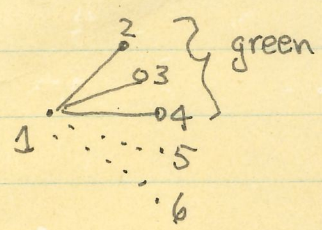
Ramsey's Theorem

Jan [illegible]31

If you ask a combinatorist what sort of object is Ramsey's Theorem he will say that it is a generalization of the pigeon-hole principle which includes the following statement: ~~An~~ In any group of six people there are either 3 people who know each other or 3 people who do not know either of the other two.

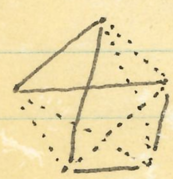
Replacing people with vertices of a the comp[illegible] graph [illegible] 6 vertices and coloring the edges red or green ~~if~~ de[illegible] on wheather the pair knows or does not know each other Ramsey theorem implies there is a triangle of one color as a subgraph.

This is easy to see. Each vertex has 5 edges so at least three are of same color (say green). Now



if any of the edges (2,3), (3,4) or (4,2) are green this gives a green $\triangle$ with 1. Otherwise $\triangle 234$ is red.

It is also easy to see that this need [illegible]t happen with 5 vertices as the following graph shows.



There is a finite and infinite version of Ramsey's Theorem (as well as a third way of stating it)

[illegible]sey Theorem: (finite form) For each k, j and integers [illegible] ,.... q_j [illegible] ≥ k there is an integer m [illegible] $(q_1, q_2, \ldots q_j, k, j)$) so tha[illegible] if f is any [illegible]om $P_k(\{1, \ldots m\})$ to $\{1, \ldots j\}$ [illegible]-element subsets) then [illegible] [illegible] and a q_i-element s[illegible] [illegible]) ≡ i [illegible]

Some observations:

(1) Let $q = \max\{q_i : 1 \le i \le j\}$, then $R(q_1, \ldots q_j, k, j) \le R(q \ldots q, k, j)$ so we may assume that the q_i's are equal, we do not because of the proof.

(2) ~~The~~ existence of $R(q_1, \ldots q_j, k, j)$ follows from the existence $\forall\, q_1, q_2 \ge k$ of $R(q_1, q_2, k, 2)$

i.e. by first grouping the values $\{2,3\}$ we have
$R(q_1, q_2, q_3, k, 3) \le R(q_1, R(q_2, q_3, k, 2), k, 2)$
etc.

Proof: Thus it suffices to show $R(q_1, q_2, k, 2)$ exists for $q_1, q_2 \ge k$. (Obviously $R(q_1, k, 1) = q_1$)

For $k = 1$, this is the pigeon-hole principle
$R(q_1, q_2, 1, 2) = q_1 + q_2 - 1$.

Two special cases if $q_1, q_2 \ge k$

(A) $R(q_1, k, k, 2) = q_1$

(B) $R(k, q_2, k, 2) = q_2$ (resp. (B))

This because in (A), if any k-tuple is mapped to 2 (resp. 1) this is the require subset, otherwise the whole set is map to 1 (resp 2) and the whole set works.

~~First~~ Induction follows from

$R(q_1, q_2, k, 2) \le R(R(q_1 - 1, q_2, k, 2), R(q_1, q_2 - 1, k, 2), k-1, 2) + 1$

To see this let f be a function $\to$ ~~{red, gr~~ $\{1,2\}$ on the k-element subsets of a set A of the size on the right (or bigger). Let $*$ be any element in this set A and let $B = A \setminus \{*\}$.
Let $g : \mathcal{P}_{k-1}(B) \to \{1,2\}$ be defined by $g(x) = f(x \cup \{*\})$.

either $\exists\, C$ with $\ge R(q_1 - 1, q_2, k, 2)$ that g maps to 1

or $\exists\, C$ with $\ge R(q_1, q_2 - 1, k, 2)$ that g maps to 2

(By symmetry it does not matter, say the first) Then

either $\exists\, D \subset C$ with $\ge q_1 - 1$ that f maps to 1

and $D \cup \{*\}$ is the required set

or $\exists\, D \subset C$ with $\ge q_2$ that f maps to 2

and D is the required set.

