ is not a complete lattice. It has all inf's except $\inf \emptyset$ and all finite sups.

1.15 line 2. replace "$A \neq \emptyset$ and" with "$A \neq \emptyset$," 1.20. Replace "isomorphic" with "isomorphism".

1.23. Replace "to" with "to:"

Lecture 2

p. 1 line 7: replace "$[\beta : \sim\!\sim \beta \leq \alpha]$" with "$[\beta : \sim\!\sim \beta < \alpha]$" (Because if $\alpha$ is finite, say $\alpha = n$, then $0, 1, \ldots, n$ are $n+1$ cardinals $\leq n$).

p 2, line 6 from bottom. Replace "$\alpha = \text{card}\ [\beta : \beta \leq \alpha]$" with "$\alpha \geq \text{card}\ [\beta : \beta \leq \alpha]$". (for the reason just mentioned above.)

# Lecture 3

<u>$b \Rightarrow c$.</u> Let $X$ be a set: If $X$ is countable, any enumeration $X = \{x_1, \cdots, x_n, \cdots\}$ amounts to a well-ordering, so Zorn's Lemma is needed only when $X$ is uncountable.

$\mathcal{R} =_{df} [(A,R) : A \subset X$ & $R \subset A \times A$ such that $(A,R)$ is a well-ordered set$]$. $\mathcal{R} \neq \emptyset$ since $(\emptyset, \emptyset) \in \mathcal{R}$.

For $(A,R), (B,S) \in \mathcal{R}$ define $(A,R) \leq (B,S) =_{df}$ $A \subset B$, $R = S \cap (A \times A)$ & $\forall b \in B \neg A\ \forall a \in A .\ (a,b) \in S$,

claim $(\mathcal{R}, \leq)$ is a partially ordered set.

$(A,R) \leq (A,R)$ is obvious.
$(A,R) \leq (B,S)$ & $(B,S) \leq (A,R) \Rightarrow A \subset B \subset A$ and then $R = S$
so that $(A,R) = (B,S)$
Suppose $(A,R) \leq (B,S)$, $(B,S) \leq (\mathbb{C},T)$. Then $A \subset B$, $B \subset \mathbb{C}$
so $A \subset \mathbb{C}$. $R = S \cap (A \times A) = (T \cap (B \times B)) \cap (A \times A) = T \cap (A \times A)$.
If $c \in \mathbb{C} \neg A$ and if $a \in A$ then $c \in \mathbb{C} \neg B$ & $a \in B$ so $(a,c) \in T$ and $\therefore$ $(A,R) \leq (\mathbb{C},T)$.

Let $\mathcal{C}$ be a chain in $(\mathcal{R}, \leq)$. $A =_{df} \bigcup [B : (B,S) \in \mathcal{C}]$
$R =_{df} \bigcup [S : (B,S) \in \mathcal{C}$. claim $(A,R) \in \mathcal{R}$.

Let $a \in A$. $\exists (B,S) \in \mathcal{C}$. $a \in B$. $\therefore$ $(a,a) \in S \subset R$.
Suppose $(a,b), (b,a) \in R$. $\exists (B,S) \in \mathcal{C}$. $(a,b), (b,a) \in S$.
$\therefore$ $a = b$,
Suppose $(a,b), (b,c) \in R$. $\exists (B,S) \in \mathcal{C}$. $(a,b), (b,c) \in S$.
$\therefore$ $(a,c) \in S \subset R$.
This proves $(A,R)$ is a partially ordered set.
Let $a, b \in A$. $\exists (B,S) \in \mathcal{C}$. $a, b \in B$. Either $(a,b) \in S \subset R$ or $(b,a) \in S \subset R$. $\therefore$ $(A,R)$ is a linearly ordered set.
Let $\emptyset \neq \Gamma \subset A$. $\exists (B,S) \in \mathcal{C}$. $\Gamma \cap B \neq \emptyset$. Let $x$ be an $S$-minimal element of $\Gamma \cap B$. We show that $x$ is an $R$-minimal element of $\Gamma$. Suppose $y \in \Gamma$ with $(y, x) \in R$,

but $\gamma \neq x$. $\exists (\mathbb{C}, T) \in \mathcal{C}$. $(B,S) \leq (\mathbb{C},T)$ and $(\gamma, x) \in T$. By the definition of $x$, $\gamma \in \mathbb{C} \neg B$. $\therefore (x,\gamma) \in T$ and then $\gamma = x$ $\cancel{X}$. Hence $(A,R) \in \mathcal{A}$.

claim $(A,R)$ is an upper bound for $\mathcal{C}$ in $\mathcal{A}$.

Let $(B,S) \in \mathcal{C}$. We must show $(B,S) \leq (A,R)$. That $B \subset A$ and that $R \cap (B \times B) = S$ are clear. Suppose $a \in A \neg B$. $\exists (\mathbb{C},T) \in \mathcal{C}$. $(B,S) \leq (\mathbb{C},T)$ & $a \in \mathbb{C} \neg B$. Hence $\forall b \in B$. $(b,a) \in T \subset R$.

By Zorn's Lemma, $(\mathcal{A}, \leq)$ has a maximal element $(M,R)$. Suppose $\exists x \in X \neg M$. $\bar{R} =_{df} R \cup \{(y,x) : y \in M \cup \{x\}\}$ $\subset (M \cup \{x\}) \times (M \cup \{x\})$.

claim $(M \cup \{x\}, \bar{R}) \in \mathcal{A}$

Reflexivity is clear. Noting that $(x,y) \in \bar{R}$ iff $y = x$ makes antisymmetry and transitivity easy to prove.

But clearly $(M \cup \{x\}, \bar{R}) > (M,R)$, $\cancel{X}$. Hence $M = X$ and then $R$ is a well-ordering for $X$.

<u>c ⇒ a.</u> Let $(A_\alpha : \alpha \in I)$ be a family of non-empty sets. Let $\leq$ be a well-ordering for $I$. $0 =_{df}$ the least element of $(I, \leq)$. $\exists x \in A_0$. $0f =_{df} x$. Suppose $\alpha \in I$ such that $\beta < \alpha \Rightarrow \beta f$ is defined in such a way that $\beta f \in A_\beta$. $\exists y \in A_\alpha$. $\alpha f =_{df} y$. By the principle of transfinite induction, $I \xrightarrow{f} \cup A_\alpha$ is well-defined and $\alpha f \in A_\alpha$ $\forall \alpha \in I$.

The proof is complete.

# Relations and Quotient sets

Let $X$ be a set.

**1 Definition.** $\Delta_X =_{df} \{(x,x) : x \in X\} \subset X \times X$. Let $R, S \subset X \times X$. $\check{R} =_{df} \{(y,x) : (x,y) \in R\}$. $RS =_{df} \{(x,z) : \exists\, y \,.\, (x,y) \in R$ & $(y,z) \in S\}$.

**2 Remarks.** Let $R \subset X \times X$. fsov.

a. $R$ is reflexive iff $\Delta_X \subset R$

b. $R$ is antisymmetric iff $R \cap \check{R} \subset \Delta_X$

c. $R$ is symmetric iff $\check{R} \subset R$.

d. $R$ is transitive iff $RR \subset R$.

e. $\Delta_X R = R = R\Delta_X$

The proofs are obvious.

**3 Definition.** Let $R \subset X \times X$. $R$ is an equivalence relation on $X$ $=_{df}$ $R$ is reflexive, symmetric and transitive. Let $R$ be an E.R. on $X$ and let $x \in X$. The equivalence class of $x$ (with respect to $R$), $=_{dn} xR$, $=_{df} [y \in X : (x,y) \in ?]$.

Let $\mathcal{P} \subset 2^X$. $\mathcal{P}$ is a partition of $X$ $=_{df}$ $\mathcal{P}$ satisfies

(a). $\forall x \in X\ \exists\, P \in \mathcal{P} \,.\, x \in P$.

(b) $\forall P, Q \in \mathcal{P} \,.\, P = Q$ or $P \cap Q = \emptyset$.

**4 Proposition.** Define functions $f, g$ by

$$[R : R \text{ is an E.R. on } X] \xrightarrow{\ f\ } [\mathcal{P} : \mathcal{P} \text{ is a partition of } X]$$

$$R \longmapsto [xR : x \in X]$$

$$[\mathcal{P} : \mathcal{P} \text{ is a partition of } X] \xrightarrow{g} [R : R \text{ is an E.R. on } X]$$

$$\mathcal{P} \longmapsto \{(x,y) : \exists P \in \mathcal{P}.\ \{x,y\} \subset P\}$$

Then $f, g$ are well-defined functions which are mutually inverse (that is $f = g^{-1}$, $g = f^{-1}$) and hence both $f, g$ are 1-to-1 and onto. In effect, partitions and E.R.'s are the same thing.

__Proof.__ $f$ is well-defined.

$\forall x \in X.\ x \in xR$. Suppose $x, y \in X$ are such that $\exists z \in xR \cap yR$. Using the symmetry of $R$ freely, we have whenever $u \in xR$ that $(u,x), (x,z), (z,y) \in R$ so that by transitivity $(y,u) \in R$ and $u \in yR$. $\therefore xR \subset yR$, and similarly $yR \subset xR$.

$g$ is well-defined.

Let $x \in X$. $\exists P \in \mathcal{P}.\ x \in P$. $\therefore \{x,x\} \subset P$. If $\{x,y\} \subset P \in \mathcal{P}$ then $\{y,x\} \subset P$. If $\{x,y\}, \{y,z\} \subset P \in \mathcal{P}$ then $\{x,z\} \subset P$.

$fg$ = identity.

Let $R$ E.R. on $X$. $R' =_{df} Rfg$. $(x,y) \in R'$ iff $\exists z \in X$ with $\{x,y\} \subset zR$ iff $xR \cap yR \neq \emptyset$ iff $(x,y) \in R$.

$gf$ = identity.

Let $\mathcal{P}$ be a partition of $X$. $\mathcal{P}' =_{df} \mathcal{P}gf$. $R =_{df} \mathcal{P}f$. $P \in \mathcal{P}'$ iff $\exists x \in X.\ P = xR$ iff $P \in \mathcal{P}$.

The proof is complete.

## Geometry 141 Seminar

Thursday, Oct. 5, 2:45 pm, 202 TG

Presentations are limited to 6 minutes

T

1. (Bellenot) If $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are functions, a coequalizer of $f, g$ is a function $Y \xrightarrow{a} Q$ with the "universal" property:
   (i) $fa = ga$
   (ii) If $fa' = ga'$ $\exists$ unique $\alpha$ . $a\alpha = a'$.

$$\begin{tikzcd} X \arrow[r, shift left, "fa"] \arrow[r, shift right, "ga"'] & Y \arrow[r, "a"] \arrow[dr, "a'"'] & Q \arrow[d, "\alpha"] \\ & & Q' \end{tikzcd}$$

$a \; Y \longrightarrow Q$

$x \in Y \quad xQ = 0 \in \{0\}$

(a) Prove two coequalizers of $f, g$ are isomorphic. [~~Hint~~: easy]

(b) Prove that every two functions $f, g$ (with same domain and range) have a coequalizer. [Hint: Divide out by an equivalence relation on the union of two disjoint copies of $Y$].

2. (George) If $X \xrightarrow{f} Z \xleftarrow{g} Y$ are functions, a DeF pullback of $f, g$ is a pair of functions $X \xleftarrow{a} P \xrightarrow{b} Y$ with the "universal" property:
   (i) $af = bg$
   (ii) If $a'f = b'g$ $\exists$ unique $\alpha$ $\alpha a = a'$ and $\alpha b = b'$.

$\alpha a.$

$x, y \in P$ st $x\alpha = y\alpha$

$xa = ya$

$$\begin{tikzcd} P' \arrow[dr, "\alpha"] \arrow[drr, bend left, "b'"] \arrow[ddr, bend right, "a'"'] & & \\ & P \arrow[r, "b"] \arrow[d, "a"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

$\boxed{\text{1-1 onto}}$

(a) Prove two pullbacks of $f, g$ are isomorphic. [Hint: easy]

(b) Prove that every two functions with the same range have a pullback.

(c) If $X \xrightarrow{f} Y$ is a function and if $A \subset Y$ construct $Af^{-1} \subset X$ as a pullback.

3. (Glassco) Let $X$ be a set.

(a) If $(R_\alpha : \alpha \in I)$ is a family of equivalence relations on $X$, show that $\bigcap R_\alpha$ is too.

(b) If $R \subset X \times X$ prove $\exists$ unique $R \subset \bar{R}$ such that $\bar{R}$ is an E.R. on $X$ and $R \subset S$ with $S$ an E.R. on $X$ implies $\underline{S \subset \bar{R}}$. [$\bar{R} =_{df}$ E.R. on $X$ generated by $R$].

(c) Prove $[R \subset X \times X : R$ is an E.R. on $X]$ is a complete lattice if $R \leq S =_{df} R \subset S$. Is this a sublattice of $2^X$?

(d) Suppose $R \subset X \times X$, and construct $\bar{R}$ as in (b). Suppose $X \xrightarrow{f} Y$ is a function such that $(x,y) \in R \Rightarrow xf = yf$. Let $X \xrightarrow{\theta} X/\bar{R}$ be the canonical projection. Prove $\exists$ unique $X/\bar{R} \xrightarrow{\bar{f}} Y$ such that $\theta \bar{f} = f$.

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "f"'] & X/\bar{R} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

4. (Greitzer) Let $A, B$ be partially ordered sets. Construct a new p.o. set "$A \times B$" equipped with order preserving maps $A \xleftarrow{p_A} A \times B \xrightarrow{p_B} B$ satisfying

the "universal" property:

If $A \xleftarrow{f_A} \mathbb{C} \xrightarrow{f_B} B$ are order preserving maps then $\exists$ unique $\alpha$ with $\alpha p_A = f_A$, $\alpha p_B = f_B$.

$$\begin{tikzcd} A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B \\ & \mathbb{C} \arrow[u, dashed, "\alpha"] \arrow[ul, "f_A"] \arrow[ur, "f_B"'] & \end{tikzcd}$$

Redo the problem for sets and functions; for topological spaces and continuous maps; for groups and homomorphisms.

5. (Hamilton) Let $X$ be a set. A <u>filter</u> <u>on</u> $X$ is a family, $\mathfrak{F}$, of subsets of $X$ satisfying
   (i) $\phi \notin \mathfrak{F}$
   (ii) $A, B \in \mathfrak{F} \Rightarrow A \cap B \in \mathfrak{F}$
   (iii) $A \in \mathfrak{F}$ & $A \subset B \subset X \Rightarrow B \in \mathfrak{F}$.

   Defining $\mathfrak{F} \leq \mathfrak{G} =_{df} \mathfrak{F} \subset \mathfrak{G}$, a maximal filter is called an <u>ultrafilter</u> <u>on</u> $X$. Prove that every filter is the intersection of the ultrafilters containing it.
   [Hints: (a) if $\mathfrak{F}$ is a filter on $X$ and if $A \subset X$ show $A \notin \mathfrak{F}$ iff $[B \subset X : \exists F \in \mathfrak{F}. F \cap A' \subset B]$ is a filter on $X$; (b) if $\mathfrak{F}$ is a filter on $X$, show $\mathfrak{F}$ is an ultrafilter on $X$ iff $\forall A \subset X$ either $A \in \mathfrak{F}$ or $A' \in \mathfrak{F}$; (c) using Zorn's Lemma, show every filter is contained in an ultrafilter.]

6. (Haney) Let $X$ be a set.
   (a) If $(R_\alpha : \alpha \in I)$ is a family of partial orders on $X$ show that $\bigcap R_\alpha$ is too.

(b) Show that $[R \subset X \times X : R \text{ is a partial order}]$ is a p.o. set via $R \leq S =_{df} R \subset S$ which has all non-empty infs, but is not a complete lattice.

(c) What are the minimal and maximal elements in (b)?

7. (Harrell) (a) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be functions. Prove

$f, g$ onto $\Rightarrow fg$ onto $\Rightarrow g$ onto

$f, g$ 1-to-1 $\Rightarrow fg$ 1-to-1 $\Rightarrow f$ 1-to-1

(b) Let $X \xrightarrow{f} Y$ be a function. $f$ is an <u>epimorphism</u> if whenever $Y \overset{g_1}{\underset{g_2}{\rightrightarrows}} Z$ such that $fg_1 = fg_2$ we have $g_1 = g_2$. $f$ is a <u>monomorphism</u> if $g_1 f = g_2 f \Rightarrow g_1 = g_2$. Prove $f$ is a monomorphism iff $f$ is 1-to-1; prove $f$ is an epimorphism iff $f$ is onto.

8. (Malcolm) Let $X$ be a set with at least two elements, and let $n > 0$ be a positive integer. $\Delta_X =_{df} \{$all constant functions from $n$ to $X\} \subset X^n$. Let $x_0 \in X$. Let $R \subset X^n$ with $\Delta_X \subset R$. Prove that $\exists$ a subset, $A$, of $X$ inclusion-maximal with the properties:

(i) $A \neq \emptyset$

(ii) $x_0 \notin A$

(iii) $A^n \subset R$.

Does your proof work if $n$ is an infinite cardinal?

$F \subset Z^n$

$\forall F \subset \bigcup_{n \in D} Z^n$

9. (Ray) If $A \subset \mathbb{Z}$, $n \in \mathbb{Z}$ define $A + n =_{df} \{a + n : a \in A\}$. Prove that $\exists$ a sequence $(A_i : i = 1, 2, \ldots)$ of disjoint subsets of $\mathbb{Z}$ (i.e. $i \neq j \Rightarrow A_i \cap A_j = \emptyset$) with the

property:

$\forall F \overset{\text{finite}}{\subset} \mathbb{Z}\ \exists\, i \in \mathbb{N},\ n \in \mathbb{Z}$ such that $A_i + n = F$.

[Hint: enumerate the finite subsets and use induction]

10. (Weeks) Show that every partially ordered set $(X, \leq)$ may be embedded in (i.e. admits a 1-to-1 order preserving function into) a complete lattice in such a way as to preserve all infs and sups that exist in $X$. [Hint: consider the function
$$\begin{aligned} X &\longrightarrow 2^X \\ x &\longmapsto \{a \in X : a \leq x\} \end{aligned}$$
]

# LECTURE 4. Quotient Sets (continued)

4.1. Definition. If $R$ is an equivalence relation on $\mathbb{X}$, then the function

$$\mathbb{X} \xrightarrow{\theta} \mathbb{X}/R$$

$$x \xrightarrow{\theta} xR$$

is called the <u>canonical</u> <u>projection</u> (with respect to $R$).

4.2. Remarks.

(a) $\theta$ is onto

(b) If $x, y \in \mathbb{X}$, then

$$(x,y) \in R \text{ iff } xR = yR \text{ iff } x\theta = y\theta.$$

4.3. Theorem. If $(R_\alpha : \alpha \in I)$ is a family of E.R.'s on $\mathbb{X}$, then $R =_{df} \bigcap_{\alpha \in I} R_\alpha$ is also an E.R. on $\mathbb{X}$.

Proof. <u>Case I:</u> $I = \phi$

$\bigcap R_\alpha =_{df}$ greatest element. $R =_{df} \mathbb{X} \times \mathbb{X}$ is an E.R. on $\mathbb{X}$, and hence $R$ is surely the greatest element (i.e., the largest E.R.).

<u>Case II:</u> $I \neq \phi$

(reflexive) $(x,x) \in R_\alpha \ \forall \alpha$, so $(x,x) \in R$

(symmetric) $(x,y) \in R \iff (x,y) \in R_\alpha \ \forall \alpha \Rightarrow (y,x) \in R_\alpha \ \forall \alpha$
$\iff (y,x) \in R.$

(transitive) $(x,y), (y,z) \in R \iff (x,y), (y,z) \in R_\alpha \ \forall \alpha$
$\Rightarrow (x,z) \in R_\alpha \ \forall \alpha \iff (x,z) \in R$

4.4. Definition. Let $R \subset \mathbb{X} \times \mathbb{X}$.

$\overline{R} =_{df} \bigcap [S \subset \mathbb{X} \times \mathbb{X} : S \text{ an E.R. on } \mathbb{X} \text{ and } R \subset S]$.

$\overline{R}$ is called the E.R. <u>generated</u> <u>by</u> $R$ ($\overline{R}$ is an E.R. by Thm. 4.3).

If $S$ is an E.R. and $R \subset S$, then $\overline{R} \subset S$.

4.5. Theorem. Let $R \subset X \times X$. $\bar{R} =_{df}$ E.R. on $X$ generated by $R$. Let $X \xrightarrow{f} Y$ such that $(x,y) \in R \Rightarrow xf = yf$. Then $\exists$ a unique $X/\bar{R} \xrightarrow{\tilde{f}} Y$ such that $\theta\tilde{f} = f$.

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "f"'] & X/\bar{R} \arrow[d, dashed, "\tilde{f}"] \\ & Y \end{tikzcd}$$

Proof.

(a) Claim $(x,y) \in \bar{R} \Rightarrow xf = yf$.

$S =_{df} [(x,y) : xf = yf]$.

$S$ is an E.R.: $(x,x) \in S$ since $xf = xf$.

If $(x,y) \in S$, then $xf = yf$; hence $yf = xf$, which implies $(y,x) \in S$.

If $(x,y), (y,z) \in S$, then $xf = yf$ and $yf = zf$. Thus $xf = zf$, and so $(x,z) \in S$.

$R \subset S$ by hypothesis.

$\therefore \bar{R} \subset S$. That is, $(x,y) \in \bar{R} \Rightarrow xf = yf$.

(b) Define $X/\bar{R} \xrightarrow{\tilde{f}} Y$, and for $\alpha \in X/\bar{R}$

$\alpha \xmapsto{\tilde{f}} xf$ where $x\theta = \alpha$

($\theta$ is onto, so every element $\alpha$ is of the form $x\theta$).

Then $\tilde{f}$ is well-defined.

Suppose $x\theta = \alpha = y\theta$. We must show that $xf = yf$. But $x\theta = y\theta$ iff $xy \in \bar{R}$

$\Rightarrow xf = yf$.

(c) $\theta\tilde{f} = f$

If $x \in X$, then $x\theta\tilde{f} = xf$ (by definition).

4.5.1. Lemma. If $A \xrightarrow{\omega} B \underset{u}{\overset{t}{\rightrightarrows}} C$ where $\omega$ is onto and $\omega t = \omega u$, then $t = u$.

Proof. Let $b \in B$. As $\omega$ is onto, $\exists a \in A$ such that $a\omega = b$. Therefore $bu = a\omega u = a\omega t = bt$. Thus $t = u$.

(d) $\tilde{f}$ is unique.

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "f"'] & X/\tilde{R} \arrow[d, shift right, dashed, "\tilde{f}"'] \arrow[d, shift left, dashed, "f'"] \\ & Y \end{tikzcd}$$

Let us assume that there exists another function $f'$ which behaves like $\tilde{f}$ : i.e., $\theta\tilde{f} = f$ and $\theta f' = f$. We must show $\tilde{f} = f'$. Since $\theta$ is onto and $\theta\tilde{f} = \theta f'$, then $\tilde{f} = f'$ by Lemma 4.5.1.

This proves Thm. 4.5.

## Group Theory

In our study of geometry we will consider a set with a collection of subsets (called lines) and a group $G$ of symmetries of geometry. It is therefore necessary to know some results of group theory.

4.6. Definition. A <u>semigroup</u> is a pair $(X, m)$ where $X$ is a set, $X \times X \xrightarrow{m} X$ is a function (binary operation, multiplication) satisfying the following axioms:

Associative Law: $\forall x, y, z \in X$
$$((x,y)m, z)m = (x, (y,z)m)m$$

Notation: $(x,y)m =_{\mathrm{dn}} xy$. Thus the Associative Law can be written as $x(yz) = (xy)z$.

4.7. Example. Let $X$ be a set. Define $xy =_{\mathrm{df}} x$. Since $x(yz) = x$ and $(xy)z = (x)z = x$, $X$ is a semigroup.

4.8. Definition.

Let $(X, m)$ be a semigroup and $e, e' \in X$.

$e$ is a <u>left identity</u> if $\forall x \in X$, $ex = x$.

$e'$ is a <u>right identity</u> if $\forall x \in X$, $xe' = x$.

(In Example 4.7, every element is a right identity).

4.9. Theorem. If $e$ is a left identity and if $e'$ is a right identity, then $e = e'$.

Proof. $e = ee' = e'$.

(Thus if an element is both a left and a right identity, it is unique).

4.10. Definition. A <u>monoid</u> is a semigroup which has both a left and a right identity.

(Hence if $(X, m)$ is a monoid, then $\exists$ unique $e \in X$ with $xe = x = ex$ $\forall x \in X$)

4.11. Examples.

1. $\{0, 1, 2, \cdots\}$ with "$+$" $= m$. This is a monoid with (left and right) identity $0$.
2. $\{1, 2, \cdots\}$ with "$+$" is a semigroup, but not a monoid.
3. $\{1, 2, \cdots\}$ with "$\times$" is a monoid with unit $1$.
4. $\{2, 3, 4, \cdots\}$ with "$\times$" is a semigroup, but not a monoid.
5. $(X, m)$ with $(x, y)m = x$ is a semigroup, but not a monoid if $X$ has more than one element (Because every element is a right identity, but there does not exist a left identity).

