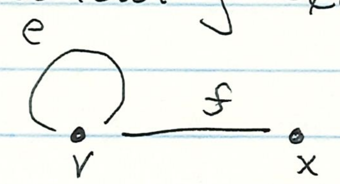
9 FEB 87.

What we would like to be true is
v a cut vertex $\Longleftrightarrow$ $\{v\}$ is a vertex cut
But the following example shows this is
false



v is a cut vertex ($E_1 = \{e\}$, $E_2 = \{f\}$) but $G-v$ is connected so $\{v\}$ isn't a vertex cut.

Lemma 1: If G is connected, v is a cut vertex and no loops are incident at v, Then $\{v\}$ is a vertex cut.

Lemma 2: If G is connected, $\{v\}$ is a vertex cut, then v is a cut vertex.

Lemma 3: If G is connected, $\nu(G) \geq 3$ and e is a cut edge of G, then one of the ends of e is a vertex cut [hence a cut vertex by Lemma 2]

Proof of 1: Let E_1 & E_2 be a partition of $E(G)$ so that $G[E_1]$ & $G[E_2]$ only have v in common as required by v being a cut vertex. Since $v \in G[E_1]$, there is an edge $e_1 \in E_1$ incident to v. Since e_1 can't be a loop, there is another vertex $x_1 \neq v$ with x_1 the other end of e_1 & hence $x_1 \in G[E_1]$. Similarly there is $x_2 \in V(G[E_2]) - v$.

~~Since G is connected~~

If $\{v\}$ isn't a vertex cut, Then $G-v$ is still connected and hence There is a path P from x_1 to x_2 say $(x_1 = u_0) f_1 u_1 f_2 \dots f_n (u_n = x_2)$. Now $f_1 \notin E_2$ or else $x_1 \in G[E_2]$ and $G[E_1] \cap G[E_2]$ wouldn't just be v. Similarly $f_n \notin E_1$. Thus for some i, $f_i \in E_1$ & $f_{i+1} \in E_2$. But this too implies $u_i \in G[E_1] \cap G[E_2]$ and $u_i \neq v$ since P is in $G-v$.

Pf of lemma 2: Since $G-v$ is disconnected let V_1 be the vertices of one of the component and $V_2 = V(G) - V_1 - v$. For each $e \in E(G)$ put e into E_1 if both ends of ~~one~~ e are in V_1 or both ends of e are v or one end of e is v and the other end is in V_1. Let $E_2 = E(G) - E_1$.

We know V_1, V_2 are non-empty (Since $G-v$ is disconnected). Let P be a path in G connecting some point in V_1 with some point in V_2. This P doesn't exist in $G-v$ (since V_1 is the set of vertices of one component. It follows that v is on P and a section of P has the form $x e v f y$ with $x \in V_1$, $e \in E_1$, $f \in E_2$ $y \in V_2$. Thus v is a vertex in $G[E_1] \cap G[E_2]$.

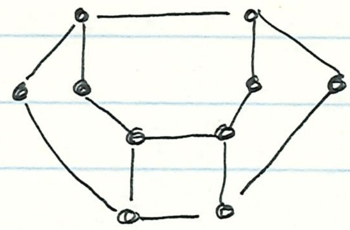
Suppose u is also in $G[E_1] \cap G[E_2]$. We know $G[V_1]$ and hence $G[E_1]$ is connected. Hence $u \in V_1$ or $u = v$. But if $u \in G[E_2]$ & $u \neq v$ then $u \in V_2$ since $G[V_2 + v] \supseteq G[E_2]$. But $V_1 \cap V_2 = \emptyset$ by construction.

Pf of lemma 3: Let e be a cut edge of G with ends x & y and let z be another ($\nu \geq 3$) vertex of G. One of x or y isn't in the same component of $G-e$, say it is x. Since G is connected, there a paths x to z. But since x & z are not connected in $G-e$, each such path uses e and hence y. Thus x & z are not connected in $G-y$. $\therefore \{y\}$ is a vertex cut & a cut vertex.

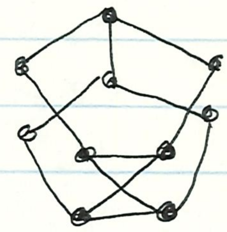
Old Discrete Tests

1 Either produce an isomorphism between G and G′ or prove none exists.

G



G′



2. Prove a cut-edge of a connected graph is in no ~~ckt~~ ~~closed~~ ~~cyke~~ cycle

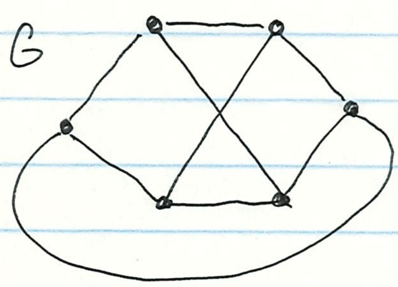
3. Prove a graph is connected if and only if it has a spanning tree.