[illegible]tion works [illegible] $(q_1, q_2, k) \ge (p_1, p_2, \ell)$ if $k > \ell$

[illegible]

The Ramseys numbers are not determined other than

$R(3,3,2,2) = 6$

$R(3,4,2,2) = 9$

$R(4,4,2,2) = 18$

$R(3,5,2,2) = 14$

$R(3,3,3,2,3) = 17.$

1963 (Latter books do not list any more)

<u>Infinite</u> <u>Ramsey</u>, for all positive integers k, If $f : \mathcal{P}_k(\mathbb{N}) \to \{1, \dots j\}$ then there is $A \subset \mathbb{N}$ with A infinite and $f | \mathcal{P}_k(A) \equiv$ constant.

<u>Proof:</u> As before the case $j > 2$ follows the case $j = 2$. and the case ~~$j=1$~~ $k = 1$ is the pigeon-hole principle. (The case $j = 1$ is obvious with $A = \mathbb{N}$).

Infinite Ramsey 2-subsets divided between red & green:

Given: $f: \mathcal{P}_2(\mathbb{N}) \to \{red, green\}$

Define: for $n \in \mathbb{N}$ R_n (resp. G_n] to be the set $\{i > n : f(\{n, i\}) = red \text{ (resp. green)}\}$.

Claim 1: If there is an infinite $A \subseteq \mathbb{N}$ so that for each $n \in A$ $A \setminus G_n$ is finite, then there is an infinite $B \subseteq A$ with $f \mid \mathcal{P}_2(B) \equiv$ green.

pf of Claim 1: Let $n_1 \in A$ and inductively choose $n_{k+1} \in A \cap (\bigcap_{i=1}^{k} G_{n_i})$. The co-finite hypothese implies that n_{k+1} can be choosen. Let $B = \{n_i\}_{i=1}^{\infty}$, if $a, b \in B$ and $a < b$ then $b \in G_a$ and $f(\{a, b\}) =$ green.

Consider the procedure, which inductively defines n_i and A_i:

(1) Choose n_1 so that R_{n_1} is infinite
(Note if n_1 does not exist claim 1 applies with $A = \mathbb{N}$)

(2) Let $A_1 = R_{n_1}$

In general choose $n_{k+1} \in A_k$ so that $R_{n_{k+1}} \cap A_k$ is infinite and let $A_{k+1} = R_{n_{k+1}} \cap A_k$.
(Note if n_{k+1} does not exist claim 1 applies with $A = A_k$)

Finally suppose we can complete the induction, let $B = \{n_i\}_{i=1}^{\infty}$ Since $a, b \in B$, $a < b \Rightarrow b \in R_a$ and $f(\{a, b\}) =$ red.

In ~~either~~ any case, we have an infinite $B \subset \mathbb{N}$ with $f \mid \mathcal{P}_2(B)$ const.

Now assume we know the Infinite Ramsey Theorem with k-subsets divided between red & green, we will procede to prove infinite Ramsey with $k+1$-subsets divide between red & green.

Given: $f: \mathcal{P}_{k+1}(\mathbb{N}) \to \{red, green\}$.

Define: for $n \in \mathbb{N}$ a function $f_n : \mathcal{P}_k(\{i \in \mathbb{N} : i > n\}) \to \{red, green\}$ given by if $x \in \mathcal{P}_k(\{i > n\})$, $f_n(x) = f(x \cup \{n\})$.

Claim 1: Suppose there is infinite $A \subseteq \mathbb{N}$ so that for each $n \in A$ and infinite $B \subset A \cap \{i : i > n\}$, green $\in f_n(\mathcal{P}_k(B))$ then there is an infinite $C \subset A$ so that $f \mid P_{k+1}(C) \equiv$ green.