4.12. Definition. Let $(X, m)$ be a monoid with unit $e$.

Let $x \in X$. A <u>left inverse</u> for $X$ is an element $x_L \in X$ such that $x_L x = e$.

$x_R =$ df a <u>right inverse</u> for $X$ if $x x_R = e$.

4.13. Theorem. If $x_L x = e$ and $x x_R = e$ then $x_L = x_R$.

Proof. $x_L = x_L e = x_L(x x_R) = (x_L x) x_R = e x_R = x_R$.

If $x_L$ and $x_R$ exist, $x_L = x_R$ is called "$x$-inverse" and is written "$x^{-1}$".

4.14. Definition. A <u>group</u> is a monoid $(X, m) \ni \forall x \in X$ $x^{-1}$ exists.

4.15. Examples.

1. $(\mathbb{Z}, +)$ is a group with unit $0$ and $n^{-1} = -n$.
2. $(\mathbb{Z} \neg \{0\}, \times)$ is <u>not</u> a group, but is a monoid (There does not exist a $k \ni kn = 1$ if $n > 1$).
3. $(Q, +)$, where $Q$ is the rationals, is a group.
4. $(Q \neg \{0\}, \times)$ is a group with unit $1$ and $x^{-1} = \frac{1}{x}$.
5. Obtain a group by replacing $Q$ by $\mathbb{R}$ in 3.
6. Obtain a group by replacing $Q$ by $\mathbb{R}$ in 4.
7. Euclidean $n$-space $(\mathbb{R}^n, +)$ is a group under "+".
   $(x_1, \cdots, x_n) + (y_1, \cdots, y_n) =_{df} (x_1 + y_1, \cdots, x_n + y_n)$.
8. Complex plane $(\mathbb{C}, +)$ is a group under "+".
9. $(\mathbb{C} \neg \{0\}, \times)$ is a group.
   $r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} =_{df} r_1 r_2 e^{i(\theta_1 + \theta_2)}$
10. $S^1$ (subset of $\mathbb{C} \neg \{0\}$, circle of radius 1 about $(0,0)$)
    $(S^1, \times)$ is a group. $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$.

* 11. Let $X$ be a set. Define the set of automorphisms
$\mathrm{Aut}(X) =_{df} [f : X \to X : f \text{ is 1-1 and onto}]$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "fg"'] & X \arrow[r, "g"] & X \end{tikzcd}$$

identity $1_X : X \to X$

$x \longmapsto x$

Notation: $\mathrm{Aut}(X)$ is also denoted by $\mathrm{bij}(X)$ (bijection).

5.1. Definition. Let $G$ be a group. A <u>subgroup of $G$</u> is a subset $(\neq \emptyset)$ $H \subset G$ such that:

(a) $\forall\, h, h' \in H$ . $hh' \in H$

(b) $\forall\, h \in H$ $h^{-1} \in H$

if $H \subset G$ is a subgroup, assert this by saying "$H \leq G$".

5.2 Remark.

If $H \leq G$, $H$ is a group with same $m, e, {}^{-1}$'s (inverses) as $G$.

Since $h, h^{-1} \in H$ (property b) and their composition $hh^{-1} \in H \implies e = hh^{-1} \in H$.

5.3 Examples:

i) $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R}$

ii) $S' \leq \mathbb{C}$

iii) If $X$ is a "<u>set with structure</u>" (viz. p.o.s., groups, topological spaces, etc.) it is virtually always true that the "<u>admissible</u> <u>automorphisms</u>," (such as order preserving automorphisms, etc.) are such that they form a subgroup of $Aut(X)$ because:

(a) If $X \xrightarrow{f} X$, $X \xrightarrow{g} X$ are admissible, so is $fg$. composition of automorphic functions

(b) $X \xrightarrow{1} X$ is admissible. (identity)

5.4 Theorem.

Let $G$ be a group. Let $(H_\alpha : \alpha \in I)$ be a family of subgroups of $G$. Then $H =_{df} \bigcap_\alpha H_\alpha \leq G$.

Proof: I. $I \neq \emptyset$ . $H = G$

II. Let $h, h' \in H$ .... $hh' \in H_\alpha\ \forall \alpha$

$\therefore hh' \in H_\alpha\ \forall \alpha$ .... $hh' \in H$.

Let $h \in H$. $h^{-1} \in H_\alpha\ \forall \alpha$, $h^{-1} \in H$.

5.11 Definition. Let $G$ be a group, $H \leq G$.

$$L_H =_{df} \{(x,y) : x,y \in G \text{ and } x^{-1}y \in H\}$$

$$R_H =_{df} \{(x,y) : x,y \in G \text{ and } xy^{-1} \in H\}$$

5.12 Theorem: $L_H$, $R_H$ are E.R.s on $G$.

Proof: Show $L_H$ is an E.R.

[Reflexivity] $\to$ Let $x \in G$ since $e \in H$,

$$x^{-1}x = e \in H$$

[Symmetry] $\to$ Let $(x,y) \in L_H$ $\therefore x^{-1}y \in H$

since $H$ is a subgroup

$$(x^{-1}y)^{-1} = y^{-1}x^{-1^{-1}} = y^{-1}x \in H \Leftrightarrow (y,x) \in L_H$$

[transitivity] $\to$ Suppose $(x,y),(y,z) \in L_H$

then we have $x^{-1}y,\ y^{-1}z \in H$

and their product $x^{-1}yy^{-1}z = x^{-1}z \in H$

$$\Leftrightarrow (x,z) \in L_H$$

Show $R_H$ is an E.R.

proved similarly

5.13 Theorem.

Given $H \leq G$. Let $x \in G$

Then $xL_H = xH$ ; $xR_H = Hx$.

Proof: $xL_H =_{df} [y \in G : (x,y) \in L_H]$

$xH =_{df} [xh : h \in H]$

$$y \in xL_H \Leftrightarrow (x,y) \in L_H$$

$$\Leftrightarrow x^{-1}y \in H \Leftrightarrow \exists h \in H \ni x^{-1}y = h$$

$$\Leftrightarrow \exists h \in H. \quad y = xh \Leftrightarrow y \in xH$$

That $xR_H = Hx$ is proved similarly

5.5 Definition: Let $G$ be a group with $A \subset G$.
The subgroup of $G$ generated by $A$.

$$=_{dn} \langle A \rangle$$
$$=_{df} \bigcap [H : H \leq G \text{ and } A \subset H]$$

5.6 Remarks: (i) $\langle A \rangle$ is the smallest group containing $A$ as it is the intersection of all subgroups containing $A$.
(ii) $\langle \emptyset \rangle = \{e\}$ which is the smallest subgroup possible

5.7 Definition. Let $G, K$ be groups. The Cartesian product of $G$ and $K$

$$=_{dn} \quad G \times K \qquad \text{[said: G cross K]}$$
$$=_{df} \quad (G \times K, m)$$

$$(G \times K) \times (G \times K) \xrightarrow{m} G \times K$$
$$(g, k) \times (g', k') \longmapsto (gg', kk')$$

5.8 Theorem: $(G \times K, m)$ is a group

Only observe that $(e, e)$ is the identity and that $(g, k)^{-1} = (g^{-1}, k^{-1})$; the associative law is no problem since it is true on each coordinate.

5.9 Reason for notation $H \leq G$.

By 5.5, subgroups ordered by "$\leq$" form a complete lattice. $\text{INF}(H_\alpha) = \bigcap H_\alpha$; $\text{SUP}(H_\alpha) = \langle \bigcup H_\alpha \rangle$.

5.10 Remark: Let $G$ be a group; $x, y \in G$.
Then $(xy)^{-1} = y^{-1}x^{-1}$

Proof:
$$((xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1}$$
$$= xx^{-1}$$
$$= e$$
$$\therefore y^{-1}x^{-1} = (xy)^{-1}$$

5.14. Remark. Let $P_1, P_2$ be partitions of set $X$, each consisting of non empty subsets of $X$ with $P_1 \subset P_2$, then $P_1 = P_2$.

Proof: Show $P_2 \subset P_1$

Let $P \in P_2$, By hypothesis $P \neq \emptyset$. Let $x \in P$.

$\exists\, Q \in P_1 \ni x \in Q$ (property of partitionity)

$Q \in P_2$ by hypothesis

$x \in P \cap Q$, so $P = Q$. $\therefore P \in P_1$.

5.15 Theorem:

Let $H \leq G$. The following statements are pair-wise equivalent:

(a) $\forall\, g \in G .\ gH = Hg$

(b) $\forall\, g \in G .\ \exists\, \tilde{g} \in G \ni gH = H\tilde{g}$.

(c) $\forall\, \tilde{g} \in G\ \exists\, g \in G \ni gH = H\tilde{g}$

(d) $L_H = R_H$

(e) $\forall\, g \in G .\ gHg^{-1} \subset H$.

(f) $L_H \leq G \times G$.

(g) $R_H \leq G \times G$.

<u>Proof.</u>

<u>a⇔b⇔c⇔d</u>. By 5.13, (a) says "$gL_H = gR_H$", (b) says "$gL_H \in \{xR_H : x \in G\}$", (c) says "$gR_H \in \{xL_H : x \in G\}$." In view of 5.14 and Lecture 3, 4, (a), (b), (c), (d) are all equivalent to $\{xL_H : x \in G\} = \{xR_H : x \in G\}$.

<u>a⇔e</u>. a ⇒ e is clear, multiplying on the right by $g^{-1}$. Conversely, if $gHg^{-1} \subset H$ $\forall g \in G$ then for each fixed $g_0 \in G$, $g_0^{-1}Hg_0 \subset H$ and then $H = g_0 g_0^{-1} H g_0 g_0^{-1} \subset g_0 H g_0^{-1}$.

6.1 Thm (5.15 continued)

<u>d ⇒ f</u> $(x,a),(y,b) \in L_H \wedge L_H = R_H \Rightarrow$

i) $xyH = xHy = xHb = aHb = abH \therefore (xy, ab) \in L_H$

ii) $(x,a) \in L_H \Rightarrow (x,a) \in R_H \Rightarrow xa^{-1} \in H \Rightarrow (x^{-1}, a^{-1}) \in L_H$

$\therefore L_H \leq G \times G$

<u>f ⇒ e</u> $(x,y) \in L_H \Rightarrow (x^{-1}, y^{-1}) \in L_H$

$\Rightarrow xy^{-1} \in H \Rightarrow (x,y) \in R_H \quad \therefore L_H \subset R_H$

$\Rightarrow gH \subset Hg \Rightarrow e$

<u>f ⇔ g</u> by symmetrical argument to above □.

6.2 Def

$H \leq G$. $H$ is <u>normal</u> (invariant) in $G$ if any of the 7 equivalent conditions of the previous proposition (5.15) are valid

6.3 Remark

$H_i$ normal in $G \Rightarrow \bigcap H_i$ normal in $G$

$\therefore$ any subset generates a normal subgrp (the smallest normal subgrp containing the subset)

6.4 D

$H, K$ groups $\wedge$ $H \xrightarrow{f} K$ a fn.

$f$ is a <u>homomorphism</u> if either of the following equiv. conditions is true:

i) $\forall x, y \in H$, $(xy)f = (xf)(yf)$

ii) i) and $\forall x \in H$, $x^{-1}f = (xf)^{-1}$ and $ef = e'$

6.4 (cont.)

ii) $\Rightarrow$ i) is clear.

i) $\Rightarrow$ $ef = eef = efef \quad \therefore \; e' = (ef)^{-1}(ef) = ef$

$\wedge \; x \in H. \quad xfx^{-1}f = (xx^{-1})f = ef = e' \quad \therefore \; (xf)^{-1} = x^{-1}f$

$\therefore$ ii).

6.5 D's

a) homomorphism $H \xrightarrow{f} K$ is an <u>isomorphism</u> if $f$ is one-one onto (bijective)

b) homo $H \xrightarrow{f} H$ is an <u>endomorphism</u> of $H$

c) isomorphism $H \xrightarrow{f} H$ is an automorphism of $H$

d) $\text{AUT}\, H = [f : H \to H \ni f$ is an automorphism$]$

e) $\text{BIJ}\, H = [f : H \to H \ni f$ is a bijective fn$]$

6.6 R's

a) $H \xrightarrow{f} K$ homo $\Rightarrow$

$f$ iso $\Leftrightarrow \exists\, K \xrightarrow{g} H$ homo $\ni fg = 1_H \wedge gf = 1_K$

b) $H$ a group $\Rightarrow \text{AUT}\, H \leq \text{BIJ}\, H$

6.7 D

$x \in G$ a group. the <u>inner automorphism</u> induced by $x$ is $\varphi_x : G \to G : g \mapsto xgx^{-1}$.

$( \; gg'\varphi_x = xgx^{-1}xg'x^{-1} = g\varphi_x g'\varphi_x$

$(g\varphi_x)^{-1} = (xgx^{-1})^{-1} = xg^{-1}x^{-1} = g^{-1}\varphi_x \; )$

6.8 T (Cayley)

$G$ a grp $\Rightarrow$ $\varphi : G \to \mathrm{Bij}\, G : x \mapsto R_x$

where $R_x : G \to G : y \mapsto yx$

$\varphi$ is an isomorphism into.

("every grp is $\cong$ to a transformation grp")

---

$y R_x R_{\bar{x}} = y x \bar{x} = y R_{x\bar{x}}$ $\therefore$ $\varphi$ is homo

$R_x = R_{\bar{x}} \Rightarrow x = e R_x = e R_{\bar{x}} = \bar{x}$ $\therefore$ $\varphi$ 1-1 $\square$.

6.9 T

$H \xrightarrow{f} K$ homo.

a) $A \leq H \Rightarrow Af \leq K$

b) $B \leq K \Rightarrow Bf^{-1} \leq H$

---

$a, b \in A \Rightarrow af, bf \in Af$, $(af)^{-1} = a^{-1}f \in Af$,

$af\, bf = abf \in Af$ $\therefore$ a)

$a, b \in Bf^{-1} \Rightarrow af, bf, abf \in B \Rightarrow ab \in Bf^{-1}$

$b \in Bf^{-1} \Rightarrow b^{-1}f = (bf)^{-1} \in B \Rightarrow b^{-1} \in Bf^{-1}$ $\therefore$ b) $\square$

6.10 T

$H$ normal in $G \Rightarrow G/H =_{df} G/L_H = G/R_H$ is a group

with $G/H \times G/H \xrightarrow{m} G/H : xH, yH \mapsto xyH$

---

$xH = x'H$, $yH = y'H \Rightarrow xyH = xHyH = x'Hy'H = x'y'H$

$\therefore$ $m$ is well defined.

$x(yz)H = xHyzH = xHyHzH = xyHzH = (xy)zH$

$H = eH$ is identity

$(xH)^{-1} = x^{-1}H$ since $xHx^{-1}H = xx^{-1}H = H$ $\square$.

# Geometry 141 Seminar.

## WEDNESDAY, OCTOBER 18, 4:15 pm

1. (Bellenot) Prove $\exists$ only one three-element group (i.e. show every two such are isomorphic.)

2. (George) Let $G$ be a group, $a \in G$. Define $H =_{df} \{g \in G : ag = ga\}$. Show that $H$ is a subgroup of $G$.

   $g_1, g_2 \in G$ $\quad g_1 g_2 a = g_1 a g_2 = a g_1 g_2$

   $g_1 \in G$ $\quad g_1^{-1}$ $\quad g^{-1} a \cancel{g g^{-1}} = \cancel{g^{-1} g} a g^{-1}$

3. (Greitzer) Prove that the group of integers is characterized by the following two properties:
   (a) it is infinite
   (b) it is isomorphic to each of its non-0 subgroups.

That is, show $\mathbb{Z}$ satisfies (a), (b) and that whenever $G$ satisfies (a), (b) then $G \cong \mathbb{Z}$. [Hint: Use the fact, which we will prove in class, that every subgroup of $\mathbb{Z}$ has form $n\mathbb{Z} = \{0, \pm n, \pm 2n, \pm 3n, \dots\}$ for some $n$.]

4. (Gulik) A metric space $=_{df}$ a pair $(X, d)$ with $X$ a set, $d$ a "distance function" $X \times X \xrightarrow{d} \mathbb{R}$ such that
   (a) $\forall x, y \in X \,.\, d(x,y) \geq 0$
   (b) $\forall x, y \in X \,.\, x \neq y \Rightarrow d(x,y) > 0$
   (c) $\forall x, y \in X .\ d(x,y) = d(y,x)$
   (d) $\forall x, y, z \in X .\ d(x,y) + d(y,z) \geq d(x,z)$.

If $(X, d)$, $(Y, e)$ are metric spaces, an isometry from $(X,d)$ to $(Y,e)$ $=_{df}$ a bijection $X \xrightarrow{f} Y$ such that $\forall x, y \in X.\ d(x,y) = e(xf, yf)$. Prove that if $(X,d)$ is a metric space then the subset $[f : f$ is an isometry from $(X,d)$ to $(X,d)] \subset bij(X)$ is a subgroup.

5. (Hamilton) Let $G$ be a group. The commutator subgroup of $G$ =df the subgroup of $G$ generated by $\{ghg^{-1}h^{-1} : g, h \in G\}$. Denote this subgroup by "$\mathbb{C}$".

   (a) show $\mathbb{C}$ is a normal subgroup.

   (b) show $G/\mathbb{C}$ is abelian.

   [Hint: assume the result of 6]

6. (Malcolm) Let $G$ be a group, $A \subset G$, $\bar{A}$ = do subgroup generated by $A$. Show $\bar{A}$ is a normal subgroup iff $\forall x \in G.\ xAx^{-1} \subset \bar{A}$. $\cancel{\bar{A} = \bigcup AUA^{-1}}$

7. (Ray) Construct an example of a group $G$ and a subgroup $H$ such that $H$ is abelian but is not a normal subgroup of $G$.

$G = S_3$ $\quad \{e, a\}$ $\quad ea = ae$

$a^2 = a^2$

$\bar{A} \supset AUA^{-1}$

$x \in \bar{A}$ & $x \notin AUA^{-1}$

Lecture #7

7.1 Def Let $G \xrightarrow{f} H$ be a group homomorphism. Then the Kernel of $f =_{df} \mathrm{Ker}\, f =_{df} \{g : g \in G \wedge gf = e\}$. $\mathrm{Ker}\, f \leq G$, since $\mathrm{Ker}\, f = ef^{-1}$.

7.2 Thm Let $G$ be a group, $H \subset G$. Then tfsae:

a) $H$ is a normal subgroup of $G$.

b) $\exists$ group homomorphism $G \xrightarrow{f} K \ni H = \mathrm{Ker}\, f$.

Proof: $a \Rightarrow b$ Suppose $H$ is a normal subgroup of $G$. Then if $G \xrightarrow{\Theta} G/H$ is defined by $x\Theta = Hx \ \forall x \in G$, $\Theta$ is trivially a homomorphism & $\mathrm{Ker}\,\Theta = \{x \in G : xH = H\} = H$.
$b \Rightarrow a$ Suppose $\exists\ G \xrightarrow{f} K \wedge H = \mathrm{Ker}\, f$. Let $x \in G$, $h \in H$. Then $(xhx^{-1})f = xf\, hf\, (xf)^{-1} = xf\, e\, (xf)^{-1} = e$. Then $xhx^{-1} \in \mathrm{Ker}\, f = H$, and $H$ is normal.

7.3 Def Let $G$ be a group, and let $\mathfrak{C} = \{g : \forall_{a \in G}\ ag = ga\}$, ie., $\mathfrak{C}$ is the set of elements of $G$ which commute with everything. $\mathfrak{C}$ is called the center of $G$.

7.4 Thm $\mathfrak{C}$ is a normal subgroup of $G$.

Proof: Let $\mathfrak{J}_A$ be the set of inner automorphisms of $G$, ie., for $g \in G$ define $T_g \in \mathfrak{J}_A : G \to G$ by $xT_g = g^{-1}xg\ \forall x \in G$. $\mathfrak{J}_A$ is easily shown to be a group, and $\mathfrak{J}_A \leq \mathfrak{A}(G)$, the set of all automorphisms of $G$. $xT_{gh} = h^{-1}g^{-1}xgh = h^{-1}(g^{-1}xg)h = g^{-1}xg\,T_h = xT_g T_h$. Define $G \xrightarrow{\psi} \mathfrak{A}(G)$ by $g\psi = T_g\ \forall g \in G$. Then $gh(\psi) = T_{gh} = T_g T_h = g\psi\, h\psi$. Then $\psi$ is a homomorphism of $G$ into $\mathfrak{A}(G)$ with image $\mathfrak{J}_A$. $\mathrm{Ker}\,\psi =_{df} \{g \in G : g\psi = e\} = \{g \in G : T_g = e\}$. Then $g \in \mathrm{Ker}\,\psi \Rightarrow \forall_{x \in G}\ xT_g = g^{-1}xg \Rightarrow xg = gx$. This is precisely the condition $x \in \mathfrak{C}$. $\therefore \mathfrak{C} = \mathrm{Ker}\,\psi$, and $\mathfrak{C}$ is normal.

7.5 Thm Let $G \xrightarrow{f} K$ be a group homomorphism. Then the image of $f$, $\mathrm{im}\, f \cong G/\mathrm{Ker}\, f$.

Proof:
$$\begin{tikzcd} G \arrow[r, "\Theta"] \arrow[dr, "f"'] & G/\mathrm{Ker}\, f \arrow[d, dashed, "\tilde{f}"] \\ & K \end{tikzcd}$$
Let $x, y \in L_{\mathrm{Ker}\, f}$. From def. 5.11, $x^{-1}y \in \mathrm{Ker}\, f$. Then $e = x^{-1}yf = (xf)^{-1}yf$ since $f$ is a group homomorphism. Then $xf = yf$. From Thm 4.5, $\exists$ unique function $\tilde{f} \ni \Theta\tilde{f} = f$. That is, we define $G/\mathrm{Ker}\, f \xrightarrow{\tilde{f}} K : g(\mathrm{Ker}\, f) \mapsto gf$; $\tilde{f}$ is well defined and clearly a homomorphism. To show $\tilde{f}$ 1-1, suppose $g(\mathrm{Ker}\, f)\tilde{f} = [g'(\mathrm{Ker}\, f)]\tilde{f}$. Then $gf = g'f$ iff $(gf)^{-1}(g'f) = e$ iff $(g^{-1}g')f = e$. Then $g^{-1}g' \in \mathrm{Ker}\, f$, or $g\,\mathrm{Ker}\, f = g'\,\mathrm{Ker}\, f$. Then $\tilde{f}$ is an isomorphism, and from def. $\mathrm{Im}\,\tilde{f} = \{gf : g \in G\} = \mathrm{Im}\, f$, and $\mathrm{im}\, f \cong G/\mathrm{Ker}\, f$.

Cor Let $G$ be a group, and let $\mathfrak{C}$ be the center of $G$. Then $G/\mathfrak{C} \cong \mathfrak{J}_A$.

7.6 Thm Let $G \xrightarrow{f} K$ be a group homomorphism. Then $f$ 1-1 iff $\mathrm{Ker}\, f = e$.

Proof: Let $x, y \in G$. $xf = yf$ iff $(xf)^{-1}yf = e$ iff $(x^{-1}y)f = e$ iff $x^{-1}y \in \mathrm{Ker}\, f$. Suppose $f$ 1-1. Let $x \in \mathrm{Ker}\, f$. Then $xf = ef = e$, and $x = e$. Conversely, suppose $\mathrm{Ker}\, f = e$. If $xf = yf$, $x^{-1}y \in \mathrm{Ker}\, f = e$. Then $x = y$.

# Lecture 7 - cont.

7.7 **Def** Let $G$ be a group, $A \subset G$. $A$ is a basis for $G =_{df} \forall$ groups $K$ and $\forall_f\ A \xrightarrow{f} K$, $\exists$ unique homomorphism $\tilde{f}$ $G \xrightarrow{\tilde{f}} K \ni f = \tilde{f}/A$.
$$\begin{tikzcd} A \arrow[r, phantom, "\subset"] \arrow[dr, "f"'] & G \arrow[d, "\tilde{f}"] \\ & K \end{tikzcd}$$
If 7.7 is satisfied by $A \subset G$, $G$ is said to be free on $A$. $L$ is a free group if $L$ has a basis.

7.8 **Thm** $\mathbb{Z}$ is free on one generator.

**Proof**: claim $\{1\}$ is basis for $\mathbb{Z}$. Let $L$ be a group, and $\{1\} \xrightarrow{x} L$. Define $\mathbb{Z} \xrightarrow{\tilde{x}} L$ by $n \overset{\tilde{x}}{\mapsto} x^n$. $\tilde{x}$ homomorphism, and since all homomorphisms of $\mathbb{Z}$ are of this form & determined by their action on $\{1\}$, $\tilde{x}$ is unique.

7.9 **Def**: $\mathbb{Z}_n$ is the cyclic group of order $n$, i.e. a group isomorphic to rotation of $n$-gon.

$(\mathbb{Z}_3 = {}_e\overset{a}{\Delta}{}_{a^2},\ \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z})$.