4 Prove a connected graph G with $k = \varepsilon - \nu + 2 \geq 0$ then G has at least k spanning trees. [Hint induction on k (delete an edge in a circuit)]

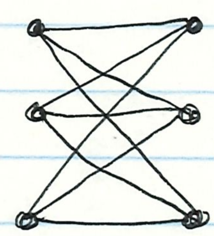
5. Draw all trees with 5 edges. No two trees in your list can be isomorphic

6 Between all pairs G, G′, G″ below, either produce an isomorphism or show none exists

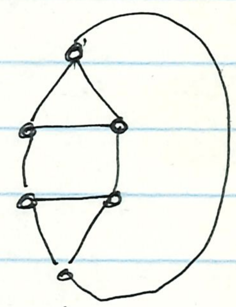
G



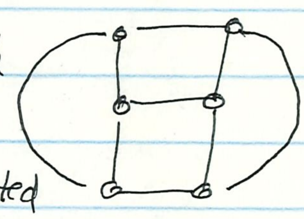
G′



G″



7. Give examples. A. Two non-isomorphic loop-free graphs with degree sequence (1, 1, 2, 2, 3, 3)
B A maximal disconnected subgraph of graph H with 5 edges

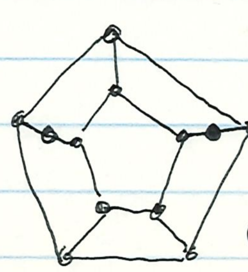


H

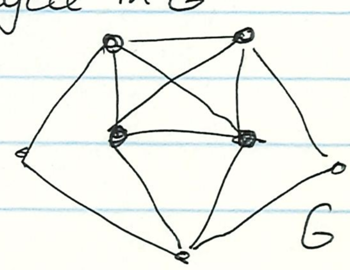
8. Give counterexamples: A. Each connected graph with a cut-edge has a cut-vertex. B. A tree with 5 or more edges has either at least 4 vertices of degree 1 or all vertices of degree ≤ 2

9. Find an Euler tour and a Hamilton cycle in G

10 Carefully show G′ has no Hamilton cycle



G′



G

Other Problems

11. Find a 3-regular graph with $k=2$ (Draw it)
12. Prove e is a cut edge of G if and only if there exist vertices x and y s.t. every path P from x to y use e.
13. Prove a connected graph with only cycles of even length is bipartite.
14. Give an example of a 2-connected graph with an even number of vertices and with a maximal matching which ~~its~~ isn't a ~~pre~~ perfect matching.
15. ~~two~~ Derive a formula between ε, ν, ω for a forest.
16. Draw all blocks with 5 edges.
17. Let G be a connected graph with ~~non-ad~~, x & y two vertices which are non-adjacent in G. Define

    $\delta_{xy} = \min\{d(x), d(y)\}$.

    K_{xy} = min number of vertices in $V-\{x,y\}$ whose removal disconnects x from y.

    K'_{xy} = min number of edges whose removal disconnects x from y.

    Prove $K_{xy} \leq K'_{xy} \leq \delta_{xy}$

    Give examples to show $K_{xy} < K'_{xy}$ and $K'_{xy} < \delta_{xy}$ are possible.
18. Show a graph is 2-edge connected if and only if any two vertices are connected by at least two edge-disjoint paths.
19. Give an example to show if P is a (u,v)-path in a 2-connected graph ~~G~~ G, then G does not necessarily contain a (u,v)-path Q internally disjoint from P.
20. Give counterexamples: A. If for each $v \in V$, $\omega(G) = \omega(G-v)$, then G is connected B. If there are two distinct (x,y)-paths in G then x and y lie on a common cycle. C. Every loop-free graph is a subgraph of some K_n. D. A tree

If e & f are edges of a connected graph, then there is a path P whose first edge is e and whose last edge is f.

Disprove:
A vertex on a cycle isn't a cut vertex
A vertex on a cycle with degree 2 isn't a cut-vertex.

A k-regular bipartite graph is ~~b~~ 2-connected for $k \geq 2$ (1-connected.)

Draw a graph which isn't K_n

If G has n vertices, and