Pf of Claim 1: Let $n_1 \in A$ by inductive hypothesis there is infinite $B_1 \subset A \cap \{i : i > n_1\}$ so that $f_{n_1}(\mathcal{P}_k(B_1))$ is constant and the hypothesis of Claim 1 implies that this constant is green.

In general let $n_{j+1} \in B_j$ and let B_{j+1} be an infinite $\subset B_j \cap \{i : i > n_{j+1}\}$ so that $f_{n_{j+1}}(\mathcal{P}_k(B_{j+1})) \equiv$ green.
Let $C = \{n_j\}_{j=1}^{\infty}$ if $a_1 < \dots < a_{k+1}$ are in C
then $f(\{a_1, \dots a_{k+1}\}) = f_{a_1}(\{a_2, \dots a_{k+1}\}) =$ green.

General Proceedure: Pick n_1 so that there is an infinite $A_1 \subset \{i : i > n_1\}$ with $f_{n_1}(\mathcal{P}_k(A_1)) \equiv$ red.
(note if n_1 cannot be chosen then claim 1 applies with $A = \mathbb{N}$)

In general pick $n_{j+1} \in A_j$ so there is an infinite $A_{j+1} \subset A_j \cap \{i : i > n_{j+1}\}$ with $f_{n_{j+1}}(P_k(A_{j+1})) \equiv$ red.
(note if n_{j+1} cannot be chosen then claim 1 applies with $A = A_j$)

Now let $C = \{n_j\}_{j=1}^{\infty}$ if $a_1 < \dots < a_{k+1}$ are in C
then $f(\{a_1, \dots a_{k+1}\}) = f_{a_1}(\{a_2, \dots a_{k+1}\}) =$ red

In any case there is an infinite $C \subset \mathbb{N}$ with $f \mid \mathcal{P}_{k+1}(C) \equiv$ constant

To complete the proof of the infinite Ramsey theorem we note that the case $k=1$ is a special case of the pigeon-hole principle.

The usual application to Banach spaces (Brunnel & Sucheston)

Let $\{e_i\} \subset X$ be a sequence of norm one elements in a norm space let $\varphi: \mathbb{N} \to \bigcup_n \mathbb{Q}^n$ be an infinite to one onto mapping.

Pick subsequences $\mathbb{N}(i+1) \subset \mathbb{N}(i)$ such that if $\varphi(i+1) = (\alpha_1, \dots \alpha_m)$ then if $n_1 < \dots < n_m$ & $n_1' < \dots < n_m'$ s.t. integers in $\mathbb{N}(i+1)$ then

$$\left| \| \sum \alpha_i e_{n_i} \| - \| \sum \alpha_i e_{n_i'} \| \right| < \frac{1}{i+1}$$

this can be done by Ramsey's theorem since

$$\| \sum \alpha_i e_{n_i} \| \le \sum |\alpha_i| = R \quad \& \quad \{\alpha : |\alpha| \le R\}$$

has a $\frac{1}{2(i+1)}$-net.

Diagonalize and define $||| \sum \alpha_i e_i ||| = \lim_{\{n_i\} \to \infty} \| \sum \alpha_i e_{n_i} \|$

then $(\operatorname{span}\{e_i\}, ||| \cdot |||)$ is finitely representable in X and is invariant under speading

$$||| \sum_{i=1}^{n} \alpha_i e_i ||| = ||| \sum_{i=1}^{n} \alpha_i e_{m_i} ||| \quad \text{if } m_1 < m_2 < \dots m_n.$$

If $\{e_i\}$ $\| \cdot \|$ is basic, then so is $\{e_i\}, ||| \cdot |||$.
But it need not be subsymmetric
example: bdd complete basis for J $||| \sum \alpha_i e_i ||| = \sup \left(\sum (\sum_{B(i)} \alpha_i)^2 \right)^{\frac{1}{2}}$
By $\{e_{2i} - e_{2i+1}\}_{i}$ is subsymmetric.

Applications yields ℓ_p^n's in X

Applications in nuclear spaces
I. trouble with zero.
(Infinite partitions are invalid)