7.10 **Thm** Let $G$ be a singly generated group; ie, $\exists_{g\in G}\ \langle g\rangle = G$. Then $G \cong \mathbb{Z}$ or $\exists_n\ G \cong \mathbb{Z}_n$.

**Proof**:
$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\tilde{h}"] & G \\ \{1\} \arrow[u, phantom, "\cup"] \arrow[ur, "h"'] & \end{tikzcd}$$
Let $h$ be the function $\{1\} \xrightarrow{h} g$, and let $\tilde{h}$ be the homo. defined by 7.7. Then $g \in \operatorname{Im} \tilde{h} \subset \langle g\rangle$, and $\therefore \langle g\rangle \subset \operatorname{Im}\tilde{h}$. $\therefore \langle g\rangle = \operatorname{Im}\tilde{h} = \mathbb{Z}/\operatorname{Ker}\tilde{h} \cong \mathbb{Z}/n\mathbb{Z}$ since $\exists_n \operatorname{Ker}\tilde{h} = n\mathbb{Z}$.

## Group Actions

7.11 **Def.** Let $X$ be a set, $G$ be a group. Let $X \times G \xrightarrow{\pi} X$. Then $(X, G, \pi)$ is a group action if
a) (identity axiom) $\forall_{x\in X}\ (x,e)\pi = x$
b) (homomorphism axiom) $\forall_{x\in X}\forall_{g,h\in G}\ ((x,g)\pi, h)\pi = (x, gh)\pi$.

Notation: write $xg =_{df} (x,g)\pi$. Then axioms become
a) $\forall_{x\in X}\ xe = x$
b) $\forall_{x\in X}\forall_{g,h\in G}\ (xg)h = x(gh)$

7.12 **Examples**.

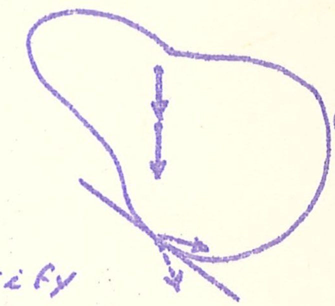
i) Let $H \le G$, $G$ any group. Define $G \times H \xrightarrow{\pi} G$ $xh \overset{\pi}{\mapsto} xh$. $(X, G, \pi)$ is a group action.
**Proof**: a) $xe = x$, and b) $(xg)h = x(gh)$, from properties of $G$.

ii) For any set $X$ and any group $G$, $\exists$ the trivial action $(X, G, \pi)$ where $x, g \overset{\pi}{\mapsto} xg =_{df} x$. Then a) $xe = x$, and b) $(xg)h = xh = x = x(gh)$.

iii) Let $X$ be a set, and let $\alpha \in \operatorname{bij} X$. The discrete flow in $X$ generated by $\alpha$ is the $\mathbb{Z}$ action $(X, \mathbb{Z}, \pi)$, where $x, n \overset{\pi}{\mapsto} x\alpha^n$. In this case, a) $x0 = x\alpha^0 = x1_X = x$, and b) $(xn)m = (x\alpha^n)m = (x\alpha^n)\alpha^m = x\alpha^{m+n} = x(m+n) = xnm$ in our notation.

7.1 Example - Billiard Ball Flow.

- Let $\mathbb{C}$ be a simple closed $C^2$ curve in the plane (i.e. $\mathbb{C}$ has a tangent at all points.)




- Let a "point" be an element of the plane interior or on the boundary of $\mathbb{C}$ together with a direction (a "unit vector"). If the point is on the boundary of $\mathbb{C}$, identify its direction with its reflection thru the tangent. (Thus all points are in the closure of $\mathbb{C}$.)

- $\mathbb{C}$ is a group action, called the Billiard Ball Flow induced by $\mathbb{C}$:

Let $X$ = set of all "points" = balls with position and velocity that bounce off edge.

$G = \mathbb{R}$ the reals

$X \times \mathbb{R} \xrightarrow{\Gamma} X$

$(x,\theta),\lambda \longmapsto (y,\tau)$ . . . where $y$ is obtained by moving $\lambda$ units of distance in direction $\theta$ (taking reflections into account) and $\tau$ is the resultant direction.

$\mathbb{C}$ is actually a group action, for

1) identity axiom: let $e=0$ then $x'0 = x'$ where $x' = (x,\theta)$

2) homomorphism axiom: note that distances are additive, i.e.

$(x'g)h = x'(g+h) \equiv x'(gh)$, using previous notation.

7.2 Example - Geodesic Flows

- Let $S$ be a "smooth" surface (a Riemann 2-manifold) in, say, $E^3$.

Let $x, y \in S$ and $C(t)$ be a smooth curve through $x$ and $y$

with the notion of curve length $- \int_x^y c(\tau)\,d\tau$.

Define the distance from $x$ to $y$ as

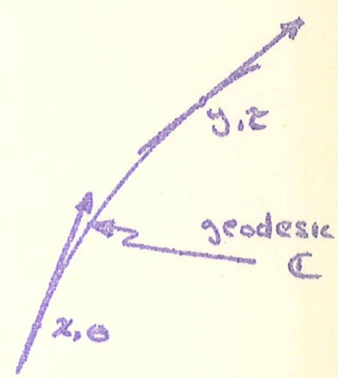
$$\underline{d}(x,y) =_{df} \inf \left\{ \int_x^y c(\tau)\,d\tau \right\} \text{ for } x,y \in S.$$

$\underline{d}$ is a metric on $S$. (the Riemann metric) and the following is true:

$\forall x \in X$ and $\forall \theta \in \mathbb{R}$ $\exists$ a unique curve $C$ through $x$ such that its tangent at $x$ has direction $\theta$ and $\forall a,b \in C$ $\underline{d}(a,b) = \int_a^b c(\tau)\,d\tau$.

$C$ is then called the geodesic through $x$ in direction $\theta$.
($C$ is the "shortest" curve with the $x$ and $\theta$ properties.)

- a group action may be formed called the <u>induced geodesic flow</u>:




Let, $X = \{(x,\theta) : x \in S,\ \theta \in \mathbb{R} = \text{direction}\}$

$G = \mathbb{R}$ the reals,

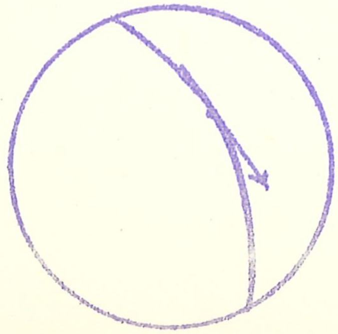
$$X \times G \xrightarrow{\Gamma} X$$

$(x,\theta),\lambda \longmapsto (y,z)$ ... where $y$ is obtained by moving $x$ $\lambda$ units along the geodesic, $\mathbb{C}$, through $x$ with direction $\theta$, and $z$ is the direction of the tangent at $y$.

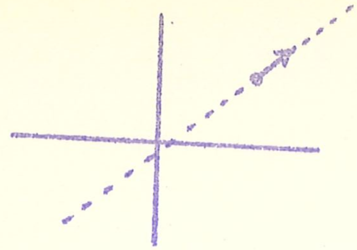
This actually a group action, for the arguments in 7.1 apply.

- Specific examples:

1)



$S$ = surface of a sphere
the geodesics are then great circles.

2) 

S = euclidean plane
geodesics are straight lines.

7.3 Example.—

Let $X$ = any set.

$G \leq \text{bij } X$

$X \times G \xrightarrow{ev} X$ (ev is just evaluation map)

$x, g \longmapsto xg$

This is a group action for

1) identity axiom: let $e = 1_X$, then $xe = x1_X = x$

2) homorphism axiom: trivially $(xg)h = x(gh)$

– Some specific examples:

1) If $(X, \leq)$ is p.o.s. and

$G =_{df}$ {order preserving bijections} $= \{f : f \in \text{bij } X$ and $x \leq y \Rightarrow xf \leq yf\} \leq \text{bij } X$

2) If $(X, \underline{d})$ is a metric space and

$G =_{df}$ group of isometries

3) If $(X, \tau)$ is a topological space and

$G =_{df}$ group of homeomorphisms of $X$

4) If $(X, \ )$ is a "geometry" and

$G =_{df}$ some relevant subgroup of bij $X$

7.3 Definition :- Let $(X, G, \pi)$ and $(Y, K, \xi)$ be group actions.

An action homomorphism from $(X, G, \pi)$ to $(Y, K, \xi)$,

$$(X, G, \pi) \xrightarrow{(f,\theta)} (Y, K, \xi)$$

is a pair of maps $f$ and $\theta$ where

$X \xrightarrow{f} Y$ is a function

$G \xrightarrow{\theta} K$ is a group homomorphism

such that

$$\begin{tikzcd} X \times G \arrow[r, "\pi"] \arrow[d, "f\times\theta"'] & X \arrow[d, "f"] \\ Y \times K \arrow[r, "\xi"] & Y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} x, g \arrow[r, mapsto] \arrow[d, mapsto] & xg \arrow[d, mapsto] \\ xf, g\theta \arrow[r, mapsto] & \begin{matrix}(xg)f \\ (xf)(x\theta)\end{matrix} \end{tikzcd}$$

i.e. $\pi f = (f \times \theta)\xi$ or $\forall x, g : (xf)(g\theta) = (xg)f$

i.e. the structure preserving mappings commute.

Observations :-

(a) $(1, 1)$, a pair of identity maps, is always a group action.

$$\begin{tikzcd} X \times G \arrow[r, "\pi"] \arrow[d, "1\times 1 = f\times\theta ="'] & X \arrow[d, "1 = f"] \\ X \times G \arrow[r, "\pi"] & X \end{tikzcd}$$

(b) The composition of two group actions is a group action:

Let $(f, \theta)(\bar f, \bar\theta) =_{df} (f\bar f, \theta\bar\theta)$, then $(f\bar f, \theta\bar\theta)$ is a group action.

$$\begin{tikzcd} X \times G \arrow[r, "\pi"] \arrow[d, "f\times\theta"'] \arrow[dd, bend right=60, "f\bar f \times \theta\bar\theta"'] & X \arrow[d, "f"] \arrow[dd, bend left=60, "f\bar f"] \\ \bar X \times \bar G \arrow[r, "\bar\pi"] \arrow[d, "\bar f\times\bar\theta"'] & \bar X \arrow[d, "\bar f"] \\ \bar{\bar X} \times \bar{\bar G} \arrow[r, "\bar{\bar\pi}"] & \bar{\bar X} \end{tikzcd}$$

4 Remark:- If $(f,\theta) : (X, G, \pi) \xrightarrow{f,\theta} (Y, K, \xi)$ is a action homomorphism, the f.s.a.e.:

a) $f$ and $\theta$ are bijections

b) $\exists\, (\bar{f}, \bar{\theta}) \ni (Y, K, \xi) \xrightarrow{\bar{f},\bar{\theta}} (X, G, \pi)$

such that $(f,\theta)(\bar{f},\bar{\theta}) = I_{(X,G,\pi)}$

and $(\bar{f},\bar{\theta})(f,\theta) = I_{(Y,K,\xi)}$

i.e. $(f,\theta)$ has an inverse.

Proof:-

1) a) $\Rightarrow$ b) Since $f$ and $\theta$ are 1-1 and onto each has a 1-1 and onto inverse $f^{-1}$ and $\theta^{-1}$. Then

(Note that $\theta^{-1}$ is still a homomorphism.)

$$\begin{tikzcd} Y \times K \arrow[r, "\xi"] \arrow[d, "f^{-1}\times\theta^{-1}"'] & Y \arrow[d, "f^{-1}"] \\ X \times G \arrow[r, "\pi"'] & X \end{tikzcd}$$

This diagram commutes as required; therefore if $\bar{f} = f^{-1}$ and $\bar{\theta} = \theta^{-1}$ we have $(f\bar{f}, \theta\bar{\theta}) = (1,1) = I_{(X,G,\pi)} = (f,\theta)(\bar{f},\bar{\theta})$ as required.

2) b) $\Rightarrow$ a) Using the definition of composition of action homomorphism:

$$\left.\begin{aligned} (f,\theta)(\bar{f},\bar{\theta}) &= I_{(X,G,\pi)} = (1,1) \\ &= (f\bar{f}, \theta\bar{\theta}) \end{aligned}\right\} \text{ or } \begin{aligned} f\bar{f} &= 1 \\ \theta\bar{\theta} &= 1 \end{aligned}$$

and similarly, $\bar{f}f = 1$, $\bar{\theta}\theta = 1$

Then the following diagram holds: [redacted]

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\bar{f}f"'] & Y \arrow[r, "\bar{f}"] & X \arrow[r, "f"] \arrow[rr, bend right, "f\bar{f}"'] & Y \arrow[r, "\bar{f}"] & X \end{tikzcd}$$

1)

2)

If a function has a two sided inverse it is 1-1 and onto. This is just the condition for a bijection. Similarly for $\theta$.

15 Definition.— Let $(X, G, \pi)$ be a group action, then $(X, G, \pi)$ is **effective** if the map

$$G \longrightarrow X^X$$
$$g \longmapsto \pi^g \quad \text{where } \pi^g = \begin{cases} X \longrightarrow X \\ x \longmapsto xg \end{cases}$$

is 1-to-1 (i.e. $g \neq g' \Rightarrow \pi^g \neq \pi^{g'}$).

Remark :- $g \longmapsto \pi^g$ is a group.

Theorem.— Let $(X, G, \pi)$ be a group action and

$$K =_{df} \{\pi^g : g \in G\} \leq \text{bij } X \quad \text{and}$$

$$X \times K \xrightarrow{ev} X \qquad \text{(ev is evaluation group)}$$

then f.s.a.e.:

a) $(X, G, \pi)$ is effective

b) $(X, G, \pi) \xrightarrow{(1_X, \lambda)} (X, K, ev)$ is an action isomorphism.

Proof.—

1) a) $\Rightarrow$ b) let $\lambda$ be the effective map. Then

$$\begin{array}{ccc} X \times G & \xrightarrow{\pi} & X \\ {\scriptstyle 1_X, \lambda} \downarrow & & \downarrow {\scriptstyle 1_X} \\ X \times K & \xrightarrow{ev} & X \end{array}$$

for $G \xrightarrow{\lambda} K$. This forms a group isomorphism for —

- $\lambda$ is 1-1 as in $1_X$ $\therefore$ $(1_X, \lambda)$ is 1-1
- $\lambda$ and $1_X$ are onto $\therefore$ " is onto
- $\lambda$ is a homomorphism.

2) b) $\Rightarrow$ a)

Since $(1_X, \lambda)$ is an action isomorphism $\pi$ is 1-1 and onto and maps, from hypotheses $G \longrightarrow K = X^X$, i.e., $(X, G, \pi)$ is effective.



(last page)

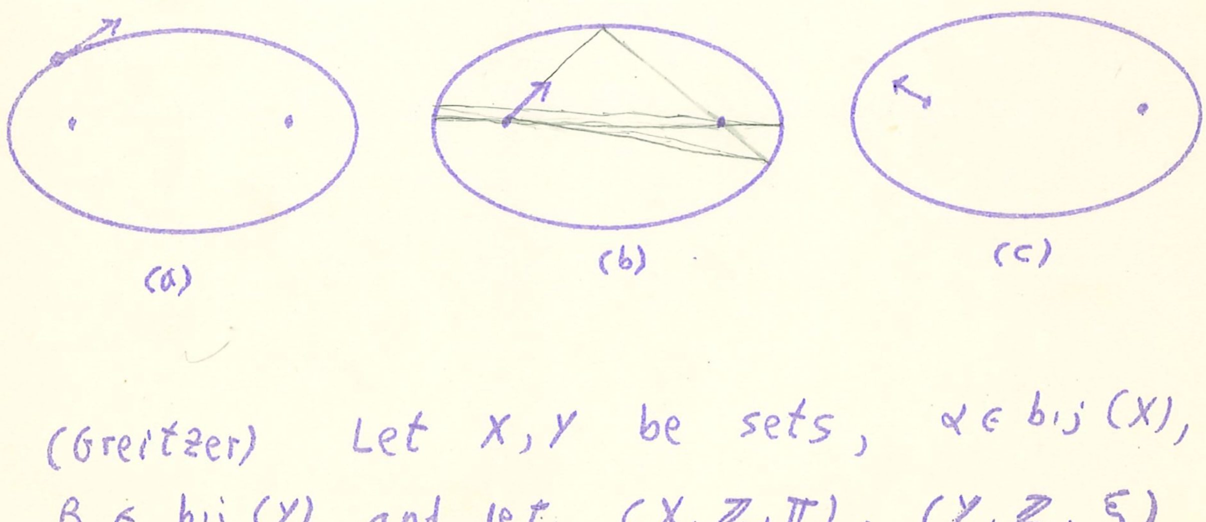
Geometry 141 Seminar
Wednesday, Oct. 25, 4:15 pm

1. (Bellenot) Let $X$ be a set, $G$ a group,
So $X \times G \xrightarrow{\pi} X$ a function. $\forall g \in G.\ \pi^g =_{df}$
the function from $X$ to $X$ defined by $x\pi^g = (x, g)\pi$.
Prove (a) $\Leftrightarrow$ (b):
   a. $(X, G, \pi)$ is a group action.
   b. $\forall g \in G.\ \pi^g \in bij(X)$ and $\begin{array}{ccc} G & \xrightarrow{f} & bij(X) \\ g & \longmapsto & \pi^g \end{array}$

   is a group homomorphism.

$f(gh) = \pi^{gh}$
$\forall_{x \in X}\ x f(gh) = x\pi^{gh} = x\pi^g\pi^h$
$= x f(g) f(h)$

2. (George) Let $(X, G, \pi)$ be the billiard
Jr ball flow induced by an ellipse with distinct
foci. Describe the orbits of the following
elements of $X$.



(a) (b) (c)

3. (Greitzer) Let $X, Y$ be sets, $\alpha \in bij(X)$,
S $\beta \in bij(Y)$ and let $(X, \mathbb{Z}, \pi)$, $(Y, \mathbb{Z}, \sigma)$
be the discrete flows generated by $\alpha, \beta$.

Assume further that there exists a bijection $X \xrightarrow{g} Y$ such that $\beta^{-1} = g\beta g^{-1}$. Prove (a) $\Leftrightarrow$ (b):

a. $(X, \mathbb{Z}, \pi) \cong (Y, \mathbb{Z}, \xi)$

b. There exists a bijection $X \xrightarrow{f} Y$ such that $\alpha f = f\beta$.

4. S? (Gulik) Let $G \xrightarrow{\varphi} K$ be a group homomorphism. Show that
$$\begin{aligned} K \times G &\xrightarrow{\pi} K \\ x, g &\longmapsto x(g\varphi) \end{aligned}$$
is a group action.

Use this result to prove that when $N$ is a normal subgroup of $G$,
$$\begin{aligned} G/N \times G &\longrightarrow G/N \\ xN,\ g &\longmapsto xgN = xNg \end{aligned}$$
is a group action.

$$\begin{aligned} G &\xrightarrow{\varphi} G/N \\ g &\longmapsto Ng \end{aligned}$$

5. S (Hamilton) Let $(X, G, \pi)$ be a group action and let $N$ be a normal subgroup of $G$. $\forall x \in X.\ N_x =_{df} [g \in G : xg \in xN]$. Prove

a. $\forall x \in X.\ N_x \leq G$ (Subgroup)

b. $\forall x \in X\ \forall g \in G.\ N_{xg^{-1}} = gN_x g^{-1}$

6. S (Malcolm) Let $(X, G, \pi)$ be a group action. The period of $(X, G, \pi)$ $=_{df}$ the subset $P = [g \in G : \forall x \in X.\ xg = x]$ of $G$. $x\, gpg^{-1} = ((xg)p)g^{-1}$

a. Prove $P$ is a normal subgroup of $G$. $= (xg)g^{-1} = x$

b. Prove
$$X \times \frac{G}{P} \xrightarrow{\pi_P} X \qquad \text{is well-defined}$$
$$x, gP \longmapsto xg$$
and that $(X, \frac{G}{P}, \pi_P)$ is an <u>effective</u> group action.

7. (Ray) Let $P$ be a finite set. If $A \subset P^{\mathbb{Z}}$, $\bar{A} =_{df} [\mathbb{Z} \xrightarrow{f} P \in P^{\mathbb{Z}} : \forall F \overset{\text{FINITE}}{\subset} \mathbb{Z}$ $\exists\, \mathbb{Z} \xrightarrow{a} P \in A$ such that $f$ and $a$ agree on elements of $F$]. Say that $A \subset P^{\mathbb{Z}}$ is

20 dense $=_{df}$ $\bar{A} = P^{\mathbb{Z}}$. [Note: If $P$ is considered as a discrete space then "$\bar{\ }$" is precisely the closure operator of $P^{\mathbb{Z}}$ with the product topology.]

a. Verify that $(P^{\mathbb{Z}}, \mathbb{Z}, \pi)$, where
$$P^{\mathbb{Z}} \times \mathbb{Z} \xrightarrow{\pi} P^{\mathbb{Z}}$$
$$\mathbb{Z} \xrightarrow{f} P,\ n \longmapsto fn =_{df} \begin{array}{c} \mathbb{Z} \xrightarrow{fn} P \\ m \longmapsto (m+n)f \end{array}$$
is a group action. (It is called the <u>symbolic flow on $P$ symbols</u>).

b. Prove there exists an element of $P^{\mathbb{Z}}$ whose orbit is dense.

(Hint: For (b), rely on problem 9 of the oct. 5 seminar.)

141 Seminar

No meeting Nov. 22

This seminar Wednesday Nov. 29, 4:15 pm, TG 202

1. (Bellenot) Let $(X, \mathcal{L})$ be a preaffine plane such that the set of points on any line has cardinal $\alpha$. Let $\mathcal{R} \subset \mathcal{L}$ with crd $\mathcal{R} < \alpha$. Show that $\bigcup \mathcal{R} \neq X$. Conclude, using problem 2 of the previous seminar, that if $X$ is infinite it is not the union of finitely-many lines.

2. (George) Prove that all isometries of the Euclidian plane are collineations, but not conversely.

3. (Greitzer) Let $(X, \mathcal{L})$ be a preaffine plane. Describe any dilatation that interchanges two distinct points. What does this amount to for the Euclidian plane?

4. (Gulik) Show that the translations of the Euclidian plane in the sense of 13.1 are the usual translations.

5. (Hamilton) Let $(X, \mathcal{L})$ be the Euclidian plane with $G, D, T$ as in 13.1. Let $d$ be the "sum of the coordinates" metric on $X$ as in lec. II Seminar, #5, #6. Let $I$ be the group of $d$-isometries.

a. Prove $T \leq I \leq G$.

b. How does this help to analyze $I$; what is $I \cap D$?

6. (Malcolm) Let $(X, \mathcal{L})$ be a preaffine plane such that there exists a translation in every direction. what can you prove about $T$ as an abstract group? e.g. prove $T$ is not singly generated.

7. (Ray) Let $(X, \mathcal{L})$ be a prealine plane such that there exists a translition connecting any two points. Let $d$ be a metric on $X$ such that translations are $d$-isometries. Prove that opposite sides of a parallelogram have equal $d$-lengths.

# Lectures 9, 10

D. Let $(X, G, \pi)$ be a group action, $H \subset G$, $A \subset X$. $AH =_{df} \{ah : a \in A, h \in H\}$. If $x \in X$, $xH =_{df} \{x\}H$. If $h \in H$, $Ah =_{df} A\{h\}$.

$\forall x \in X$, the orbit of $x$ under $G$ $=_{df}$ $xG$.

$A$ is $H$-invariant $=_{df}$ $AH \subset A$.

T. Let $(X, G, \pi)$ be a group action, $A \subset X$, $H \leq G$ such that $A$ is $H$-invariant.

Then
$$\begin{aligned} A \times H &\xrightarrow{\pi_0} A \\ a, h &\longmapsto ah \end{aligned}$$
is a group action and

$$(A, H, \pi_0) \xrightarrow{(inc_A,\, inc_H)} (X, G, \pi)$$
is an action homomorphism.

<u>Proof.</u> Easy. $\square$

T. Let $(X, G, \pi)$ be a group action. Then $X/G =_{df} [xG : x \in X]$ is a partition of $X$.

Proof. $\forall x \in X$, $x = xe \in xG$. Suppose $z \in xG \cap yG$. $\exists g \in G$. $z = xg$. $\therefore$ $zG = (xg)G = x(gG) = xG$. Similarly $zG = yG$. $\Box$

# Geometries

<u>10.1 D.</u> A <u>geometry</u> is a quadruple $(X, G, \pi, \mathcal{L})$ such that $(X, G, \pi)$ is a group action, $\mathcal{L} \subset 2^X$ is a family of subsets of $X$ whose elements are called the <u>lines</u> of the geometry, all subject to the following five <u>geometry</u> <u>axioms</u>:

G1. $\forall L, L' \in \mathcal{L}.\ L \subset L' \Rightarrow L = L'$.

G2. $\forall x, y \in X\ \exists L \in \mathcal{L}.\ x, y \in L$.

G3. $\forall L \in \mathcal{L}\ \forall g \in G.\ Lg \in \mathcal{L}$.

G4. $\forall L, L' \in \mathcal{L}\ \exists g \in G.\ Lg = L'$.

