FUNCTIONAL ANALYSIS II
BELENOT

LET K BE THE SET OF REALS OR COMPLEX NUMBERS.

DEF: A <u>TVS</u> IS A VECTOR SPACE OVER K WITH RESPECT TO THE TOPOLOGY ξ SUCH THAT:

① VECTOR ADDITION $+: E \times E \longrightarrow E$
$(x,y) \longmapsto x+y$

AND ② SCALAR MULTIPLICATION $K \times E \longrightarrow E$
$(\lambda, x) \longmapsto \lambda x$

ARE CONTINUOUS.

DEF: IF E IS A VECTOR SPACE, $\rho : E \longrightarrow \mathbb{R}^+$ IS A <u>SEMINORM</u> IF:

① $\rho(x) \geq 0$ AND $x = 0 \Longrightarrow \rho(x) = 0$
② $\rho(\lambda x) = \lambda \rho(x)$
③ $\rho(x+y) \leq \rho(x) + \rho(y)$

DEF: <u>ABSOLUTELY CONVEX</u> IS CONVEX AND BALLANCED.

EX: LET $U \subseteq E$, E A V.S., AND U IS ABSOLUTELY CONVEX AND ABSORBING. THEN THE FUNCTION $\|\cdot\|_U$ DEFINED BY $\|x\|_U = \inf\{\lambda > 0 : x \in \lambda U\}$ IS A SEMINORM.

IF ρ IS A SEMINORM, LET $W = \{x ; \rho(x) < 1\}$
$V = \{x : \rho(x) \leq 1\}$

THEN W & V ARE ABSOLUTELY CONVEX AND ABSORBING.

PF:

$x, y \in W$ $\quad t, s \geq 0 \ni s + t = 1$

$\rho(tx+sy) \le \rho(tx)+\rho(sy) \le t\rho(x)+s\rho(y) < s+t=1$

$\therefore tx+sy \in W$. AND W IS CONVEX.

SIMILAR FOR V.

$|\lambda| \le 1,\ x \in W \Rightarrow \lambda x \in W$

$\rho(\lambda x) = |\lambda|\rho(x) < 1$

$\therefore W$ IS BALLANCED $\therefore W$ IS ABSOLUTELY CONV.

V IS SIMILAR.

W IS ABSORBING: LET $x \in E$, $\lambda > \rho(x)$

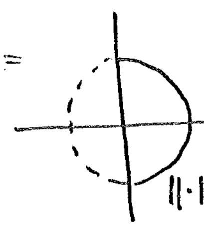
CONSIDER $\rho(\frac{1}{\lambda}x) = \frac{1}{\lambda}\rho(x) < 1$ SO THAT $\frac{1}{\lambda}x \in W$ FOR $\lambda > \rho(x)$

OR $x \in \lambda W$ FOR $\lambda > \rho(x)$.

SUPPOSE WE START WITH U ABSOLUTELY CONVEXED AND ABSORBING. IF $\|\cdot\|_U = \rho$, WE GET V & W ABSOLUTE CONVEX AND ABSORBING.

CLAIM: $\|\cdot\|_W = \|\cdot\|_V = \|\cdot\|_U$

IF $U =$



HALF OPEN
HALF CLOSED

$\|\cdot\|_W$ is the SO CALLED MINTOWSKI GAUGE FUNCTIONAL OF W

STEP 1: $W \subseteq U \subseteq V$

SUPPOSE $x \in W$, $1 > \rho(x) = \|x\|_U = \inf\{\lambda > 0 : x \in \lambda U\}$

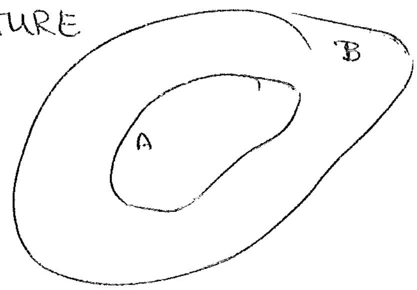
SO THERE IS A $\lambda_0 < 1$ WITH $x \in \lambda_0 U$ BUT U IS BALANCED, $\lambda_0 U \subseteq U$. HENCE $x \in U$ & $W \subseteq U$.

SUPPOSE $x \in U$, THEN THE SET $\{\lambda > 0 : x \in \lambda U\}$ INCLUDES 1.

$\|x\|_U = \rho(x) \le 1$ SO $x \in V$.

STEP 2: LEMMA: IF $A \subseteq B$ ARE ABSOLUTE CONVEX AND ABSORBING THEN $\|\cdot\|_A \geq \|\cdot\|_B$

PICTURE



FIND $\lambda \ni x \in \lambda A \Rightarrow x \in \lambda B$

COR: $\|\cdot\|_V \leq \|\cdot\|_U \leq \|\cdot\|_W$

TO SHOW GAUGE FUNCTION OF W & U HAVE TO BE THE SAME

WANT $\|\cdot\|_V \geq \|\cdot\|_W$

SUPPOSE $x \in E \quad \|x\|_W = 1$

FOR $\mu < 1 \quad x \notin \mu W$ i.e. $x/\mu \notin W$

$\Rightarrow \rho(x/\mu) \geq 1$

$\Rightarrow \rho(x) \geq \mu$. SINCE μ IS ARBITRARY POSITIVE REAL < 1 SO $\rho(x) \geq 1$.