G5. $\forall L \in \mathcal{L}\ \forall x, y \in L\ \exists g \in G.\ xg = y \ \&\ Lg = L$.

<u>10.2 E.</u> The Euclidian plane is a geometry as follows. $X =_{df} \mathbb{R}^2$. Think of $X$ equipped with the usual Euclidian metric $d$, $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and define $G =_{df}$ isometry group of $(X, d)$. $G$ acts on $X$ by evaluation, $X \times G \xrightarrow{ev} X$. $\mathcal{L} =_{df}$ set of all straight lines. Then $(X, G, ev, \mathcal{L})$ is a geometry, called <u>Eulidian plane geometry</u>. The proof is as follows:

G1 and G2 are clear since two points determine a line.

To prove G5, $g = 1_X$ works if $x = y$; otherwise consider a vector based at $x$ pointing toward $y$ of length $d(x, y)$. This vector induces a translation $g$ leaving $L$ invariant and taking $x$ to $y$.

To prove G4, $g = 1_X$ works if $L = L'$. If $L \parallel L'$ let $M$ be the line parallel to both and midway between them. Otherwise, $L, L'$ meet and form a pair of vertical angles so let $M$ be the angle bisector. "Reflection in the mirror $M$" is a $g$ such that $Lg = L'$ (and, in fact, $L'g = L$).

Finally, we prove G3. If $A \subset \mathbb{R}^2$, say that $A$ is <u>collinear</u> if $\{x, y, z\} \subset A \Rightarrow x, y, z$ are collinear. Observe that a line is characterized by being a maximal collinear set. If $A$ is collinear and if $g \in G$ then $Ag$ is collinear because $x, y, z$ are collinear iff $x, y, z$ can be ordered (say as $(x, y, z)$) such that $(x, y)d + (y, z)d = (x, z)d$, and $g$ is an isometry. In particular, if $L \in \mathcal{L}$, $g \in G$ then $Lg$ is collinear. Suppose $Lg \subset A$ and $A$ is collinear. Then $L = Lgg^{-1} \subset Ag^{-1}$ and $Ag^{-1}$ is collinear. As $L$ is a line, $L = Ag^{-1}$. $\therefore Lg = Ag^{-1}g = A$. $\therefore Lg$ is a maximal collinear subset, i.e. $Lg \in \mathcal{L}$. $\square$

<u>10.3 Example.</u> Let $X$ be the surface of a 3-dimensional sphere. Let $d$ be the "great-circle" metric on $X$. $G$ = df isometry group of $(X,d)$, so $(X, G, ev)$ is a group action. $\mathcal{L}$ = df the set of great circles. Then $(X, G, ev, \mathcal{L})$ is a geometry.

G1 and G2 are clear, although if $x, y$ are antipodal points there are uncountably-many distinct lines through $x, y$.

G3 is proved by exactly the same argument used in 10.2.

If $L \neq L' \in \mathcal{L}$, $L \cap L' = \{x, y\}$ where $x, y$ are antipodal points. There is a rotation with axis $\overline{xy}$ which carries $L$ into $L'$, proving G4.

For G5, let $L \in \mathcal{L}$. Any rotation with axis perpendicular to the plane of $L$ leaves $L$ invariant, and if $x, y \in L$ rotate through $\measuredangle xoy$ ($o$ is the center) to take $x$ to $y$. $\square$

**10.4 D.** A geometry homomorphism

$$(X, G, \Pi, \mathcal{L}) \xrightarrow{(f, \varphi, \alpha)} (X', G', \Pi', \mathcal{L}')$$

from the geometry $(X, G, \Pi, \mathcal{L})$ to the geometry $(X', G', \Pi', \mathcal{L}')$ is a triplet $(f, \varphi, \alpha)$ such that $(X, G, \Pi) \xrightarrow{(f, \varphi)} (X', G', \Pi')$ is an action homomorphism and $\mathcal{L} \xrightarrow{\alpha} \mathcal{L}'$ is a function such that $\forall L \in \mathcal{L}.\ Lf \subset L\alpha$.

A subgeometry is a geometry homomorphism $(f, \varphi, \alpha)$ such that $f, \varphi, \alpha$ are 1-1 (into).

**10.5 T.** $(X, G, \Pi, \mathcal{L}) \xrightarrow{(1_X, 1_G, 1_{\mathcal{L}})} (X, G, \Pi, \mathcal{L})$ is a geometry homomorphism. If

$$(X, G, \Pi, \mathcal{L}) \xrightarrow{(f, \varphi, \alpha)} (X', G', \Pi', \mathcal{L}'), \quad (X', G', \Pi', \mathcal{L}') \xrightarrow{(f', \varphi', \alpha')} (X'', G'', \Pi'', \mathcal{L}'')$$

are geometry homomorphisms, so is

$$(f, \varphi, \alpha) \circ (f', \varphi', \alpha') =_{df} (ff', \varphi\varphi', \alpha\alpha').$$

$(f, \varphi, \alpha)$ has an inverse iff $f, \varphi, \alpha$ are all bijective; in either case, $(f, \varphi, \alpha)$ is called a geometry isomorphism. $\square$

Let $(X, G, \pi, \mathcal{L})$ be a geometry.

**10.6 D.** $\forall L \in \mathcal{L}.\ G_L =_{df} [g \in G : Lg = L]$

**10.7 Prop.** The following are valid.

a. $\forall L \in \mathcal{L}.\ G_L \leq G.$

b. $\forall L \in \mathcal{L}\ \forall g \in G.\ G_{Lg} = g^{-1} G_L g$

**Proof.**

a. Let $g, g' \in G_L$. $Lgg' = (Lg)g' = Lg' = L$.

$Lg^{-1} = (Lg)g^{-1} = Le = L$.

b. Let $g' \in G_{Lg}$. $\therefore Lgg'g^{-1} = Lgg^{-1} = L$ proving $gg'g^{-1} \in G_L$, so $g' \in g^{-1}G_L g$. So far $G_{Lg} \subset g^{-1} G_L g$.

Now let $g' \in G_L$. $(Lg)(g^{-1}g'g) = (Lg')g = Lg$;

hence $g^{-1}G_L g \subset G_{Lg}$. $\square$

**10.8 Prop.** Let $L \in \mathcal{L}$. Then, defining $\pi_0 =_{df} \pi|_{L \times G_L}$, $(L, G_L, \pi_0, \{L\})$ is a subgeometry of $(X, G, \pi, \mathcal{L})$.

**Proof.** $L$ is $G_L$-invariant by the definition of $G_L$, so by the theorem on p. 9.1, $(L, G_L, \pi_0)$ is a group action. G1, G2, G3, G4 are obvious. To prove G5, let $x, y \in L$. By G5 for $(X, G, \pi, \mathcal{L})$ $\exists g \in G.\ xg = y$ and $Lg = L$. But then $g \in G_L$. It is clear that $(inc_L, inc_{G_L}, inc_{\{L\}})$ is a subgeometry. $\square$

141 SEMINAR

Wednesday, Nov 1, 3:15, TG 202

1. (Bellenot) $X =_{df} \mathbb{R}^2$, $G =_{df}$ bij $X$. $\forall a, b \in \mathbb{R}^2$, $[a,b] =_{df}$ line segment joining $a$ and $b$ including endpoints. $\mathcal{L} =_{df} \{[a,b] : a \neq b,\ a, b \in \mathbb{R}^2\}$. Show that there exists no subset $\mathfrak{M}$ of $\mathcal{L}$ such that $(X, G, ev, \mathfrak{M})$ is a geometry.

2. (George) Construct a geometry on $\mathbb{Z}$ in which all lines are finite.

3. (Greitzer) Let $r, n$ be integers, $1 < r \leq n$. $X =_{df} \{1, \dots, n\}$, $G =_{df}$ bij $X$. $\mathcal{L} =_{df}$ $\{A \subset X : A$ has $r$ elements$\}$.

   a. Show $(X, G, ev, \mathcal{L})$ is a geometry.

   $\binom{n-2}{r-2}$ b. How many lines pass through two distinct points?

   c. Through a point $x$ not on a line $L$, how many lines $L'$ are there through $x$ parallel to (that is not intersecting) $L$?

$n \leq r$ 0

$\frac{n}{2}$

$r \leq \frac{n}{2}$

$\binom{x-r-1}{r-1}$

4. (Gulik). $X =_{df} \mathbb{R}^3$, $G =_{df}$ isometry group, $\mathcal{L} =_{df}$ set of all planes. Show that $(X, G, ev, \mathcal{L})$ is a geometry. G3

P is equidistant from two pts

5. (Hamilton) Find all non-isomorphic 3-element geometries with effective group action.

6. (Malcolm). Let $(X, \leq)$ be an infinite (reflexive, ... ) partially ordered set. $G =_{df}$ set of order-preserving (isomorphisms) bijections. $\mathcal{L} =_{df}$ set of all inclusion-maximal linearly ordered subsets of $X$.

$\{A \subset X\}$ $\forall x, y \in A$ $xRy \vee yRx \wedge A$ is inclusion maximal with resp to the 1st prop.

a. Prove $G \leq bij\, X$.

b. Which of G1, ..., G5 does $(X, G, ev, \mathcal{L})$ always satisfy? G1, G3

c. Give enough examples to show that the remaining axioms may or may not be satisfied. G5, G2, G4 may

7. (Ray) Using 10.2 as a hint, define <u>Euclidian</u> <u>line</u> geometry on $\mathbb{R}$, and prove it is a geometry. How does this compare with the 1-line subgeometries (as in 10.8) of the Euclidian plane geometry?

# Lecture 11

**11.1 T&D.** Let $g \in G$. The map $(f, \varphi, \alpha)$ defined by $f =_{df} \begin{array}{c} X \xrightarrow{\pi^g} X \\ x \longmapsto xg \end{array}$, $\varphi =_{df} \begin{array}{c} G \xrightarrow{g^{-1} \circ \circ g} G \\ h \longmapsto g^{-1}hg \end{array}$,

$\alpha =_{df} \begin{array}{c} \mathcal{L} \xrightarrow{\alpha} \mathcal{L} \\ L \longmapsto Lg \end{array}$ is an automorphism (meaning geometry isomorphism) of $(X, G, \pi, \mathcal{L})$ with itself. It is called the inner automorphism induced on $(X, G, \pi, \mathcal{L})$ by $g$.

**Proof.** $\pi^g$ is bijective (its inverse is $\pi^{g^{-1}}$). By 6.7, $\varphi$ is a group automorphism; $\alpha$ is well-defined by G3 and is bijective because $\pi^g$ is. $\forall L \in \mathcal{L}$ we have $L\pi^g = L\alpha$ and in particular $L\pi^g \subset L\alpha$. To complete the proof, we observe $(\pi^g, \varphi)$ is an action homomorphism:

$$\begin{array}{ccc} X \times G & \xrightarrow{\pi^g \times \varphi} & X \times G \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow{\pi^g} & X \end{array} \qquad \begin{array}{ccc} (x, g') & \longmapsto & (xg, g^{-1}g'g) \\ \big\downarrow & & \big\downarrow \\ & & xgg^{-1}g'g \\ & & \| \\ xg' & \longmapsto & xg'g \end{array} \qquad \square$$

**11.2 T.** Let $L_1, L_2 \in \mathcal{L}$. Consider $L_1, L_2$ as subgeometries $(L_1, G_{L_1}, \pi_1, \{L_1\})$, $(L_2, G_{L_2}, \pi_2, \{L_2\})$ as in 10.5. Then there exists an inner automorphism on $(X, G, \pi, \mathcal{L})$ which, by restriction, maps $(L_1, G_{L_1}, \pi_1, \{L_1\})$ isomorphicly onto $(L_2, G_{L_2}, \pi_2, \{L_2\})$. In short, all lines in a geometry are isomorphic as geometries.

**Proof.** By G4 $\exists\, g \in G$. $L_1 g = L_2$. Let $(f, \varphi, \alpha)$ be the inner automorphism induced on $(X, G, \pi, \mathcal{L})$ by $g$. $\pi^g$ maps $L_1$ onto $L_2$. $\alpha$ maps $\{L_1\}$ onto $\{L_2\}$. As $G_{L_2} = g^{-1} G_{L_1} g$ (by 10.7 b), $\varphi$ maps $G_{L_1}$ onto $G_{L_2}$. $\square$

11.2 says that all lines in a geometry are, in some sense, playing the same rôle. Observe further that if $L_0 \in \mathcal{L}$ then $L_0$ "generates" $\mathcal{L}$ in the sense that, by G3 and G4, $\mathcal{L} = [L_0 g : g \in G]$. This suggests that a geometry could be defined by specifying a group action $(X, G, \pi)$ and a subset $L_0 \subset X$ to serve as a "typical" line. This is in fact the case; we begin by axiomitizing $L_0$ as follows:

<u>11.3 D.</u> Let $(X, G, \Pi)$ be a group action and let $L_0 \subset X$. $L_0$ is a <u>base line for</u> $(X, G, \Pi)$ if $L_0$ satisfies the following three axioms:

BL1. $\forall g \in G.\ L_0 g \subset L_0 \Rightarrow L_0 g = L_0$.
BL2. $\forall x, y \in X\ \exists g \in G.\ \{x, y\} \subset L_0 g$.
BL3. $\forall x, y \in L_0\ \exists g \in G.\ xg = y\ \&\ L_0 g \subset L_0$.

The following theorem says that, to all intents and purposes, a geometry is a group action with base line.

<u>11.4 T.</u> Let $(X, G, \Pi)$ be a group action and let $\emptyset \neq \mathcal{L} \subset 2^X$. The following statements are equivalent.

a. $(X, G, \Pi, \mathcal{L})$ is a geometry.
b. $\forall L \in \mathcal{L}.\ L$ is a base line for $(X, G, \Pi)$ and $\mathcal{L} = [Lg : g \in G]$.
c. $\exists L_0 \in \mathcal{L}.\ L_0$ is a base line for $(X, G, \Pi)$ and $\mathcal{L} = [L_0 g : g \in G]$.

<u>Proof.</u> <u>$a \Rightarrow b$.</u> Let $L \in \mathcal{L}$. By G3, G4, $\mathcal{L} = [Lg : g \in G]$. BL1 follows from G3 and G1. BL2 is clear from G2 and G4. BL3 is immediate from G5.

<u>$b \Rightarrow c$.</u> obvious, since $\mathcal{L} \neq \emptyset$.

<u>$c \Rightarrow a$.</u> We must prove G1, ..., G5.

G1. If $L_0 g \subset L_0 h$, $L_0 g h^{-1} \subset L_0$. By BL1, $L_0 g h^{-1} = L_0$ so $L_0 g = L_0 h$.

G2. Immediate from BL2.

G3. $(L_0 g) h = L_0 (gh)$.

G4. If $L_0 g, L_0 h \in \mathcal{L}$, $(L_0 g)(g^{-1} h) = L_0 h$.

G5. Let $x, y \in L_0 g$. $\therefore$ $x g^{-1}, y g^{-1} \in L_0$. By BL3 $\exists h \in G$. $x g^{-1} h = y g^{-1}$ & $L_0 h \subset L_0$. By BL1, $L_0 h = L_0$. $\therefore$ $x(g^{-1} h g) = y g^{-1} g = y$ and $(L_0 g)(g^{-1} h g) = (L_0 h) g = L_0 g$. $\square$

11.4 is used to construct geometries as follows. Let $(X, G, \pi)$ be a group action, and let $L_0 \subset X$ be a base line. Define $\mathcal{L} =_{df} [L_0 g : g \in G]$. By 11.4 ($c \Rightarrow a$), $(X, G, \pi, \mathcal{L})$ is a geometry. This replaces the five axioms G1, ..., G5 about a bunch of subsets of $X$ with the three axioms BL1, BL2, BL3 about a single subset of $X$, a considerable simplification.

**11.5 Lemma.** Let $X$ be a topological space, let $X \xrightarrow{g} X$ be a homeomorphism of $X$ onto itself and let $L_0 \subset X$ be a subspace with the following properties:

a. $L_0$ is closed in $X$.

b. $L_0$ is homeomorphic to $\mathbb{R}$.

c. $L_0 g \subset L_0$.

Then $L_0 g = L_0$.

**Proof.** Let $x \in L_0 g$, $y \in L_0 - L_0 g$. We will find a contradiction to the existence of $y$. There exists a homeomorphism $L_0 \xrightarrow{f} \mathbb{R}$ with $xf = 0$ and $yf = 1$, so we assume that $L_0$ is "coordinatized" with $x = 0$, $y = 1$. Since $[t \in L_0 : t < 1]$, $[t \in L_0 : t > 1]$ are disjoint open subsets of $L_0$ whose union contains $L_0 g$, and since $L_0$ connected $\Rightarrow L_0 g$ connected, $[t \in L_0 : t > 1] \cap L_0 g = \phi$.

Now suppose $t \in L_0 g$ and $u < t$. Claim $u \in L_0 g$. Suppose not. Then, arguing just as we did above, $[v \in L_0 g : v < u] = \phi$. $\therefore L_0 g \subset [u, 1]$. $\therefore L_0 g$ is bounded and closed (since $L_0$ is closed) and hence compact. But $L_0$ is homeomorphic to $L_0 g$ and $L_0$ is not compact, $\times$.

$t_0 =_{df} \inf [t \in L_0 : t \notin L_0 g]$. Since $(-\infty, 0] \subset L_0 g$, $t_0$ is well-defined, indeed $t_0 \in [0, 1]$. Clearly $\exists$ a sequence $(u_n)$ in $L_0 g$ converging to $t_0$. As $L_0 g$ is closed, $t_0 \in L_0 g$. It follows that $L_0 g = (-\infty, t_0]$.

$\exists\, u \in L_0 \cdot ug = t_0$. $L_0 \neg \{u\} = [v : v < u]$ $\cup\, [v : v > u]$ is the union of two disjoint connected open subsets. Since $g$ is a homeomorphism, it must be the case that $L_0 g \neg \{ug\} = L_0 g \neg \{t_0\}$ $= (-\infty, t_0)$ is the union of two disjoint open subsets, which is clearly a contradiction. $\square$

__11.6 T & D.__ Let $X =_{df} \mathbb{R}^2$, $G =_{df}$ the group of homeomorphisms of $X$ onto $X$, $L_0 =_{df}$ the abcissa-axis, $\mathcal{L} =_{df} \{L_0 g : g \in G\}$. Then $(X, G, ev, \mathcal{L})$ is a geometry. It is called the __topological__ __Euclidian__ __plane__.

__Proof.__ $(X, G, ev)$ is clearly a group action. We show $L_0$ satisfies BL1, BL2, BL3.

__BL1.__ This is immediate from 11.5.

__BL2.__ By using translations and rotations for $g$'s, it is clear that all ordinary straight lines are in $\mathcal{L}$. So use them.

__BL3.__ This follows from G5 in 10.2. $\square$

**11.7 T.** Let $(X, G, \pi, \mathcal{L})$ be a geometry. Let $X \xrightarrow{f} \bar{X}$ be any bijection of $X$ onto another set $\bar{X}$. Let $G \xrightarrow{\varphi} \bar{G}$ be any isomorphism of $G$ with another group $\bar{G}$.

Define $\bar{X} \times \bar{G} \xrightarrow{\bar{\pi}} \bar{X}$

$$\bar{x}, \bar{g} \longmapsto \bar{x}\bar{g} =_{df} [(\bar{x}f^{-1})(\bar{g}\varphi^{-1})]f.$$

Define $\bar{\mathcal{L}} =_{df} \mathcal{L}f = [Lf : L \in \mathcal{L}]$.

Then $(\bar{X}, \bar{G}, \bar{\pi})$ is a group action, and $(f, \varphi)$ is an action isomorphism; $(\bar{X}, \bar{G}, \bar{\pi}, \bar{\mathcal{L}})$ is a geometry and $(f, \varphi, \alpha)$ is a geometry isomorphism where $\forall L \in \mathcal{L}.\ L\alpha =_{df} Lf$.

Proof. $\bar{x}\bar{e} = [(\bar{x}f^{-1})(\bar{e}\varphi^{-1})]f = [(\bar{x}f^{-1})e]f = \bar{x}f^{-1}f = \bar{x}$

$$(\bar{x}\bar{g})\bar{h} = \{[(\bar{x}f^{-1})(\bar{g}\varphi^{-1})]f\}\bar{h} = [[(\bar{x}f^{-1})(\bar{g}\varphi^{-1})](\bar{h}\varphi^{-1})]f$$

$$= [(\bar{x}f^{-1})[(\bar{g}\varphi^{-1})(\bar{h}\varphi^{-1})]]f = [(\bar{x}f^{-1})((\bar{g}\bar{h})\varphi^{-1})]f = \bar{x}(\bar{g}\bar{h})$$

proves $(\bar{X}, \bar{G}, \bar{\pi})$ is a group action. We have

$$\begin{tikzcd} X \times G \arrow[r, "f \times \varphi"] \arrow[d, "\pi"] & \bar{X} \times \bar{G} \arrow[d, "\bar{\pi}"] \\ X \arrow[r, "f"] & \bar{X} \end{tikzcd} \qquad \begin{tikzcd} x, g \arrow[r, mapsto] \arrow[dd, mapsto] & xf, g\varphi \arrow[d, mapsto] \\ & ((xff^{-1})(g\varphi\varphi^{-1}))f \\ xg \arrow[r, mapsto] & (xg)f \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

and $f, \varphi$ are bijective so $(f, \varphi)$ is an action isomorphism. The first statement is proved.

Now let $L_0 \in \mathcal{L}$. Claim $L_0 f$ is a base line for $(\bar{X}, \bar{G}, \bar{\pi})$.

<u>BL1.</u> Observe that we have already proved that $\forall x \in X\ \forall g \in G$, $(xf)(g\varphi) = (xg)f$.

Suppose $(L_0 f)\bar{g} \subset L_0 f$. $\therefore L_0 = L_0 f f^{-1} \supset$ $[(L_0 f)\bar{g}]f^{-1} = (L_0 f f^{-1})(\bar{g}\varphi^{-1}) = L_0(\bar{g}\varphi^{-1})$. $\therefore L_0 = L_0(\bar{g}\varphi^{-1})$. It follows quickly, $(L_0 f)\bar{g} = L_0 f$.

<u>BL2.</u> Let $\bar{x}, \bar{y} \in \bar{X}$. $\exists g \in G$, $\{\bar{x}f^{-1}, \bar{y}f^{-1}\} \subset L_0 g$. It follows quickly that $\{\bar{x}, \bar{y}\} \subset (L_0 f)(g\varphi)$.

<u>BL3.</u> Let $\bar{x}, \bar{y} \in L_0 f$. $\exists g \in G$, $\bar{x}f^{-1}g = \bar{y}f^{-1}$ and $L_0 g \subset L_0$. It follows quickly that $(L_0 f)(g\varphi) \subset L_0 f$ and $\bar{x}(g\varphi) = \bar{y}$.

$\therefore L_0 f$ is a base line. But $\bar{\mathcal{L}} = [Lf : L \in \mathcal{L}]$ $= [(L_0 g)f : g \in G] = [(L_0 f)(g\varphi) : g \in G] = [(L_0 f)\bar{g} : \bar{g} \in \bar{G}]$.

By 11.4, $(\bar{X}, G, \bar{\pi}, \bar{\mathcal{L}})$ is a geometry.

As $Lf = L\alpha$, $(f, 1_G, \alpha)$ is a geometry isomorphism. $\square$

Geometry 141 Seminar

Wednesday, Nov. 8, 4:15 pm

1. (Ray) Prepare a short exposition: "continuous maps between metric spaces" to suit the needs of the other problems below. Include

   a. a discussion of when two metrics on a set induce the same topology, and

   b. the theorem: if $(X,d) \xrightarrow{f} (\bar{X},\bar{d})$ such that $\forall x,y \,.\, d(x,y) \geq \bar{d}(xf, yf)$ then $f$ is continuous; in particular, if $f$ is an isometry, $f$ is a homeomorphism.

2. (Gulik) Show that Euclidean plane geometry is a subgeometry of the topological Euclidean plane.

3. (Bellenot) Let $(X,d)$ be a metric space, and let $X \rightarrowtail\!\!\!\xrightarrow{f}\!\!\!\twoheadrightarrow \bar{X}$ be a bijection of $X$ onto another set $\bar{X}$.

   a. Define $\bar{X} \times \bar{X} \xrightarrow{\bar{d}} \mathbb{R}$ by $(\bar{x}, \bar{y})\bar{d} =_{df} (\bar{x}f^{-1}, \bar{y}f^{-1})d$. Show that $(\bar{X},\bar{d})$ is a metric space and that $f$ is an isometry.

   b. Let $G, \bar{G}$ be the isometry groups of $(X,d)$, $(\bar{X},\bar{d})$. Show that

$$G \xrightarrow{\varphi} \bar{G}$$
$$X \xrightarrow{g} X \mapsto \bar{X} \xrightarrow{f^{-1}} X \xrightarrow{g} X \xrightarrow{f} \bar{X}$$

is a group isomorphism and that $(X, G, ev) \xrightarrow{(f, \varphi)} (\bar{X}, \bar{G}, \bar{ev})$ is an action isomorphism.

4. (George) Let $(X, d)$ be a metric space. Define $X \times X \xrightarrow{e} \mathbb{R}$ by $e(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$.

   a. Show $(X, e)$ is a metric space.
   b. Show that $(X, d)$, $(X, e)$ have the same topology.
   c. Show that the isometry group of $(X, d)$ is a subgroup of the isometry group of $(X, e)$

5. (Greitzer) Let $(X, d)$, $(\bar{X}, \bar{d})$ be metric spaces. Define $(X \times \bar{X}) \times (X \times \bar{X}) \xrightarrow{e} X \times \bar{X}$ by $e[(x,\bar{x}),(y,\bar{y})] =_{df} d(x,y) + \bar{d}(\bar{x},\bar{y})$.

   a. Show $(X \times \bar{X}, e)$ is a metric space.
   b. Show that the topology of $(X \times \bar{X}, e)$ is the product topology of $(X, d)$, $(\bar{X}, \bar{d})$.

6. (Hamilton) Let $(X, d) = (\bar{X}, \bar{d}) = \mathbb{R}$ with usual metric. Metrize $\mathbb{R}^2$ as in #5. Let $\mathcal{L} =_{df}$ set of all usual straight lines. Let $G$ be the isometry group. Is $(X, G, ev, \mathcal{L})$ a geometry? If so, is it isomorphic to Euclidian plane geometry?

7. (Malcolm). Let $(\mathbb{R}^2, d)$ be the usual metric plane. Consider the metric space $(\mathbb{R}^2, e)$ where $e = \frac{d}{d+1}$ as in #4. Say that three points are <u>collinear</u> if $e(x,y) + e(y,z) = e(x,z)$ for some ordering $(x,y,z)$ of these three points. If $A \subset \mathbb{R}^2$, say that $A$ is <u>collinear</u> if $A$ has at least 3 points and if every three points in $A$ are collinear. Define a <u>line</u> to be a maximal collinear subset. Let $G$ be the isometry group of $(\mathbb{R}^2, e)$. Let $\mathcal{L}$ be the set of all lines.

Is $(X, G, \Pi, \mathcal{L})$ a geometry, and, if so, is it isomorphic to Euclidean plane geometry?

# Lecture 12

We begin our study of affine plane geometry.

**12.1 Def.** A *preaffine plane* is a pair $(X, \mathcal{L})$ such that $X$ is a set (of *points*) and $\mathcal{L}$ is a collection of subsets of $X$ (called the set of *lines*) subject to the following three axioms.

AF1. If $x, y \in X$ with $x \neq y$ $\exists$ unique $L \in \mathcal{L}$ such that $x, y \in L$.

If $L, L' \in \mathcal{L}$, $L$ is parallel to $L'$, $=_{dn}$ $L \parallel L'$, $=_{df}$ $L = L'$ or $L \cap L' = \emptyset$.

AF2. $\forall x \in X$ $\forall L \in \mathcal{L}$ $\exists$ unique $L' \in \mathcal{L} \ni x \in L'$ & $L' \parallel L$.

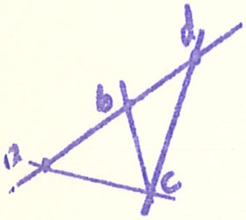
AF3. $\exists$ $x, y, z \in X$ such that $x \neq y \neq z \neq x$ & $\forall L \in \mathcal{L}$, $\{x, y, z\} \not\subset L$.

Notation: if $x \neq y \in X$ the unique line through $x$ and $y$ $=_{dn}$ $x/y$.

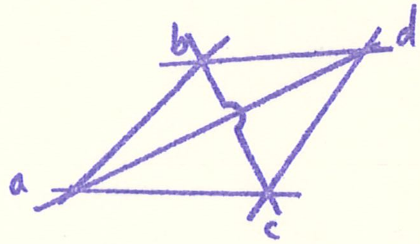
AF1 says that two points determine a line; AF2 says that through a given point there is a line parallel to a given line; AF3 says that there exist 3 non-collinear points.

**12.2 Example.** The Euclidean plane with the usual straight lines is a preaffine plane. Euclidean 3-space with the usual lines is not (AF2 fails). $(\mathbb{R}, \{\mathbb{R}\})$ is not because AF3 fails.

**12.3 Example.** There exists a unique preaffine plane with four points. By AF3, start with three non-collinear points $a, b, c$. To adjoin a fourth point $d$, the following two choices are exhaustive, in view of AF1:




choice I. 3 points lie on a line




choice II. no 3 points lie on a line

Choice I is impossible as no line through $c$ is parallel to $ab$. That choice II is a preaffine plane is easily verified by inspection.

**12.4 Prop.** "$\parallel$" is an equivalence relation on $\mathcal{L}$.

**Proof.** Reflexive and symmetric are clear. Now suppose $L_1 \parallel L_2$, $L_2 \parallel L_3$. If $L_1 \cap L_3 = \emptyset$ then $L_1 \parallel L_3$. Otherwise $\exists\, x \in L_1 \cap L_3$. But then $L_1, L_3$ are lines through $x$ each parallel to $L_2$ so that by AF2 $L_1 = L_3$. $\square$

**12.5 Proposition.** The following statements are valid.

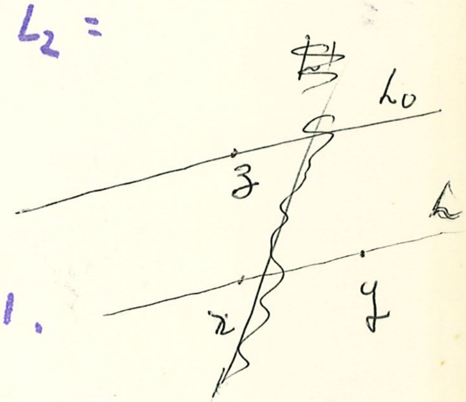
(a) crd $X \geq 4$.

(b) If $L \neq L' \in \mathcal{L}$. crd $(L \cap L') \leq 1$.

(c) If $L_1, L_2 \in \mathcal{L}$ $\exists L \in \mathcal{L}$. $L_1 \nparallel L \nparallel L_2$.

(d) If $L \in \mathcal{L}$ $\exists L_1, L_2 \in \mathcal{L}$. $L_1 \nparallel L \nparallel L_2$.

(e) If $L_1, L_2, L_3 \in \mathcal{L}$ then crd $L_1$ = crd $L_2$ = crd $\{L \in \mathcal{L} : L \parallel L_3\}$.




(f) $\forall L \in \mathcal{L}$. crd $L \geq 2$.

(g) If $L, L' \in \mathcal{L}$, $L \nparallel L'$ iff crd $L \cap L' = 1$.

(h) $\forall x, y \in X$. crd $[L \in \mathcal{L} : x \in L]$ = crd $[L \in \mathcal{L} : y \in L]$.

(i) If $\alpha$ is the number of points on a line (always the same by (e)) and if $\beta$ is the number of lines through a point (always the same by (h)) then $\alpha \leq \beta$ and, if $X$ is finite, $\alpha < \beta$.

**Proof.**

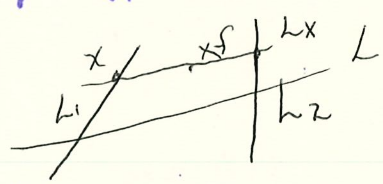
(a) By AF3, crd $X \geq 3$. If crd $X = 3$, by AF3 and AF1 the only possible choice is

which violates AF2. $\therefore$ crd $X \geq 4$. (that crd $X = 4$ is possible was shown in 12.3).

<u>(b)</u>. clear from AF1.

<u>(c)</u>. By AF3 $\exists$ $x, y, z$ non-collinear. Clearly no two of $x/y$, $y/z$, $x/z$ are parallel. Given $L_1, L_2 \in \mathcal{L}$, it follows from 12.4 that at least one of $x/y$, $y/z$, $x/z$ is parallel to neither of $L_1, L_2$.

<u>(d)</u>. Let $x, y, z$ be as in (c). Given $L \in \mathcal{L}$, it follows from 12.4 that $L$ is parallel to at most one of —hence non-parallel with at least two of— $x/y$, $y/z$, $x/z$.




(e). Let $L_1, L_2, L_3 \in \mathcal{L}$. First we will define a function $L_1 \xrightarrow{f} L_2$ and prove that $f$ is bijective. $\forall x \in L_1$ $\exists$ unique $L_x \in \mathcal{L}$ with $x \in L_x \parallel L$ where $L \in \mathcal{L}$ such that $L_1 \nparallel L \nparallel L_2$ (such $L$ exists by (d)). So for $x \in L_1$, define $xf =_{df} L_2 \cap L_x$ (noting that $\mathrm{crd}(L_2 \cap L_x) = 1$ since $L_2 \nparallel L_x$).

$f$ is 1-to-1.

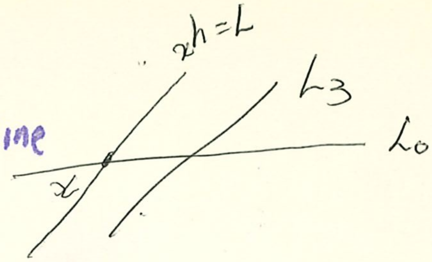
suppose $x, y \in L_1$, $xf = yf$. Define $\tilde{L}$ by $xf \in \tilde{L} \parallel L$. Since $xf \in L_x \parallel L$, $L_x = \tilde{L}$. Similarly $L_y = \tilde{L}$ because $xf = yf$. $\therefore$ $x = L_1 \cap L_x = L_1 \cap \tilde{L} = L_1 \cap L_y = y$.

$f$ is onto

Let $y \in L_2$. Define $\tilde{L}$ by $y \in \tilde{L} \parallel L$, $x =_{df} L_1 \cap \tilde{L}$. Since $x \in \tilde{L} \parallel L$, $\tilde{L} = L_x$ so $xf = L_x \cap L_2 = \tilde{L} \cap L_2 = y$.

This shows $\cancel{\mathrm{crd}\, L_1}$ crd $L_2 =$ crd $L_1$

$\exists\, L_0 \in \mathcal{L} .\ L_0 \nparallel L_3$ by (d). Define




$$L_0 \xrightarrow{\ h\ } [L \in \mathcal{L} : L \parallel L_3]$$

$$x \longmapsto \text{unique } L \in \mathcal{L} \text{ such that } x \in L \parallel L_3 .$$

$h$ is 1-to-1.

if $xh = yh$ then $x, y \in xh \cap L_0$. As $L_0 \nparallel xh$,
$\therefore\ x = y$.

$h$ is onto.

if $L \parallel L_3$ then $L \nparallel L_0$. $\therefore$ define $x =_{df} L \cap L_0$.
As $x \in L \parallel L_3$, $L = xh$.

The proof of (e) is complete.

(f) By AF1 and AF3 $\exists\, L \in \mathcal{L} .$ crd $L \geq 2$. It follows from (e) that crd $L \geq 2$ $\forall\, L \in \mathcal{L}$.

(g) Let $L, L' \in \mathcal{L}$. Clearly $L \nparallel L' \Rightarrow$ crd $(L \cap L') = 1$. Conversely, suppose crd $L \cap L' = 1$. $\therefore L \cap L' \neq \emptyset$. Suppose $L = L'$. $\therefore$ crd $L = 1$ which contradicts (f). $\therefore L \neq L'$ and $L \cap L' \neq \emptyset$, i.e. $L \nparallel L'$. $\square$

(h), (i) are relegated to the seminar.

141 Seminar

wednesday, Nov. 15, 4:15 pm, TG 202

1. (Bellenot) Prove 12.5 (h).

2. (George) Prove that if $(X, \mathcal{L})$ is a preaffine plane with $X$ infinite then the number of points on a line is infinite.

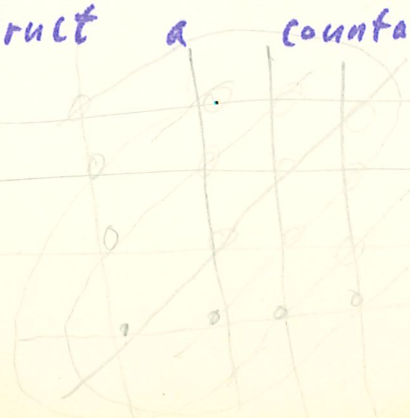
3. (Greitzer) Prove a generalization of 12.5 (i), namely $\alpha + 1 = \beta$.

4. (Gulik) Let $(X, \mathcal{L})$ be a preaffine plane with $X$ a finite set of $n$ elements. Prove that $\exists$ an integer $p$ with $p^2 = n$.

5. (Hamilton). Let $p$ be a prime. Construct a preaffine plane with $p^2$ elements.

6. (Malcolm) Let $(X, \mathcal{L})$ be a preaffine plane with $X$ a finite set of $n$ elements. Decide conclusively whether or not $\sqrt{n}$ is prime.

7. (Ray) Construct a countably infinite preaffine plane.



$\binom{p^2}{p} = \frac{p^2!}{p!\,(p^2-p)!}$

$p^2+p$

$\frac{16!}{4!\,12!}$

20

2100

# Lecture 13

Let $(X, \mathcal{L})$ be a preaffine plane.

## 13.1 Definition.

$G =_{df} [g \in bij(X) : \forall L \in \mathcal{L}.\ Lg \in \mathcal{L}]$

$D =_{df} [d \in G : \forall L \in \mathcal{L}.\ Ld \parallel L]$

$T =_{df} [t \in D : \forall x \in X.\ xt \neq x] \cup \{1_X\}.$

Elements of $G$ are called <u>collineations</u>.

Elements of $D$ are called <u>dilatations</u>.

Elements of $T$ are called <u>translations</u>.

**13.2 Proposition.** The following statements are valid.

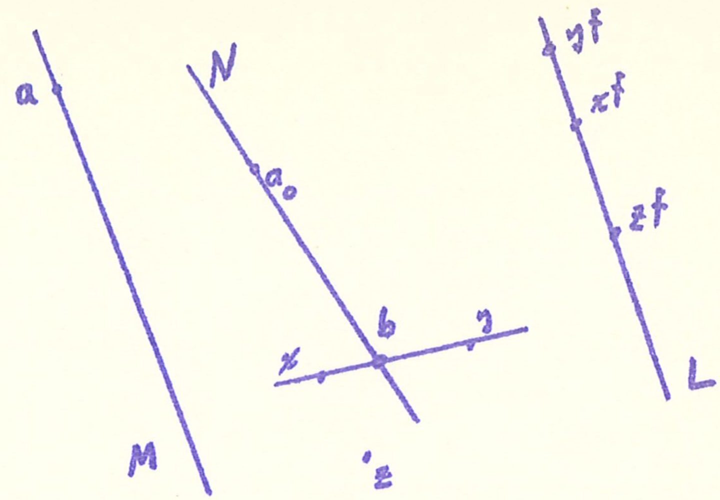
a. Collineations map noncollinear points to noncollinear points.

b. Let $f \in bij(X)$ be such that whenever $x \neq y$ then $(x/y)f \subset xf/yf$. Then $f \in G$.

c. $G$ is a subgroup of $bij(X)$.

**Proof.** Let $f$ be as in (b). We begin by showing that $f$ maps noncollinear points into noncollinear points.

Let $x, y, z$ be three noncollinear points. We must show that $xf, yf, zf$ are noncollinear. So suppose $xf, yf, zf$ are collinear. Then $\exists\, L \in \mathcal{L}$ with $xf, yf, zf \in L$.




By 12.5 (d) $\exists\, a \in X$ with $a \notin L$. Define $M \in \mathcal{L}$ by $a \in M \parallel L$. As $f$ is onto $\exists\, a_0 \in X$ with $a_0 f = a$. Let $N$ be any line through $a_0$. Since $x/y$, $y/z$ are nonparallel, $N$ intersects one of these lines at a point $b$. Since $a_0 f = a \notin L$ and $bf \in L$, $Nf \subset a/bf \nparallel M$ and $Nf \cap M = \{a\}$. By 12.5 (f) $\exists\, c \in M$. $c \neq a$. $\exists\, c_0 \in X$. $c_0 f = c$. As $f$ is 1-to-1, $c_0 \neq a_0$. If $N =_{df} a_0/c_0$, $N$ is a line through $a_0$, and $c \in Nf \cap M$, $\times$

__Proof of a.__ This follows from what we just proved since surely any collineation satisfies the property of $f$ in (b).

__Proof of b.__ Let $f$ be as in (b). Suppose $x \neq y \in X$. We have $(x/y)f \subset xf/yf$ and have only to show $(x/y)f = xf/yf$. Let $\tilde{a} \in xf/yf$. $\exists\, a \in X$. $af = \tilde{a}$. Since $xf, yf, \tilde{a}$ are collinear we have $x, y, a$ are collinear (by what we proved above) so that $a \in x/y$ and $\tilde{a} \in (x/y)f$.

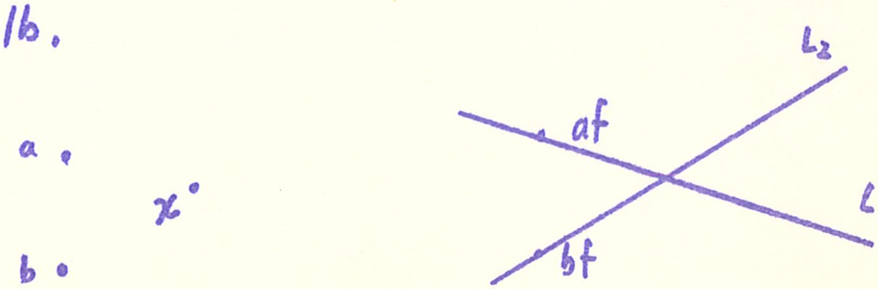
<u>Proof of c.</u> If $g, h \in G$ and $L \in \mathcal{L}$ then $L(gh) = (Lg)h \in \mathcal{L}$ which proves $gh \in G$.

Now we prove $g^{-1} \in G$. Let $x \neq y \in X$. By (b) we have only to show $(x/y)g^{-1} \subset xg^{-1}/yg^{-1}$. Let $z \in (x/y)$. As $\{x, y, z\} = \{(xg^{-1})g, (yg^{-1})g, (zg^{-1})g\}$ is a collinear set and as $g$ maps noncollinear points to noncollinear points we must have $\{xg^{-1}, yg^{-1}, zg^{-1}\}$ is collinear, and hence $zg^{-1} \in xg^{-1}/yg^{-1}$. $\square$

<u>13.3 Lemma.</u> Let $X \xrightarrow{f} X$ be any function such that $\forall x, y \in X$. $xf \neq yf \Rightarrow x/y \parallel xf/yf$. (e.g., $f$ could be constant). Then $f$ is completely determined by what it does to any two distinct points; (more precisely, if $g$ is another such function and if $\exists$ two distinct points on which $f$ and $g$ agree, then $f = g$).

<u>Proof.</u> Let $a, b$ be any two distinct points of $X$ and let $x \in X$. We will compute $xf$ from $af$ and $bf$.

<u>Case I.</u> $x \notin a/b$.




Define $L_1, L_2$ by $af \in L_1 \parallel a/x$, $bf \in L_2 \parallel b/x$. (conceivably, $af = bf$, but this won't affect the proof).

if $af = xf$ then $xf \in L_1$. otherwise $af \neq xf$ so that $xf \in af/xf \parallel a/x$ and $af/xf = L_1$ and still $xf \in L_1$. Similarly $xf = bf \in L_2$ or else $xf \neq bf$, $xf/bf = L_2$ and $xf \in L_2$. But as $a/x \nparallel b/x$, $L_1 \nparallel L_2$ and hence $xf = L_1 \cap L_2$ has been computed.

Case II. $x \in a/b$. By AF3 $\exists\, c \in X$. $c \notin a/b$. Therefore $cf$ may be computed as in case I. Since $a, b$ were arbitrary in case I and since $x \notin a/c$, $x$ may be computed as in case I. $\square$

13.4 Proposition. The following statements are valid.

a. Two dilatations agreeing on two distinct points are equal.

b. Let $X \xrightarrow{f} X$ be a nonconstant function such that $\forall x, y \in X$. $xf \neq yf \Rightarrow x/y \parallel xf/yf$. Then $f \in D$.

c. $D$ is a subgroup of $G$.

Proof.

a. clear from 13.3.

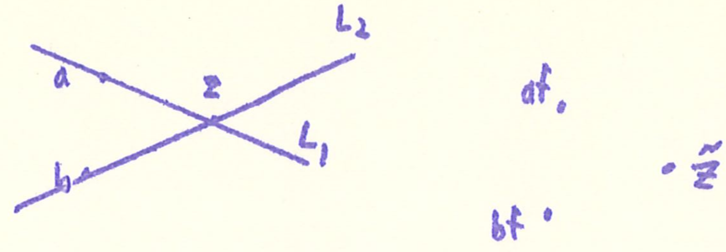
b. Let $f$ be as in (b).

$f$ is 1-to-1.

suppose $\exists\, x \neq y \in X$. $xf = yf$. Let $X \xrightarrow{g} X$ be the constant function $a \mapsto xf$. Since $g$ vacuously satifies the hypothesis $ag \neq bg \Rightarrow a/b \parallel ag/bg$ and since $g$ agrees with $f$ on $x$ and $y$, it follows from 13.4 that $f = g$, $\times$.

$f$ is onto.

Let $\tilde{z} \in X$. $\exists\, a \neq b \in X$ by 12.5 (a). As $f$ is 1-to-1, $af \neq bf$.




Case I. $\tilde{z} \notin af/bf$. Define $L_1, L_2$ by $a \in L_1 \parallel af/\tilde{z}$, $b \in L_2 \parallel bf/\tilde{z}$. As $af/\tilde{z} \nparallel bf/\tilde{z}$, $L_1 \nparallel L_2$. So define $z =_{df} L_1 \cap L_2$. Then, just as in 13.3 case I., we see $zf = \tilde{z}$.

Case II. $\tilde{z} \in af/bf$. $\exists\, \tilde{x} \notin af/bf$. By case I $\exists\, x \in X$. $xf = \tilde{x}$. As $\tilde{z} \notin af/xf$ we have from case I (since a, b were arbitrary) that $\exists\, z \in X$. $zf = \tilde{z}$.

To complete the proof of (b) we have only to show that $f$ is a collineation, and to do this we invoke 13.2 (b). That $f \in \mathrm{bij}(X)$ is already proved. Now let $x \neq y \in X$, $z \in x/y$. if $z = x$, $z \in xf/yf$. otherwise $z \neq x$ and $z \in xf/zf \parallel x/z$. But $x \in xf/zf \parallel x/z = x/y$ and $x \in xf/yf \parallel x/y \Rightarrow xf/zf = xf/yf$. $\therefore zf \in xf/yf$.

c. Let $d, d' \in D$. That $dd' \in D$ is clear from (b) and 12.4. $d^{-1} \in G$ by 13.4 c. Suppose $\exists\, L \in d$. $Ld^{-1} \nparallel L$. Then $Ld^{-1} \cap L \neq \emptyset \Rightarrow L \cap Ld \neq \emptyset$. As $L \parallel Ld$, $\therefore L = Ld$. $\therefore Ld^{-1} = L$ ✗. $\square$

**13.5 Proposition.** The following statements are valid.

a. $\forall t \in T \neg \{1_X\}\ \forall x, y \in X.\quad x/xt \parallel y/yt.$

Notation: If $t \in T \neg \{1_X\}$ we write "$\mathrm{dir}(t) = L$", read "the direction of $t$ is $L$", to mean $x/xt \parallel L$ $\forall x \in X$.

b. Two translations agreeing on any point are equal.

c. If $t \in T \neg \{1_X\}$, $\mathrm{dir}(t) = \mathrm{dir}(t^{-1})$.

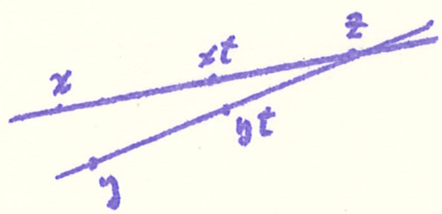
d. $T$ is a normal subgroup of $D$.

e. If $L \in \mathcal{L}$, $T_L =_{df} [t \in T : t = 1_X \text{ or } \mathrm{dir}(t) = L]$ is a subgroup of $T$.

f. $\forall t \in T \neg \{1_X\}\ \forall d \in D.\ \mathrm{dir}(t) = \mathrm{dir}(dtd^{-1})$;
$\therefore$ each $T_L$ is a normal subgroup of $D$.

Proof.

a. Let $t \in T \neg \{1_X\}$, $x, y \in X$. Suppose $z \in x/xt \cap y/yt$.

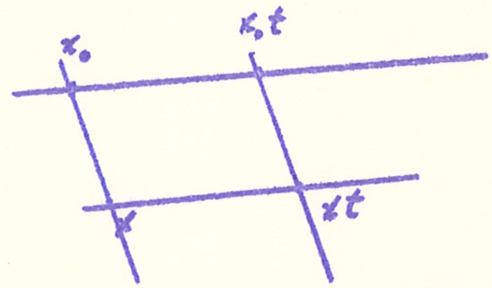



As $x/xt = x/z \parallel xt/zt$ and $xt \in x/z \cap xt/zt$, necessarily $x/z = xt/zt$ and $zt \in x/xt$. As $\{z, zt\} \subset x/xt$ and $z \neq zt$, $x/xt = z/zt$. Similarly $y/yt = z/zt$. $\therefore$ $x/xt = y/yt$.

<u>b.</u> Let $t \in T$, $x_0, x \in X$. We must construct $xt$ from knowledge of $x_0 t$.

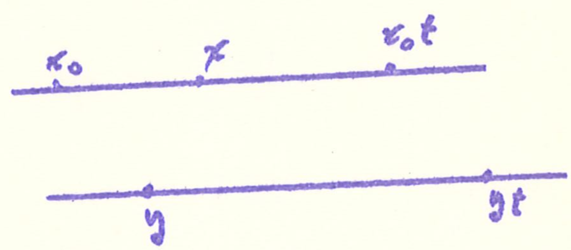
<u>Case I.</u> $x_0 t = x_0$. Then by the definition of $T$, $t = 1_X$ and $xt = x$.

<u>Case II.</u> $x_0 t \neq x_0$, $x \notin x_0 | x_0 t$.




By (a), $x | xt \parallel x_0 | x_0 t$. As $t \in D$, $x_0 t | xt \parallel x_0 | x$. As $x \notin x_0 | x_0 t$, $x | xt \nparallel x_0 t | xt$. $\therefore xt = x | xt \cap x_0 t | xt$.

<u>Case III.</u> $x_0 t \neq x_0$, $x \in x_0 | x_0 t$.




$\exists\, y \notin x_0 | x_0 t$. By case II we may construct $yt$. As $x_0 | x_0 t \parallel y | yt$ by (a), $x \notin y | yt$. Using case II with $y$ replacing $x_0$ we can construct $xt$.

<u>c.</u> Let $t \in T \neg \{1_X\}$. Then using (a) we have $\forall x \in X$ that $x | xt = xt | xt^{-1} t \parallel x | xt^{-1}$ [noting that $xt^{-1} \neq x$ since $xt^{-1} = x \Rightarrow x = xt$]

d. If $t \in T \neg \{1_X\}$ then $t^{-1} \in D$ and $\forall x \in X$ . $xt^{-1} \neq x$ (as was just observed) so $t^{-1} \in T$. If $t, s \in T \neg \{1_X\}$ then $ts \in D$; if $\exists\, x \in X$ . $xts = x$ then $xt = xs^{-1}$; since $s^{-1} \in T \neg \{1_X\}$ it follows from (b) that $t = s^{-1}$ and then $ts = 1_X$, $\cancel{X}$. $\therefore$ $ts \in T \neg \{1_X\}$. So far we have $T \leq D$.

Suppose $t \in T \neg \{1_X\}$, $d \in D$. $\forall x \in X$ we have $xdt \neq xd$ and so $xdtd^{-1} \neq x$. $\therefore$ $dtd^{-1} \in T$.

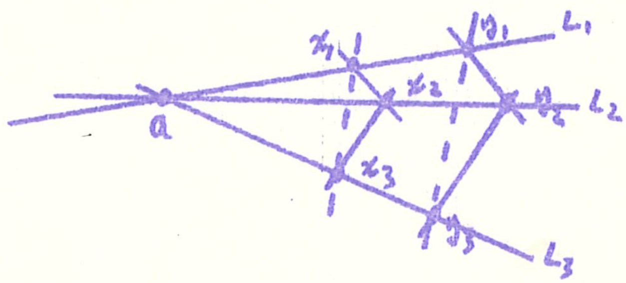
e. Let $L \in \mathcal{L}$, $t, s \in T_L$. $t^{-1} \in T_L$ by (d) and (c). Let $x \in X$. As $x/xt \parallel L \parallel x/xs^{-1}$. we have using (a) and the fact that $x, xt, xs^{-1}$ are collinear that $x/xts \parallel xs^{-1}/xt = x/xs^{-1} \parallel L$, and $ts \in T_L$.

f. Let $t \in T \neg \{1_X\}$, $d \in D$, $x \in X$. Using (a), $x/xt \parallel xd/xdt \parallel x/xdtd^{-1}$. $\square$

# Lecture 14

**14.1 Proposition.** The following statements are equivalent.

a. Let $L_1, L_2, L_3$ be distinct lines with a common point $a \in L_1 \cap L_2 \cap L_3$ and let $x_i \neq a \neq y_i$ $(i=1,2,3)$ with $x_i, y_i \in L_i$ in such a way that $x_1/x_2 \parallel y_1/y_2$ and $x_2/x_3 \parallel y_2/y_3$. Then $x_1/x_3 \parallel y_1/y_3$.



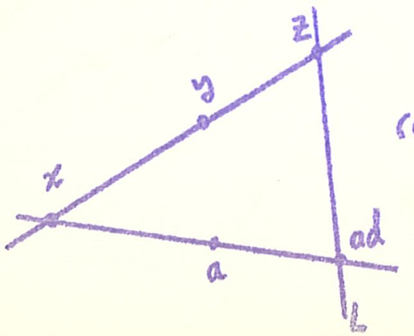

b. If $x, y, z$ are distinct collinear points then $\exists\, d \in D$ with $xd = x$ and $yd = z$

c. If $x, y, z$ are distinct collinear points then $\exists$ unique $d \in D$ with $xd = x$ and $yd = z$.

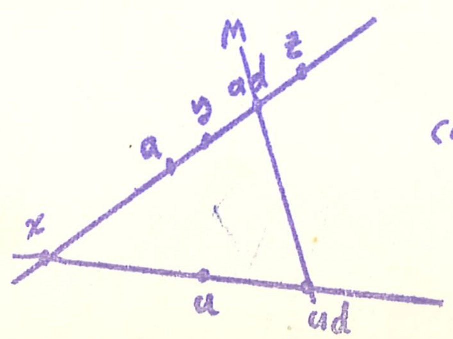
**Proof.**

a $\Rightarrow$ b. Let $x, y, z$ be distinct collinear points. Fix a point $u \notin x/y$ and define $X \xrightarrow{d} X$ by

$$ad = \begin{cases} L \cap x/a \text{ where } z \in L \parallel a/y & (a \notin x/y) \\ M \cap x/y \text{ where } ud \in M \parallel u/a & (a \in x/y) \end{cases}$$




$(a \notin x/y)$




$(a \in x/y)$

clearly $d$ is well-defined and has the following properties:

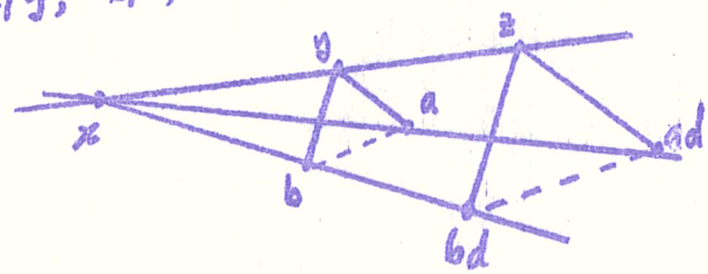
(i) $xd = x$

(ii) $yd = z$

(iii) $\forall a \in X.\ ad \in x/a$.

To prove (b) we must show $d \in D$ and to this end we employ 13.4 (b). Let $a, b \in X$ with $ad \neq bd$. We must show $a/b \parallel ad/bd$.

Case I. $x/y$, $x/a$, $x/b$ are distinct.




By the construction of $d$, $y/a \parallel z/ad$, $y/b \parallel z/bd$, so by (a) we have $a/b \parallel ad/bd$.

Case II. $x/y = x/a \neq x/b \neq x/u$

Case III. $x/y = x/a \neq x/b = x/u$

Case IV. $x/a = x/b$

relegated to the seminar

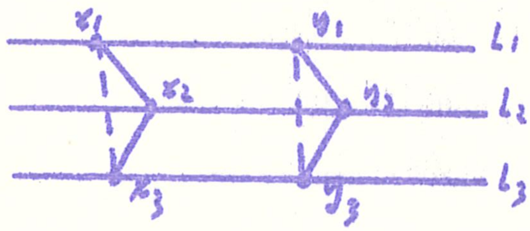
$b \Rightarrow c$. Immediate from 13.4 (a).

$c \Rightarrow a$. We refer to the figure of (a) on the previous page. By (c) $\exists\, d \in D$. $ad = a$ and $x_2 d = y_2$. As $y_2 \in y_2/x_1 d = x_2 d/x_1 d \parallel x_2/x_1$, $y_2/x_1 d = y_2/y_1$. $\therefore x_1 d = a/x_1 \cap y_2/x_1 d = a/x_1 \cap y_2/y_1 = y_1$. Similarly $x_3 d = y_3$. $\therefore$ $y_1/y_3 = x_1 d/x_3 d \parallel x_1/x_3$.

$\square$

**14.2 Proposition.** fsae

a. if $L_1, L_2, L_3$ are distinct parallel lines and if $x_i \neq y_i \in L_i$ $(i=1,2,3)$ in such a way that $x_1|x_2 \parallel y_1|y_2$ and $x_2|x_3 \parallel y_2|y_3$ then $x_1|x_3 \parallel y_1|y_3$.




b. $\forall x,y \in X \; \exists \; t \in T \,.\; xt = y$.

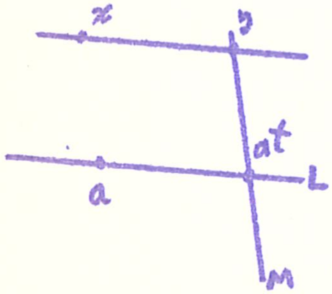
c. $\forall x,y \in X \; \exists$ unique $t \in T \,.\; xt = y$.

Proof.

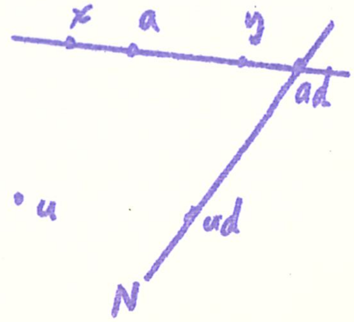
a ⇒ b. Let $x, y \in X$. If $x = y$, $t = 1_X$ is the desired translation, otherwise assume $x \neq y$. Define $X \xrightarrow{t} X$ by

$$at = \begin{cases} L \cap M \text{ where } a \in L \parallel x|y,\; y \in M \parallel a|x & (a \notin x|y) \\ N \cap x|y \text{ where } ud \in N \parallel u|a & (a \in x|y) \end{cases}$$

(where $u \notin x|y$ is fixed before defining $t$)




$(a \notin x|y)$

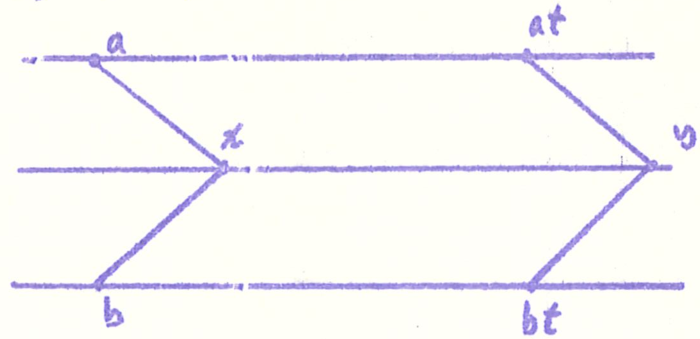



$(a \in x|y)$

clearly $t$ is well-defined and has the properties
(i) $t$ has no fixpoints, (ii) $xt = y$, (iii) $\forall a \in X \,.\; a|at \parallel x|y$.

To prove (b) we use 13.4 (b) to show $t \in D$. Clearly $t$ is not constant. Let $a, b \in X$ with $at \neq bt$. We must show $a/b \parallel at/bt$.

Case I. $x/y$, $a/at$, $b/bt$ are distinct.




By the construction of $t$, $a/at \parallel x/y \parallel b/bt$, $a/x \parallel at/y$, $x/b \parallel y/bt$. By (a) we conclude $a/b \parallel at/bt$.

Case II. $x/y = a/at \neq b/bt \neq u/ut$
Case III. $x/y = a/at \neq b/bt = u/ut$
Case IV. $x/y = a/at = b/bt$

} relegated to the seminar.

$b \Rightarrow c$. Use 13.5 (b).

$c \Rightarrow a$. Refer to the figure of (a) on the previous page.
By (c) $\exists\, t \in T$, $x_2 t = y_2$. It is easy to check that $x_1 t = y_1$, $x_3 t = y_3$ and so $y_1/y_3 = x_1 t/x_3 t \parallel x_1/x_3$.

$\square$

**14.3 Definition.** Any of the three equivalent conditions of 14.1 on a preaffine plane $(X, \mathcal{L})$ is called the Desargue Axiom. An affine plane is a preaffine plane which satisfies the Desargue axiom.

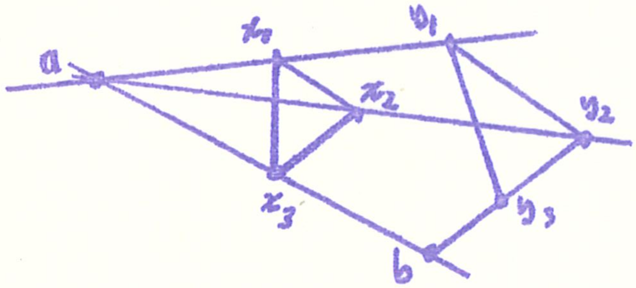
**14.4 Proposition.** Let $(X, \mathcal{L})$ be an affine plane. The following statements are valid.

a. Given six distinct points $x_1, x_2, x_3, y_1, y_2, y_3$ such that $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$ are noncollinear and such that $x_i|x_j \parallel y_i|y_j$ $(\{i,j\} = \{1,2\}, \{2,3\}, \{1,3\})$ then the three lines $x_i|y_i$ $(i=1,2,3)$ are either concurrent or parallel.

b. all of the equivalent conditions of 14.2 are valid.

Proof.

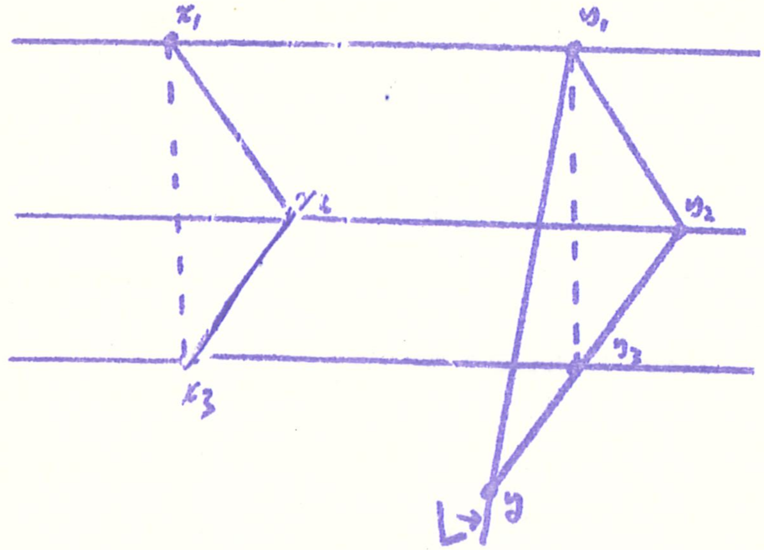
a. Suppose $x_i|y_i$ $(i=1,2,3)$ are not all parallel. Then (say) $x_1|y_1 \nparallel x_2|y_2$. $a =_{df} x_1|y_1 \cap x_2|y_2$. To show: $a \in x_3|y_3$.




if $x_2|x_3 = a|x_3$ then $a, x_2, x_3, y_2, y_3$ are collinear and $a \in x_3|y_3$. otherwise $y_2|y_3 \parallel x_2|x_3 \nparallel a|x_3$, so define $b =_{df} a|x_3 \cap y_2|y_3$. By 14.1 (a), $y_1|y_3 \parallel x_1|x_3 \parallel y_1|b$, and so $y_1|y_3 = y_1|b$.

if $b \neq y_3$, $y_1|b = b|y_3 = y_2|y_3$ and $\{y_1, y_2, y_3\}$ are collinear, $\cancel{X}$. $\therefore$ $y_3 = b$ and $a \in a|b = x_3|b = x_3|y_3$.

<u>b.</u> We verify 14.2 (1). We consider




$x_1|y_1 \parallel x_2|y_2 \parallel x_3|y_3$

$x_1|x_2 \parallel y_1|y_2$

$x_2|x_3 \parallel y_2|y_3$

To show: $x_1|x_3 \parallel y_1|y_3$. Clearly $\{x_1, x_2, x_3\}$ collinear iff $\{y_1, y_2, y_3\}$ collinear and, in either case, $x_1|x_3 \parallel y_1|y_3$. Otherwise assume $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$ are noncollinear. Define $L$ by $y_1 \in L \parallel x_1|x_3$. As $L \parallel x_1|x_3 \nparallel x_2|x_3 \parallel y_2|y_3$, $y =_{df} L \cap y_2|y_3$ is well-defined, and $\{y_1, y_2, y\}$ is noncollinear. By (a), $x_1|y_1$, $x_2|y_2$, $x_3|y$ are parallel. $\therefore$ $x_3|y = x_3|y_3$. As $y \in x_3|y_3 \cap y_3|y_2$, $y = y_3$ and $y_1|y_3 = L \parallel x_1|x_3$. $\square$

<u>Remark.</u> That 14.1 a $\Rightarrow$ 14.2 a is a relatively recent discovery. Older texts list 14.1 a, 14.2 a together as the Desargue axiom.

<u>14.5 Roundup of 14.1, 14.2, 14.4</u>. Let $(X, \mathcal{L})$ be an affine plane. The following are valid.

a. Given six distinct points $x_1, x_2, x_3, y_1, y_2, y_3$ with $x_1|y_1$, $x_2|y_2$, $x_3|y_3$ distinct lines, $x_1|x_2 \parallel y_1|y_2$, $x_2|x_3 \parallel y_2|y_3$,

then $x_1|x_3 \parallel y_1|y_3$ iff $x_i|y_i$ $(i=1,2,3)$ are concurrent or parallel.

b. $\forall$ $x, y, z$ distinct collinear points $\exists$ unique $d \in D$ with $xd = x$ and $yd = z$.

c. $\forall$ $x, y$ $\exists$ unique $t \in T$ with $xt = y$.

<u>14.6 Example.</u> The Euclidian plane is an affine plane.

# 141 Seminar

## Wednesday Dec. 6, 4:15 pm, TG 202

1. (Bellenot) Prove cases II, III, IV in 14.1.

2. (George-Hamilton; present jointly). Let $(X, \mathcal{L})$ be an affine plane. A <u>point-reflection</u> is a dilatation $p \in D$ such that $\exists\ x, y, z \in X$, $y \neq z$ with $xp = x$, $yp = z$, $zp = y$.
   a. If $p$ is as above, show $x \in y/z$.
   b. If $p$ is as above show that $p$ is the only point reflection fixing $x$.
   c. If $d$ is any dilatation interchanging two distinct points, show that $d$ is a point reflection.
   d. What are necessary and sufficient (geometric) conditions for either of the properties (i) $\forall x\ \exists$ point reflection $p$ with $xp = x$, (ii) $\forall x \neq y\ \exists d \in D$. $xd = y$ & $yd = x$. How are (i) and (ii) related?

3. (Greitzer). Show that for the Euclidean plane, $[d \in D : (0,0)d = (0,0)]$ is a subgroup of $D$, isomorphic to the group of multiplicative non-0 reals.

4. (Gulik) Let $(X, \mathcal{L})$ be an affine plane, $x \in X$, $d \in D$. Show $\exists\ \tilde{d} \in D$, $t \in T$ with $x\tilde{d} = x$ and $\tilde{d}t = d$.

5. (Malcolm) If $(X, \mathcal{L})$ is an affine plane prove $T$ is abelian.

6. (Ray) Prove cases II, III, IV of 14.2 and expound on 14.2 $\Leftarrow$ a.

# Lecture 15

We interrupt our study of affine planes to make a brief study of vector spaces over a division ring.

**15.1 Definition.** A <u>division ring</u> is a triple $(F, +, \cdot)$ such that $(F, +)$ is an abelian group (with identity element written "$0$"), $(F, \cdot)$ is a monoid such $(F \setminus \{0\}, \cdot)$ is a group and such that the <u>distributive laws</u>

$$\forall x, y, z \in F \qquad x(y+z) = xy + xz$$
$$(y+z)x = yx + zx$$

are valid.

A <u>Field</u> is a division ring $(F, +, \cdot)$ such that $(F \setminus \{0\}, \cdot)$ is an abelian group.

The identity of the monoid $(F, \cdot)$ is written "$1$". "$x^{-1}$" means $xx^{-1} = 1 = x^{-1}x$ whereas $+$-inverses are written "$-x$".

**15.2 Remarks.** Let $(F, +, \cdot)$ be a division ring.

a. $0x = 0 = x0 \ \forall x$ [as $0x = (0+0)x = 0x + 0x \Rightarrow 0x = 0$].

b. $xy = 0 \Rightarrow x = 0$ or $y = 0$ [if $y \neq 0$, $x = x \cdot 1 = xyy^{-1} = 0 \cdot y^{-1} = 0$].

<u>15.3 Example; the field $\mathbb{Z}_p$.</u> Let $p$ be a prime. Let $(\mathbb{Z}_p, +)$ be the cyclic group of order $p$. For the sake of a specific model set $\mathbb{Z}_p = \{0, \dots, p-1\}$ and $+$ = addition modulo $p$. Let $\mathbb{Z}_p \times \mathbb{Z}_p \xrightarrow{\cdot} \mathbb{Z}_p$ be multiplication modulo $p$. It is easy to check that $(\mathbb{Z}_p, \cdot)$ is a monoid and that the distributive laws hold. We show that if $x \neq 0$, $x^{-1}$ exists. Suppose $xy_1 = xy_2$. $\therefore 0 = xy_1 - xy_2 = x(y_1 - y_2)$

$\Rightarrow x(y_1 - y_2) \equiv 0 \pmod p \Rightarrow p \mid x(y_1 - y_2) \Rightarrow p \mid x$ or $p \mid (y_1 - y_2)$

(as $p$ is prime) $\Rightarrow x \equiv 0 \pmod p$ or $(y_1 - y_2) \equiv 0 \pmod p$

$\Rightarrow x = 0$ or $y_1 - y_2 = 0 \Rightarrow y_1 = y_2$. (as we assumed $x \neq 0$).

$\therefore y_1 \neq y_2 \Rightarrow xy_1 \neq xy_2$ so $\mathbb{Z}_p = \{0, x, 2x, \dots, (p-1)x\}$ and $\exists\, y$ . $xy = yx = 1$.

<u>15.4 Theorem.</u> Every finite division ring is isomorphic to $\mathbb{Z}_p$ for some unique prime $p$.

<u>Proof.</u> "isomorphism" = bijection preserving "+" and "$\cdot$", and units. The proof is too long and specialized for inclusion here, but may be found in most texts that treat elementary field theory.

## 15.5 Examples.

a. The rational numbers, $\mathbb{Q}$, form a field.

b. The real numbers, $\mathbb{R}$, form a field.

c. The complex numbers, $\mathbb{C}$, form a field.

$(x+iy)+(\bar{x}+i\bar{y}) = (x+\bar{x}+i(y+\bar{y}))$

$(x+iy)\cdot(\bar{x}+i\bar{y}) = ((x\bar{x}-y\bar{y})+i(y\bar{x}+x\bar{y}))$

## 15.6 Example. The real quaternions is a division ring not a field.

Notice that to define $\mathbb{C}$ as above, take a two-dimensional $\mathbb{R}$-vectorspace $\mathbb{R}^2$ with basis $\{\mathbb{1}, i\}$ and define a multiplication by $\mathbb{1}\cdot x = x = x\cdot\mathbb{1}$ $\forall x \in \mathbb{R}^2$, $i^2 = -1$. This forces the definition $(x, yi)(\bar{x}, \bar{y}i) = x\bar{x} + y\bar{x}i + x\bar{y}i + y\bar{y}i^2$ $= (x\bar{x}-y\bar{y}, (y\bar{x}+x\bar{y})i)$. Similarly, define the real quaternions $\mathcal{Q}$, by taking $\mathcal{Q} = \mathbb{R}^4$ with basis $\{\mathbb{1}, i, j, k\}$ and defining

$$\mathbb{1}\cdot x = x = x\cdot\mathbb{1} \quad \forall x \in \mathbb{R}^4$$
$$i^2 = j^2 = k^2 = -\mathbb{1}$$
$$ij = k = -ji$$
$$jk = i = -kj$$
$$ki = j = -ik$$

The definition of multiplication $\mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is similarly forced, and is provably a division ring (routine exercise). $\mathcal{Q}$ is not a field since $jk \neq kj$.

15.7 Definition. Let $(F,+,\cdot)$ be a division ring, $A \subset X$. $A$ is a subdivision ring of $(F,+,\cdot)$ if $\{0,1\} \subset A$, $\forall\, a,b \in A$. $a+b \in A$, $-a \in A$, $ab \in A$, $a \neq 0 \Rightarrow a^{-1} \in A$.

Clearly, a subdivision ring is a division ring and any intersection of subdivision rings is a division ring. Hence every subset $B \subset F$ generates a subdivision ring $\langle B \rangle = \bigcap [A \supset B : A \text{ is a subdivision ring}]$.

15.8 Example. 15.7 gives rise to lots of examples. For instance consider the subfield of $\mathbb{R}$ generated by $\mathbb{Q} \cup \{\sqrt{2}\}$.

15.9 Definition. Let $(F,+,\cdot)$ be a division ring. A vectorspace over $F$ is a triple $(V,+,\pi)$ such that $(V,+)$ is an abelian group and

$$\begin{aligned} V \times F &\xrightarrow{\pi} V \\ x, \lambda &\longmapsto x\lambda \end{aligned}$$

is a function satisfying the axioms

$$\begin{aligned} (x+y)\lambda &= x\lambda + y\lambda \\ x(\lambda+\mu) &= x\lambda + x\mu \\ x(\lambda\mu) &= (x\lambda)\mu \\ x1 &= x \end{aligned}$$

**15.10 Definitions.** Let $(V, +, \pi)$ be a vectorspace over $(F, +, \cdot)$.

$A \subset V$ is a subspace if $\forall x, y \in A \; \forall \lambda \in F$, $x+y \in A$, $-x \in A$, $x\lambda \in A$. $A$ is itself a vectorspace. By the usual argument every $B \subset V$ generates a subspace $\langle B \rangle$.

Let $B \subset V$. $B$ is independent if whenever $b_1 \neq \cdots \neq b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in F$ then $\sum_1^n b_i \lambda_i = 0 \Rightarrow \lambda_1 = \cdots = \lambda_n = 0$. Otherwise, $B$ is dependent.

If $V, W$ are vectorspaces over $F$ (we drop the triplet notation if there is no confusion), a function $V \xrightarrow{f} W$ is linear if $(x+y)f = xf + yf$ and $(x\lambda)f = (xf)\lambda$ $\forall x, y \in V$ $\forall \lambda \in F$. Clearly $1_V$ is linear and $f, g$ linear $\Rightarrow fg$ linear. $\exists g$ with $g$ linear and $fg = 1_V$, $gf = 1_W$ iff $f$ is bijective and linear.

**15.11 Theorem.** Let $V$ be a vectorspace over $F$, $B \subset V$. The following statements are equivalent.

a. $B$ is independent and $\langle B \rangle = V$.

b. $\forall\, 0 \neq x \in V$ $\exists$ unique $b_1, \ldots, b_n \in B$, $\lambda_1, \ldots, \lambda_n \in F$ such that $x = \sum_1^n b_i \lambda_i$.

c. For every vectorspace $W$ over $F$, every function $B \xrightarrow{g} W$ extends uniquely to a linear map $V \xrightarrow{\tilde{g}} W$.

Proof.

a ⇒ b. Let $0 \neq x \in V$. It is easy to check that $\{ \sum_1^n b_i \lambda_i : b_1, \dots, b_n \in B, \lambda_1, \dots, \lambda_n \in F \}$ is a subspace and hence is equal to $\langle B \rangle$. Since $\langle B \rangle = V$, $\therefore \exists$ a representation $x = \sum_1^n b_i \lambda_i$. Suppose also $x = \sum_1^m \tilde{b}_j \tilde{\lambda}_j$. Write $\sum_1^n b_i \lambda_i = \sum_{b \in B} b \lambda_b$ (where $\lambda_b = \begin{cases} \lambda_i & b = b_i \\ 0 & \text{otherwise} \end{cases}$)

Similarly $\sum_1^m \tilde{b}_j \tilde{\lambda}_j = \sum_{b \in B} b \tilde{\lambda}_b$. $0 = \sum_{b \in B} b \lambda_b - \sum_{b \in B} b \tilde{\lambda}_b$ $= \sum_{b \in B} b (\lambda_b - \tilde{\lambda}_b)$. As $B$ is independent, $(\lambda_b - \tilde{\lambda}_b) = 0$ $\forall b$ and $\lambda_b = \tilde{\lambda}_b$ $\forall b$, which proves the uniqueness of the representation.

b ⇒ c. If $x \in V$, let $x = \sum_{b \in B} b \lambda_b$ be the unique representation. Define $x\tilde{g} =_{df} \sum_{b \in B} (bg) \lambda_b$. Clearly $\tilde{g}$ is linear and agrees with $g$ on $B$. Moreover linearity forced this definition of $\tilde{g}$, so $\tilde{g}$ is unique.

c ⇒ a. Seminar problem.

<u>15.12 Definition.</u> A subset of $V$ satisfying any of the equivalent conditions of 15.11 is called a <u>basis for</u> $V$.

<u>15.13 Theorem.</u> Let $V$ be a vectorspace over $F$. Then $\exists\, B \subset V$ such that $B$ is a basis for $V$.

<u>Proof.</u> Let $\mathcal{L} =_{df} \{B \subset V : B \neq \emptyset \text{ and } B \text{ is independent}\}$.

<u>Case I.</u> $\nexists\, x \in V$, $x \neq 0$. Then $\emptyset$ is a basis. (Seminar problem).

<u>Case II.</u> $\exists\, x \in V$, $x \neq 0$. Claim $\{x\} \in \mathcal{L}$. Surely $\{x\} \neq \emptyset$. Suppose $x\lambda = 0$. If $\lambda \neq 0$ then $x = x \cdot 1 = x(\lambda\lambda^{-1})$ $= (x\lambda)\lambda^{-1} = 0\lambda^{-1}$; but $0\lambda^{-1} = (x-x)\lambda^{-1} = x\lambda^{-1} - x\lambda^{-1} = 0$. $\therefore$

This proves indeed that $\{x\} \in \mathcal{L}$. It is routine to check that any nested union of elements of $\mathcal{L}$ is itself in $\mathcal{L}$. By Zorns lemma, $\mathcal{L}$ has a maximal element $B$. As $B$ is independent, we have only to show $\langle B \rangle = X$. This is left to the seminar. $\square$

<u>15.14 Theorem.</u> Let $B$, $\mathbb{C}$ be bases for a vectorspace $V$ over $F$. Then card $B$ = card $\mathbb{C}$.

<u>Proof.</u> Difficult; can be found in many texts; easier to prove ! at least one has finite cardinality. $\square$

<u>15.15 Definition.</u> Let $V$ be a vectorspace over $F$. The <u>dimension of</u> $V$, <u>$\dim(V)$</u>, is the cardinality of any of its bases.

<u>15.16 Theorem.</u> Let $V, W$ be vectorspaces over $F$. Then $V \cong W$ iff $\dim(V) = \dim(W)$.

<u>Proof.</u> Left to the seminar.

<u>15.17 Definition.</u> Let $(V_i : i \in I)$ be a family of vectorspaces over $F$. Take the abelian group $\prod V_i$, with action

$$\prod V_i \times F \xrightarrow{\pi} \prod V_i$$
$$(x_i), \lambda \longmapsto (x_i \lambda)$$

Then $\prod V_i$ is a vectorspace called the <u>cartesian product of</u> the family $(V_i)$. The subspace

$$\bigoplus V_i =_{df} \{ (x_i) \in \prod V_i : x_i = 0 \text{ for all except finitely many } i \in I \}$$

is called the <u>direct</u> <u>sum</u> of the $V_i$.

Notice that $\bigoplus V_i = \prod V_i$ if $I$ is finite.

**15.18 Theorem.** Let $\alpha$ be a cardinal number. Then $\exists$ a vectorspace $V$ over $F$ with card $V = \alpha$.

**Proof.** $V =_{df} \bigoplus_{i \in I} F$ where card $I = \alpha$ and $F$ is a vectorspace over itself via $F \times F \xrightarrow{\cdot} F$, $x, \lambda \longmapsto x\lambda$.

$B =_{df} \{ (x_i)_j \ni x_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} ; j \in I \}$ is a subset of cardinality $\alpha$. It is trivial to verify $B$ is a basis (use 15.11 (b)).

**15.19 Remark.** If $F' \subset F$ is a subdivision ring, $F \times F' \xrightarrow{\pi} F$, $x, \lambda \longmapsto x\lambda$ makes $F$ a vectorspace over $F'$.

# 141 Seminar

**Wednesday, Dec. 13, 4:15 pm, MG 202**

1. (Bellenot) Prove 15.11 $c \Rightarrow a$.

2. (George) Prove 15.13 case I.

3. (Greitzer) Proof 15.16.

4. (Gulik). Let $F$ be a division ring, $V$ a vectorspace over $F$. Prove that if $\lambda \neq 0 \in F$ that $\begin{array}{l} V \xrightarrow{f} V \\ x \longmapsto x\lambda \end{array}$ is a linear isomorphism.

5. (Hamilton) Let $\mathbb{R} \xrightarrow{f} \mathbb{R}$ be a function such that $f(1) = 1$, $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$, $f(\frac{1}{n}) = \frac{1}{f(n)}$ $\forall$ integers $n > 0$, and $f(x^2) = f(x) \cdot f(x)$ $\forall x > 0$. Prove $f = \mathrm{id}_{\mathbb{R}}$.

$f(1+y) = f(1) + f(y)$

$f(2) = 2f(1)$

$f(1) = f(1) f(1)$

$f(1) = 0, 1$

6. (Malcolm). Prove that $\mathbb{R}$ and $\mathbb{R}^n$ are isomorphic as abelian groups for any positive integer $n$.

if $f(1) = 0$

$f(1+y) = f(y)$

$f(y) = 0$ if $y$ integer

7. (Ray) Finish 15.13 Case II.

# Lecture 16

For this section, let $F$ be a division ring. Let $X$ be a vectorspace over $F$ with $\dim X = 2$. (By 15.18, one model for $X$ is $F \oplus F$, but we don't need to be so explicit).

If $x, y \in X$ with $x \neq y$, $x/y =_{df} \{x + (y-x)\lambda : \lambda \in F\}$. $\mathcal{L} =_{df} \{x/y : x, y \in X, x \neq y\}$.

**16.1 Proposition.** $(X, \mathcal{L})$ is a preaffine plane. [More is true, see 16.5]

**Proof.**

**AF1.** Let $x, y \in X$, $x \neq y$. As $x = x + (y-x)0$, $y = x + (y-x)1$ we have $x, y \in x/y$ so that every pair of distinct points belongs to at least one line. Now suppose $x, y \in a/b$. We must show $a/b = x/y$.

Since $x, y \in a/b$ $\exists\, \lambda_x, \lambda_y \in F$ with $x = a + (b-a)\lambda_x$, $y = a + (b-a)\lambda_y$. $\forall \lambda \in F$ we have

$$
\begin{aligned}
x + (y-x)\lambda &= a + (b-a)\lambda_x + (b-a)(\lambda_y - \lambda_x)\lambda \\
&= a + (b-a)(\lambda_x + \lambda_y\lambda - \lambda_x\lambda) \in a/b
\end{aligned}
$$

which proves $x/y \subset a/b$.

Now observe that $y-x = (b-a)(\lambda_x - \lambda_y)$. $y-x \neq 0 \Rightarrow \lambda_x - \lambda_y \neq 0$.

$\forall \lambda \in F$ we have

$$a+(b-a)\lambda = [x-(b-a)\lambda_x] + (b-a)\lambda$$
$$= x + (b-a)(\lambda - \lambda_x)$$
$$= x + (y-x)(\lambda_x - \lambda_y)^{-1}(\lambda - \lambda_x) \quad \in x/y$$

proving $a/b \subset x/y$.

<u>AF2.</u> Let $a/b \in \mathcal{L}$, $x \in X$. We will show that $x|x+b-a$ is the unique line through $x$ parallel to $a/b$.

<u>Case I.</u> $x \in a/b$. Then $x+b-a = (a+(b-a)\lambda) + (b-a)$ [for some $\lambda \in F$] $= a + (b-a)(\lambda+1) \in a/b$. As $a \neq b$, $x \neq x+b-a$ and hence $x|x+b-a = a/b$. Of course $a/b$ is the only line through $x$ parallel to $a/b$ [because parallel intersecting lines are equal by definition of "parallel"].

<u>Case II.</u> $x \notin a/b$.

First claim $x|x+b-a \cap a/b = \phi$.

If not $\exists \lambda, \mu \in F$ with $x+(b-a)\lambda = a+(b-a)\mu$ and then $x = a+(b-a)\mu - (b-a)\lambda$

$$= a + (b-a)(\mu - \lambda)$$
$$\in a/b \quad \cancel{\times}.$$

Claim $x|y \cap a|b = \emptyset \Rightarrow x|y = x|x+b-a$

suppose $\{y-x, b-a\}$ is independent. Since every independent subset may be extended to a basis [proof of 15.13] whereas every basis has two elements [15.14], $\{y-x, b-a\}$ is a basis, $\therefore \exists$ (unique) $\lambda, \mu \in F$, $x-a = (y-x)\lambda + (b-a)\mu$, and then $x+(y-x)(-\lambda) = a+(b-a)\mu$ is a common point of $x|y$ and $a|b$, $\times$.

This proves $\{y-x, b-a\}$ is dependent, so that there exist $\lambda, \mu \in F$ not both zero with $(y-x)\lambda + (b-a)\mu = 0$. Since $y-x \neq 0$ and $b-a \neq 0$ it follows that neither $\lambda$ nor $\mu = 0$ [Here we are using the fact that $\forall z \in X \; \forall \alpha \in F$. $z\alpha = 0 \Rightarrow z = 0$ or $\alpha = 0$; this was proved in 15.13 case II].

$\therefore y = x + (b-a)(-\mu\lambda^{-1}) \in x|x+b-a$

AF3. Let $\{x, y\}$ be a basis for $X$. Then $x, y, 0$ are distinct noncollinear points. [Proof is a seminar problem.] $\square$

16.3 Proposition. $x|x+b-a$ is the unique line through $x$ parallel to $a|b$.

Proof. already done in 16.2 AF2. $\square$

<u>16.4 Proposition.</u> For each $\lambda \in F, \lambda \neq 0$ and for each $a \in X$ define $d_{\lambda,a}$ to be the function

$$\begin{array}{ccc} X & \xrightarrow{d_{\lambda,a}} & X \\ x & \longmapsto & x\lambda + a \end{array}$$

Let $D, T$ be the dilatation, translation groups of the preaffine plane $(X, \mathcal{L})$.

Then $D = \{ d_{\lambda,a} : \lambda \in F, \lambda \neq 0, a \in X \}$

$T = \{ d_{\lambda,a} : \lambda = 1, a \in X \}$

<u>Proof.</u> Let $\lambda \in F, \lambda \neq 0, a \in X$. We show $d_{\lambda,a} \in D$. To do so we use 13.4 b. $\forall x \in X$,

$$x\, d_{\lambda,a}\, d_{\lambda^{-1},-a\lambda^{-1}} = (x\lambda + a)\lambda^{-1} - a\lambda^{-1} = x$$

$$x\, d_{\lambda^{-1},-a\lambda^{-1}}\, d_{\lambda,a} = (x\lambda^{-1} - a\lambda^{-1})\lambda + a = x$$

proves that $d_{\lambda,a}$ is bijective and non-constant in particular.

Let $x/y \in X$. We must show $x/y \parallel x\lambda + a \,/\, y\lambda + a$.

But $x\lambda + a / y\lambda + a = \{ x\lambda + a + (y-x)\lambda\mu : \mu \in F \}$

$= \{ x\lambda + a + (y-x)\mu : \mu \in F \}$ [because as $\lambda^{-1}$ exists the set $\{\lambda\mu : \mu \in F\} = F$]

$= x\lambda + a \,/\, x\lambda + a + y - x$

$=$ unique line through $x\lambda + a$ parallel to $x/y$ [by 16.3]

as we wished to show.

Conversely, let $d \in D$. $a =_{df} od$. By 16.3, $(o|x)d = a|a+x$ $\forall x \neq o$. Fix $x \neq o$. As $d$ is 1-1, $xd \neq a$ so $\exists \lambda \neq o$, $xd = a + x\lambda$. Since $d$ agrees with $d_{\lambda,a}$ on $o$ and $x$, it follows from 13.4 a that $d = d_{\lambda,a}$. This proves the statement about $D$. The statement about $T$ is a seminar problem.

**16.5 Proposition.** $(X, \mathcal{L})$ is an affine plane.

**Proof.** We have only to verify 14.1 (b). Let $z \in x/y$, $z$ distinct from $x, y$. $\exists \lambda \neq o$, $z = x + (y-x)\lambda$. $d =_{df} d_{\lambda, x(1-\lambda)}$. Then $xd = x\lambda + x(1-\lambda) = x$ and $yd = y\lambda + x(1-\lambda) = (y-x)\lambda + x = z$. $\square$

**16.6 Proposition.** If $X \xrightarrow{f} X$ is linear and bijective then $f$ is a collineation.

**Proof.**
$$
\begin{aligned}
(x/y)f &= \{x + (y-x)\lambda : \lambda \in F\}f \\
&= \{(x + (y-x)\lambda)f : \lambda \in F\} \\
&= \{xf + (yf - xf)\lambda : \lambda \in F\} \\
&= xf/yf. \quad \square
\end{aligned}
$$

<u>16.7 Proposition.</u> Let $\{x, y\}$ be a basis for $X$, and let $\bar{x}, \bar{y} \in X$. Then

a. $\exists$ unique linear $X \xrightarrow{f} X$ with $xf = \bar{x}$, $yf = \bar{y}$.

b. $f$ as above is bijective iff $\{\bar{x}, \bar{y}\}$ is a basis.

<u>Proof.</u> Seminar problem.

<u>16.8 Proposition.</u> Let $G$ be the collineation group of $(X, \mathcal{L})$. Then $(X, G, ev, \mathcal{L})$ is a geometry.

<u>Proof.</u> G1 and G2 are obvious from AF1 and G3 is clear from the definition of collineation.

<u>G4.</u> Let $L, L' \in \mathcal{L}$.

<u>Case I.</u> $L = L'$. Use $g = 1_X$.

<u>Case II.</u> $L \parallel L'$. Let $x \in L$, $x' \in L'$ and let $t$ be the unique translation such that $xt = x'$. Then $Lt = L'$.

<u>Case III.</u> $L \cap L' = \{0\}$. Let $x \in L$, $x \neq 0$, $x' \in L'$, $x' \neq 0$. By extending $x$ and $x'$ to bases and applying 16.7, $\exists$ linear bijection $X \xrightarrow{f} X$ with $xf = x'$. As $0f = 0$ and $f \in G$ (by 16.6), $Lf = L'$.

<u>Case IV.</u> $L \cap L' = \{x\}$. Define $t \in T$ by $xt = 0$.
$\therefore Lt \cap L't = \{0\}$. By case III $\exists f \in G$. $Ltf = L't$.
$\therefore Ltft^{-1} = L'$ so define $g = tft^{-1}$!

<u>G5</u>. Let $x, y \in L \in \mathcal{L}$. $\therefore L = xly$. Define $t \in T$ by $xt = y$. Then $Lt = L$. $\square$

Clearly, our affine plane $(X, \mathcal{L})$ has special nice properties. In the next section we will prove that, in fact, all affine planes are induced as a 2-dimensional vectorspace. This will prove, for instance, that 16.8 is true for any affine plane.

141 Seminar

Wednesday, Jan. 3, 1968, TG 202, 4:15 pm

1. (Bellenot) Prove 16.7.

2. (George) Prove the statement about T in 16.4.

3. (Greitzer) Find all collineations of the 4-element affine plane.

4. (Gulik). Prove AF3 in 16.2.

5. (Hamilton). Let $p$ be prime, $p>1$. Construct an affine plane with $p^2$ elements. Make some group-theoretic comments about G, D, T.

6. (Malcolm). What is G for the affine plane of this section.

7. (Ray). For the affine plane of this section is the following true? Two collineations agreeing on 3 non-collinear points are equal?

Merry Christmas

# Lecture 17

For this section let $(X, \mathcal{L})$ be an arbitrary affine plane with $G, D, T$ as in 13.1.

**17.1 Proposition.** $T$ is abelian.

**Proof.** We make free use of 13.5.

Let $t, u \in T$, $x \in X$. We show $xtu = xut$.

Case I. $t = 1_X$ or $u = 1_X$. This case is obvious.

Case II. $t \neq 1_X \neq u$, $dir(t) \neq dir(u)$. Then $xt|xtu \parallel x|xu \parallel xt|xut \Rightarrow xt|xtu = xt|xut$. Similarly, $xu|xtu = xu|xut$. $\therefore$ $xtu = xt|xtu \cap xu|xtu = xt|xut \cap xu|xut = xut$.

Case III. $t \neq 1_X \neq u$, $dir(t) = dir(u)$. $\exists\, s \in T - \{1_X\}$ with $dir(s) \neq dir(t)$. Since $dir(u^{-1}) = dir(u) = dir(t)$, if $dir(us) = dir(t)$ then $dir(t) = dir(us) = dir(u^{-1}(us)) = dir(s)$ $\times$, so indeed $dir(us) \neq dir(t)$. By Case II,

$$tus = t(us) = (us)t = u(st) = u(ts) = uts$$

$$tu = tuss^{-1} = utss^{-1} = ut. \quad \square$$

**17.2 Definition.** A scalar is a function $T \xrightarrow{\lambda} T$ satisfying

SC1. $\lambda$ is a group homomorphism.

SC2. $\forall t \in T.\ t\lambda \neq 1_X \Rightarrow dir(t\lambda) = dir(t)$.

The set of all scalars $=_{dn}$ $F$.

**17.3 Proposition.** The following are scalars.

a. $$\begin{array}{ccc} T & \xrightarrow{1} & T \\ t & \longmapsto & t \end{array}$$

b. $$\begin{array}{ccc} T & \xrightarrow{-1} & T \\ t & \longmapsto & t^{-1} \end{array}$$

c. $$\begin{array}{ccc} T & \xrightarrow{0} & T \\ t & \longmapsto & 1_X \end{array}$$

d. $$\begin{array}{ccc} T & \xrightarrow{d\circ - \circ d^{-1}} & T \\ t & \longmapsto & dtd^{-1} \end{array}$$ (any fixed $d \in D$).

**Proof.**

a. obvious

b. $(tu)^{-1} = u^{-1}t^{-1} = t^{-1}u^{-1}$ (by 17.1). $Dir(t) = dir(t^{-1})$ whenever $t^{-1} \neq 1_X$. $\therefore -1 \in F$.

c. obvious. (SC2 is vacuous).

d. Let $d \in D$. $\forall t, s \in T$ we have $d(ts)d^{-1}$ $= dt(d^{-1}d)sd^{-1} = (dtd^{-1})(dsd^{-1})$ which proves SC1. As $\operatorname{dir}(dtd^{-1}) = \operatorname{dir}(t)$ by 13.5, we have SC2.

**17.4 Proposition.** Let $\lambda \in F$, $\lambda \neq 0$ (the scalars "0", "1", "−1" are defined in 17.3 and will be used from now on). Let $x \in X$. Then $\exists$ unique $d \in D$ such that $xd = x$ and $\lambda = d^{-1}0 - od$.

**Proof.** $\forall y \in X$ let $t_{x,y}$ denote the unique translation such that $x t_{x,y} = y$. Define

$$\begin{aligned} X &\xrightarrow{\ d\ } X \\ y &\longmapsto \langle x, t_{xy}\lambda \rangle \end{aligned}$$

$d$ is not constant, and $xd = x$.

By hypothesis, $\lambda \neq 0$. $\therefore \exists\, t \in T$. $t\lambda \neq 1_X$. Since $t\lambda$ has no fixpoints (definition of $T$) $\langle x, t\lambda \rangle \neq x$. Clearly $t = t_{x,y}$ where $y =_{df} xt$. $\therefore yd = \langle x, t_{x,y}\lambda \rangle = \langle x, t\lambda \rangle$ $\neq x$. But $xd = \langle x, t_{x,x}\lambda \rangle = \langle x, 1_X \lambda \rangle$ $= \langle x, 1_X \rangle$ ($1_X\lambda = 1_X$ by SC1) $= x$. $\therefore yd \neq xd$ proving $d$ is not constant.

$d$ is a dilatation.

We use 13.4 (b). Let $y, z \in X$, $yd \neq zd$. We must show $y/z \parallel yd/zd$. $t_{xy} t_{yz} = t_{xz}$ because they both agree on $x$. By SC1 $\therefore$ $t_{xz}\lambda = (t_{xy}\lambda)(t_{yz}\lambda)$.

$\therefore$ $zd = \langle x, t_{x,z}\lambda \rangle = \langle x, (t_{x,y}\lambda)(t_{y,z}\lambda) \rangle$

$= \langle \langle x, t_{x,y}\lambda \rangle, t_{y,z}\lambda \rangle = \langle yd, t_{y,z}\lambda \rangle$

$\therefore$ $yd/zd = yd / (yd)(t_{y,z}\lambda) \parallel y/y(t_{y,z}\lambda)$ [By 13.5a]

Now $zd \neq yd \Rightarrow t_{y,z}\lambda \neq 1_X$. $\therefore$ By SC2,

$y/y(t_{y,z}\lambda) \parallel y/y\,t_{y,z} = y/z$.

$\lambda = d^{-1} \circ - \circ d$

Let $t \in T$. $t = t_{x,y}$ where $y =_{df} xt$.

$\therefore$ $yd = \langle x, t_{x,y}\lambda \rangle = \langle xd, t_{x,y}\lambda \rangle$ (as $xd = x$)

so $y = \langle xd, t\lambda \rangle d^{-1}$.

$\therefore$ $\langle x, d(t\lambda)d^{-1} \rangle = \langle xd, t\lambda \rangle d^{-1} = y$. But

$d(t\lambda)d^{-1} \in T$ (by 13.5 d) so $d(t\lambda)d^{-1} = t_{x,y} = t$.

$\therefore$ $t\lambda = d^{-1} t d$ as desired.

$d$ is unique

Suppose also $h \in D$, $xh = x$, $\lambda = h^{-1} \circ - \circ h$. Let $y \in X$.

Then $yh = \langle x, t_{x,y}\rangle h = \langle xh^{-1}, t_{x,y}\rangle h$ (as $xh^{-1} = x$)

$= \langle x, h^{-1}t_{x,y}h\rangle = \langle x, t_{x,y}\lambda\rangle = yd.$

The proof is complete. $\square$

Note that in view of 17.3 d and 17.4 we have completely characterized F: $F - 0$ is in 1-to-1 correspondence with the dilatations leaving any point of X fixed. À propos of this, look back at Dec. 6 seminar, problem 3.

<u>17.5 Proposition.</u> For $\lambda \in F$, $T \xrightarrow{-1} T \xrightarrow{\lambda} T = T \xrightarrow{\lambda} T \xrightarrow{-1} T$ and both are denoted "$-\lambda$". Define

$$\begin{array}{ccc} F \times F & \xrightarrow{+} & F \\ \lambda, \mu & \longmapsto & \lambda + \mu \end{array} \quad \text{where} \quad \begin{array}{ccc} T & \xrightarrow{\lambda+\mu} & T \\ t & \longmapsto & X \xrightarrow{t(\lambda+\mu)} X = X \xrightarrow{t\lambda} X \xrightarrow{t\mu} X \end{array}$$

$$\begin{array}{ccc} F \times F & \xrightarrow{\cdot} & F \\ \lambda, \mu & \longmapsto & \lambda\mu \end{array} \quad \text{where } \lambda\mu = T \xrightarrow{\lambda} T \xrightarrow{\mu} T.$$

Then $(F, +, \cdot)$ is a division ring and the symbols "0", "$-\lambda$", "1" are as usual.

Proof. We first show $(F, +)$ is an abelian group.

Since the product of translations is a translation, $\lambda + \mu$ is at least a well-defined function from $T$ to $T$. $\lambda + \mu = \mu + \lambda$ is immediate from 17.1. Since $\langle tu, \lambda+\mu \rangle = \langle tu, \lambda \rangle \circ \langle tu, \mu \rangle = \langle t, \lambda \rangle \circ \langle u, \lambda \rangle \circ \langle t, \mu \rangle \cdot \langle u, \mu \rangle = \langle t, \lambda \rangle \circ \langle t, \mu \rangle \circ \langle u, \lambda \rangle \circ \langle u, \mu \rangle$ (By 17.1) $= \langle tu, \lambda \rangle \circ \langle tu, \mu \rangle$, $\lambda + \mu$ satisfies SC1. That $\lambda + \mu$ satisfies SC2 is immediate from 13.5d.

$\therefore \lambda + \mu \in F$. $(\lambda + \mu) + \nu = \lambda + (\mu + \nu)$ is immediate from the fact that $T$ is a group (that is because composition of functions from $X$ to $X$ is associative). That $\lambda + 0 = \lambda = 0 + \lambda$ is obvious. Accepting for the moment that "$-\lambda$" is well-defined and in $F$, we have that $\forall t \in T$. $\langle t, \lambda + (-\lambda) \rangle = t\lambda \circ (t\lambda)^{-1} = 1_X = \langle t, 0 \rangle$ so $\lambda + (-\lambda) = 0$.

$(F, \cdot)$ is a monoid

$\langle tu, \lambda\mu \rangle = \langle \langle tu, \lambda \rangle, \mu \rangle = \langle \langle t, \lambda \rangle \circ \langle u, \lambda \rangle, \mu \rangle =$
$\langle t, \lambda \rangle \mu \circ \langle u, \lambda \rangle \mu = \langle t, \lambda\mu \rangle \circ \langle \mu, \lambda\mu \rangle$ shows $\lambda\mu$ satisfies SC1. [This is only the usual proof that a composition of two group homomorphisms is a group homomorphism]. If $\langle t, \lambda\mu \rangle \neq 1_X$ then $\langle t, \lambda \rangle \neq 1_X$ so $\operatorname{dir}(t) = \operatorname{dir}(t\lambda) = \operatorname{dir}((t\lambda)\mu) = \operatorname{dir}(t(\lambda\mu))$. $\therefore \lambda\mu \in F$.

[In particular $(-1)\lambda$, $\lambda(-1) \in F$. To say $(-1)\lambda = \lambda(-1)$ is equivalent to saying $\forall t \in T$. $t^{-1}\lambda = (t\lambda)^{-1}$ which is true if $\lambda \in F$ by SC1. This fills in the gap of the first paragraph]. That "$\cdot$" is associative is clear since composition of functions from $T$ to $T$ is associative. That $1\lambda = \lambda = \lambda 1$ is obvious. $\therefore$ $(F, \cdot)$ is a monoid with unit $1$.

The distributive laws are valid

$\langle t, \alpha(\lambda+\mu)\rangle = \langle t\alpha, \lambda+\mu\rangle = \langle t\alpha, \lambda\rangle \circ \langle t\alpha, \mu\rangle$
$= \langle t, \alpha\lambda\rangle \circ \langle t, \alpha\mu\rangle = \langle t, \alpha\lambda + \alpha\mu\rangle$ so

$$\alpha(\lambda+\mu) = \alpha\lambda + \alpha\mu.$$

$\langle t, (\lambda+\mu)\alpha\rangle = \langle t, \lambda+\mu\rangle\alpha = (t\lambda \circ t\mu)\alpha$
$= \langle t, \lambda\alpha\rangle \circ \langle t, \mu\alpha\rangle$ (by SC1 for $\alpha$)
$= \langle t, \lambda\alpha + \mu\alpha\rangle$

so $(\lambda+\mu)\alpha = \lambda\alpha + \mu\alpha$.

$\lambda \neq 0 \Rightarrow \lambda^{-1}$ exists.

By 17.4 $\exists d \in D$. $\lambda = d^{-1}\circ - \circ d$. $\lambda^{-1} =_{df} d \circ - \circ d^{-1}$.
Then $\lambda^{-1} \in F$ (17.3 d) and clearly $\lambda\lambda^{-1} = 1 = \lambda^{-1}\lambda$.

$\square$

17.6 Propositions. Fix $a \in X$. $\forall x \in X$ let $t_x$ be the unique translation sending $a$ to $x$ (using 17.2 c). Define functions

$$\begin{array}{ccc} X \times X & \xrightarrow{\quad + \quad} & X \\ x, y & \longmapsto & \langle x, t_y \rangle \end{array}$$

$$\begin{array}{ccc} X \times F & \xrightarrow{\quad \pi \quad} & X \\ x, \lambda & \longmapsto & x\lambda =_{df} \langle a, t_x \lambda \rangle \end{array}$$

The following statements are valid.

a. $t_{x+y} = t_x t_y$ and $t_{x\lambda} = t_x \lambda$.

b. $(X, +, \pi)$ is a vectorspace over $F$. The additive $0$ is $a$, $-x = x(-1) = \langle a, (t_x)^{-1} \rangle$.

c. The dimension of $(X, +, \pi)$ is $2$.

d. The affine plane structure induced on $X$ as in lecture 16 is exactly the original $(X, \mathcal{L})$.

Hence every affine plane arises as in lecture 16 and may be studied by "analytic geometry" methods.

Proof.

a. $\langle a, t_x t_y \rangle = \langle x, t_y \rangle = x+y \Rightarrow t_x t_y = t_{x+y}$

$\langle a, t_x \lambda \rangle = x\lambda \Rightarrow t_x \lambda = t_{x\lambda}$.

We use these identities freely during the rest of the proof.

**b.** $(X,+)$ is an abelian group with $0 = a$, $-x = \langle a, t_x^{-1}\rangle$

$x+y = \langle a, t_x t_y\rangle = \langle a, t_y t_x\rangle$ (by 17.1) $= y+x$

$(x+y)+z = \langle\langle x, t_y\rangle, t_z\rangle = \langle x, t_y t_z\rangle = \langle x, t_{y+z}\rangle = x+(y+z)$

$a+x = \langle a, t_x\rangle = x$

$x + \langle a, (t_x)^{-1}\rangle = \langle\langle a, (t_x)^{-1}\rangle, t_x\rangle = \langle a, t_x^{-1} t_x\rangle = \langle a, 1_X\rangle = a$

$(x+y)\lambda = x\lambda + y\lambda$

$(x+y)\lambda = \langle a, t_{x+y}\lambda\rangle = \langle a, (t_x t_y)\lambda\rangle = \langle a, t_x\lambda \circ t_y\lambda\rangle$ (SC1)

$= \langle\langle a, t_{x\lambda}\rangle, t_{y\lambda}\rangle = \langle x\lambda, t_{y\lambda}\rangle = x\lambda + y\lambda$.

$x(\lambda+\mu) = x\lambda + x\mu$

$x(\lambda+\mu) = \langle a, t_x(\lambda+\mu)\rangle = \langle a, t_x\lambda \circ t_x\mu\rangle$ (definition of "$\lambda+\mu$")

$= \langle x\lambda, t_{x\mu}\rangle = x\lambda + x\mu$.

$x(\lambda\mu) = (x\lambda)\mu$

$x(\lambda\mu) = \langle a, t_x(\lambda\mu)\rangle = \langle a, (t_x\lambda)\mu\rangle$ (definition of "$\lambda\mu$")

$= \langle a, t_{x\lambda}\mu\rangle = (x\lambda)\mu$.

$x1 = x$

$x1 = \langle a, t_x 1\rangle = \langle a, t_x\rangle = x$

**c.** Let $x, y \in X$ with $\{a, x, y\}$ noncollinear. We will show that $\{x, y\}$ is a basis.

Suppose $x\lambda + y\mu = a$. $\therefore$ $a = \langle x\lambda, t_{y\mu}\rangle$ $= \langle a, t_x\lambda \circ t_y\mu\rangle$ so that $t_x\lambda \circ t_y\mu = 1_X$ and $t_x\lambda = (t_y\mu)^{-1}$.

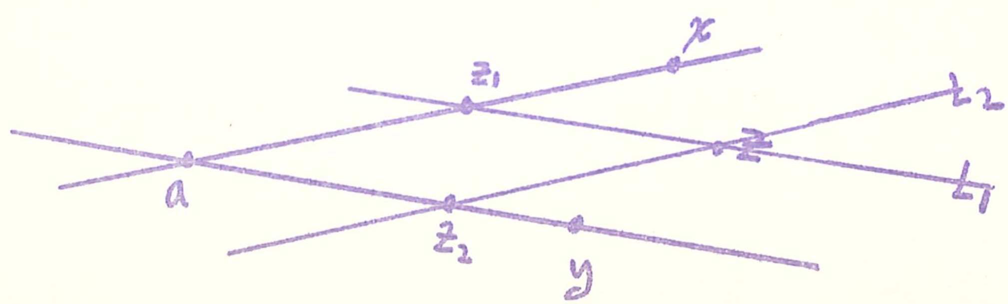
Suppose $\lambda \neq 0$. It follows from 17.4 and 6.7 that $T \xrightarrow{\lambda} T$ is a group automorphism and in particular $\lambda$ is 1-to-1. $\therefore$ $t_x \neq 1_X \Rightarrow t_x\lambda \neq 1_X$ and then $t_y\mu = (t_x\lambda)^{-1} \neq 1_X$. By SC 2 we have $\operatorname{dir}(t_x) = \operatorname{dir}(t_x\lambda) = \operatorname{dir}(t_x\mu)^{-1} = \operatorname{dir}(t_y\mu) = \operatorname{dir}(t_y)$ $\times$.
$\therefore$ $\lambda = 0$. Similarly $\mu = 0$. This proves $\{x, y\}$ is independent.

Now let $z \in X$. We must find $\lambda, \mu$ with $z = x\lambda + y\mu$. For intuition in the following proof think of the way two non-parallel lines ($a|x$ and $a|y$) coordinatize the Euclidean plane.

<u>Case I.</u> $z = a$, set $\lambda = 0 = \mu$.

<u>Case II.</u> $z \neq a$, $z \in a|x$ or $z \in a|y$. Suppose $z \in a|x$ (the proof for $z \in a|y$ is similar). By 19.1 $\exists$ unique $d \in D$ with $ad = a$ and $zd = x$ (if $z = x$ use $d = 1_X$). $\lambda \stackrel{df}{=} d \circ - \circ d^{-1} \in F$. Then $x\lambda = \langle a, t_x\lambda\rangle = \langle a, d \circ t_x \circ d^{-1}\rangle = \langle a, t_x\rangle d^{-1} = xd^{-1} = z$.

<u>Case III.</u> $z \notin a|x \cup a|y$



Define $L_1, L_2$ by $z \in L_1 \parallel a/y$, $z \in L_2 \parallel a/x$

and then $z_1 =_{df} a/x \cap L_1$, $z_2 =_{df} a/y \cap L_2$.

By this construction, $z = \langle z_1, t_{z_2} \rangle = z_1 + z_2$.

By case II $\exists\, \lambda, \mu \in F$ with $z_1 = x\lambda$, $z_2 = y\mu$.

$\therefore$ $z = z_1 + z_2 = x\lambda + y\mu$.

<u>d.</u> For $x \neq y \in X$ let $x/_{\mathcal{L}}y =_{dn}$ the line $x/y \in \mathcal{L}$

and let $x/_F y =_{dn}$ the line $\{x + (y-x)\lambda : \lambda \in F\}$ as in

lecture 16. To prove d amounts to proving $x/_{\mathcal{L}} y = x/_F y$

for all $x \neq y \in X$. We prove this first for lines through

$a = 0$.

$$a/_F x = \{0 + (x-0)\lambda : \lambda \in F\} = \{x\lambda : \lambda \in F\} = \{\langle a, t_x\lambda \rangle : \lambda \in F\}.$$

$a/_F x \subset a/_{\mathcal{L}} x$ : if $\lambda = 0$, $\langle a, t_x\lambda \rangle = \langle a, t_x \rangle = x \in a/_{\mathcal{L}} x$

if $\lambda \neq 0$, $t_x\lambda \neq 1_X$ & $\operatorname{dir}(t_x\lambda) = \operatorname{dir}(t_x)$ so

that $a/_{\mathcal{L}} at_x \parallel a/_{\mathcal{L}} at_x\lambda \Rightarrow a/_{\mathcal{L}} x = a/_{\mathcal{L}} x\lambda$ so $x\lambda \in a/_{\mathcal{L}} x$.

$a/_{\mathcal{L}} x \subset a/_F x$ : Let $y \in a/_{\mathcal{L}} x$. If $y = a$, $y \in a/_{\mathcal{L}} x$. otherwise

$\exists$ unique $d \in D$, $ad = a$, $yd = x$. As $y = xd^{-1} = \langle a, dt_x d^{-1} \rangle$,

$y = x\lambda$ where $\lambda =_{df} d \circ \neg d^{-1}$ and so $y \in a/_F x$.

Otherwise consider the case $x \neq a \neq y$. Define $t \in T$

by $xt = a$. We have

$$(x|_{\mathcal{L}}y)t = a|_{\mathcal{L}}yt = a|_F yt$$, and hence

$$x|_{\mathcal{L}}y = (a|_{\mathcal{L}}yt)t^{-1} = (a|_F yt)t^{-1} = (a|_F yt)t_x$$

$$= a|_F yt + x = a+x|_F yt+x$$

$$= x|_F ytt^{-1} = x|_F y.$$

The proof is complete. $\square$

<u>17.7 Proposition.</u> Let $(X, \mathcal{L})$ be an affine plane. Let $G$ be the collineation group of $(X, \mathcal{L})$. Then $(X, G, ev, \mathcal{L})$ is a geometry.

<u>Proof.</u> Immediate from 17.6 and 16.8 $\square$

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Meet Jan. 10, wednesday, TG 202, 4:15 pm.

No seminar. General lecture-discussion to fill in some intuitive comments left out of the notes. Final set of notes handed out. Last class on Jan. 17 when 6 hour take-home final (due Jan. 29) will be handed out.

Geometry 141
December 13, 1967
Final Exam part I.

Due date: Wednesday, January 17, 1968, 4:15 pm
in 103 TG.

Work all seven problems. You will probably want to solve them in an order different from the one given. You may use the course notes or any published book or manuscript. No communication of course. All statements not appearing in the course notes must be proved before being used. Grade will be based 50% on mathematical content, 50% on expository content and readability. Logical gaps in proofs will be sharply scanned for. Good luck.

Definition. A projective plane is a pair $(X, \mathcal{L})$ such that $X$ is a set, $\mathcal{L}$ is a collection of subsets of $X$ and the following three axioms hold:

AF1 PR1. If $x, y \in X$ and $x \neq y$ then $\exists$ unique $L \in \mathcal{L}$ with $x, y \in L$.

NAF2 PR2. If $L, M \in \mathcal{L}$ and $L \neq M$ then $\exists$ unique $x \in X$ with $x \in L \cap M$.

AF3 < PR3. $\exists$ four distinct points in $X$ no three of which lie on any line.

PR4. $\exists$ four distinct lines in $\mathcal{L}$ no three of which have a common point.

<u>Definition.</u> Let $(X, \mathcal{L})$ be a projective plane. A <u>projective collineation of</u> $(X, \mathcal{L})$ is a bijection $X \xrightarrow{f} X$ such that $\forall L \in \mathcal{L}$. $Lf \in \mathcal{L}$.

<u>Definition.</u> Let $(X, \mathcal{L})$, $(Y, \mathcal{M})$ be projective planes. $(X, \mathcal{L})$ and $(Y, \mathcal{M})$ are <u>isomorphic</u> if $\exists$ a bijection $X \xrightarrow{f} Y$ such that $\forall L \subset X$, $L \in \mathcal{L}$ iff $Lf \in \mathcal{M}$.

<u>Definition.</u> Let $(X, \mathcal{L})$, $(Y, \mathcal{M})$ be preaffine planes. $(X, \mathcal{L})$ and $(Y, \mathcal{M})$ are <u>isomorphic</u> if $\exists$ a bijection $X \xrightarrow{f} Y$ such that $\forall L \subset X$, $L \in \mathcal{L}$ iff $Lf \in \mathcal{M}$.

## <u>TEST PROBLEMS</u>

1. Let $(X, \mathcal{L})$ be a projective plane such that $X$ has $n$ elements for $n$ an integer.

   a. Find the number of points on a line.

   b. Find the number of lines through a point.

   c. Discuss the possible values of $n$.

2. Let $(X, \mathcal{L})$ be a projective plane and let $L \in \mathcal{L}$ be fixed. Define $X_0 =_{df} X \neg L$, $\mathcal{L}_0 =_{df} [L \cap X_0 : L \in \mathcal{L}]$.

   ~~a. Prove~~ $(X_0, \mathcal{L}_0)$ is a preaffine plane.

   ?b. Show that any two $(X_0, \mathcal{L}_0)$'s defined by two $L$'s in $\mathcal{L}$ are isomorphic.

3. Let $(X, \mathcal{L})$ be a preaffine plane. Show that $\exists$ a projective plane $(\bar{X}, \bar{\mathcal{L}})$ such that $((\bar{X})_0, (\bar{\mathcal{L}})_0) = (X, \mathcal{L})$ and show that any two such projective planes are isomorphic. [Hint: adjoin a "point at infinity" for each equivalence class of parallel lines.]

4. Let $(X, \mathcal{L})$ be the four-element preaffine ~~plane~~ (course notes 12.3). construct $(\bar{X}, \bar{\mathcal{L}})$ as in problem 3.

5. Let $(X, \mathcal{L})$ be a projective plane and define $G =_{df}$ the set of projective collineations of $(X, \mathcal{L})$. Assume that $(X_0, \mathcal{L}_0)$ as in problem 2 is not only a preaffine plane but in fact an affine plane.

→ a. Prove $G$ is a subgroup of $bij(X)$.

b. Prove $(X, G, ev, \mathcal{L})$ is a geometry in the sense of course notes 10.1.

C4, C5?

6. Let $(X, \mathcal{L})$, $G$, $(X_0, \mathcal{L}_0)$ be exactly as in problem 5. $D_0 =_{df}$ the dilatation group of $(X_0, \mathcal{L}_0)$. Prove that $G_0 =_{df} [g \in G : \forall x \in L .\ xg = x]$ (where $L$ is the element of $\mathcal{L}$ used to construct $(X_0, \mathcal{L}_0)$) is a subgroup of $G$ isomorphic to $D_0$.

7. $S =_{df}$ the surface of a sphere of radius 1 in Euclidian 3-space. $R =_{df} \{(x, y) : x = y$ or the points $x, y$ are collinear with the center of the sphere (i.e. lie on a diameter)$\} \subset S \times S$

a. Show $R$ is an equivalence relation on $S$.

Let $X =_{df} S/R$ and let $S \xrightarrow{\theta} X$ be the canonical projection. $\mathcal{L} =_{df} [\mathbb{C}\theta : \mathbb{C} \subset S, \mathbb{C}$ is a great circle$]$.

→ b. Show $(X, \mathcal{L})$ is a projective plane.

? c. Show $(X_0, \mathcal{L}_0)$ is isomorphic to the Euclidian affine plane.

## PART A.

1. Which of the following are preaffine planes?

   a. $X = \mathbb{R}, \mathcal{L} = \{\mathbb{R}\}$.

   b. $X = \mathbb{R}^2, \mathcal{L} =$ usual straight lines.

   c. $X =$ surface of sphere, $\mathcal{L}$ = great circles.

   d. $X = \{(x,y,z): x^2 + y^2 = 1\} \subset \mathbb{R}^3$ (hence $X$ is the surface of an infinitely-long right circular cylinder), $\mathcal{L} = \{P \cap X: P$ is a plane in $\mathbb{R}^3\}$.

2. Which of the following $(X,G,\Pi,\mathcal{L})$'s are geometries?

   a. $X = \mathbb{R}$, $G$ = all d-isometries where $d(x,y) = |x - y|$, $\Pi$ = evaluation, $\mathcal{L} = \{\mathbb{R}\}$.

   b. $X = \mathbb{R}$, $G$ = any subgroup of $\mathrm{bij}(X)$ such that if $x,y \in X$ then $\exists\, g \in G$ with $xg = y$, $\Pi$ = evaluation, $\mathcal{L} = \{\mathbb{R}\}$.

   c. $X$ = surface of sphere, $G$ = group of all rotations (about all axes, through all angles), $\Pi$ = evaluation, $\mathcal{L}$ = great circles.

   d. $X = \mathbb{R}^2$, $G = \mathrm{bij}(X)$, $\Pi$ = evaluation, $\mathcal{L}$ = all countable subsets of $X$.

END OF PART A.

continued on next sheet -

PART B.

## Problem I. Midpoints and half-turns.

Let $(X, \mathcal{X})$ be an affine plane with dilatation group $D$ and translation group $T$. $h \in D$ is a half-turn if there exist $x \neq y \in X$ with $xh = y$ and $yh = x$. If $x \neq y \in X$, a midpoint of $(x,y)$ is a point $m \in X$ such that there exists $t \in T$ with $xt = m$, $mt = y$.

a. Let $m$ be a midpoint for $(x,y)$. Prove $m$ is a midpoint for $(y,x)$ and prove $m \in x/y$.

b. Prove that if $m \neq y \in X$ then there exists $x \in X$ with $m$ a midpoint of $(x,y)$.

c. If $x \neq y$ prove there exists a unique half-turn with $xh = y$ and $yh = x$. [Hint: A short proof follows from (b.)].

Now we make the following group-theoretic assumptions about $T$:

1. $T$ has square roots (that is if $t \in T$ there exists $u \in T$ with $u^2 = t$).
2. $T$ has no elements of order 2 (that is $t = t^{-1}$ is impossible unless $t = 1_X$).

d. Prove that if $x \neq y \in X$ then there exists unique $m \in X$ with $m$ a midpoint of $(x,y)$.

e. Let $x \neq y \in X$, let $m$ be the midpoint of $(x,y)$ and let $h$ be the half-turn which interchanges $x$ and $y$. Prove that $mh = m$. What other point of $X$ does $h$ leave fixed?

Porblem II. Uniqueness of collineations in the Euclidian plane.

a. ~~Let~~ $(X, \mathcal{L})$ be an affine plane, let $L \in \mathcal{L}$ and let $x \in X - L$. Prove that two collineations agreeing on $L \cup \{x\}$ are equal.

b. Let $X = \mathbb{R}^2$, $\mathcal{L}$ = usual straight lines. If $x \neq y \in X$ let $\overline{xy}$ denote the segment of points between $x$ and $y$ as shown below:


Assume without proof

Darboux's Lemma. If $g$ is a collineation of $(\mathbb{R}^2, \mathcal{L})$ then whenever $x \neq y \in X$, $(\overline{xy})g = \overline{(xg)(yg)}$.

Prove that if two collineations of $(\mathbb{R}^2, \mathcal{L})$ agree on three noncollinear points then they are equal.