CASE II: $\rho(x) = 1$. SHOW $\mu < 1 \Rightarrow \mu \notin \{\lambda > 0 : x \in \lambda U\}$

SUPPOSE NOT. i.e. $\exists \mu < 1$ WITH $x \in \mu V$

$x/\mu \in V$.

$\Rightarrow \rho(x/\mu) \leq 1$

$\Rightarrow \rho(x) \leq \mu < 1$ CONTRADICTION $\Rightarrow \|x\|_V = 1$

NOTE THAT $x \in E$, $\left\| \frac{x}{\|x\|_W} \right\|_W = 1$

$$\left\| \frac{x}{\|x\|_W} \right\|_V = 1 \Rightarrow \frac{1}{\|x\|_W} \|x\|_W = \frac{1}{\|x\|_W} \|x\|_V$$

THEOREM: IF U IS OPEN, ABSOLUTE CONVEX & ABSORBING, THEN $\|\cdot\|_U = \rho$ IS CONTINUOUS & IF ρ IS CONTINUOUS THEN

INVERSE IMAGE OF CONT. MAP $\begin{cases} W = \{x : \rho(x) < 1\} \text{ IS OPEN} \\ V = \{x : \rho(x) \leq 1\} \text{ IS CLOSED NHBD OF } 0. \end{cases}$

$\rho : E \longrightarrow \mathbb{R}^+$
$x \longmapsto \rho(x)$

PICK $\epsilon > 0$ $D_\epsilon(\rho(x)) = \{ r \in \mathbb{R}^+ : |r - \rho(x)| < \epsilon \}$. FIND AN OPEN SET ABOUT x SUCH THAT $\rho(\ \ ) \subseteq D_\epsilon(\rho(x))$.

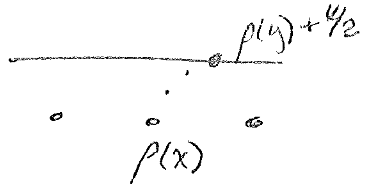
LET $G \subseteq U$ BE OPEN SET THEN $\epsilon/2\, G$ IS OPEN.

$\Rightarrow$ $x + \epsilon/2\, G$ IS OPEN

LET $y \in x + \epsilon/2\, G$.

$\Rightarrow$ $y = x + \epsilon/2\, g$ WHERE $g \in G$.

$$\rho(y) = \rho(x + \epsilon/2\, g) \leq \rho(x) + \epsilon/2\, \rho(g) \leq \rho(x) + \epsilon/2$$
$$\rho(x) = \rho(y - \epsilon/2\, g) \leq \rho(y) + \epsilon/2\, \rho(g) \leq \rho(y) + \epsilon/2$$



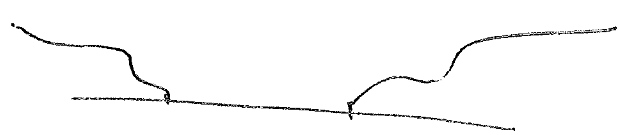
SO $y \in D_\epsilon(\rho(x))$ AND $\rho(x)$ IS CONTINUOUS

EXAMPLES OF SEMINORMS INCLUDE NORMS.

ALSO

EX: CONCIDER $C(\mathbb{R})$

$\rho_n(f) = \sup_{\|x\| < n} f(x)$



NOT A NORM

EX: IN $\omega = \{(x_n) : x_n \in \mathbb{K}\}$

$\rho_N(x_n) = |x_N|$ IS SEMINORM BUT NOT NORM

EX: B SPACE, X. LET $f \in X^*$

$\rho_f(x) = |f(x)|$ IS SEMINORM ON X.

AND $x \in X$, $\rho_x(f) = |f(x)|$ IS SEMINORM ON X^*

MAT 535 Notes of Jan 7

<u>Definition</u> If A is a subset of topological space, the interior of A is any of the following:

(1) Union of all open sets contained in A

(2) $\{x \in A : \exists \text{ open } U \text{ with } x \in U \subseteq A\}$

(3) the set of interior points

<u>Definition</u> A TVS $(E, \mathcal{E})$ is a locally convex space (LCS) if $\exists$ base for the neighborhoods of the origin consisting of convex sets. (i.e. $\exists$ a collection $\mathcal{U}$ of open sets about 0 s.t. $\forall$ open sets V $\exists\, U \in \mathcal{U} \ni U \subseteq V$)

<u>Example</u> Normed spaces are LCS, since

$$\mathcal{U} = \{ B_\varepsilon = \{\|x\| < \varepsilon\} \mid \varepsilon > 0 \} \text{ is such a base.}$$

<u>Theorem</u> If E is a TVS and U is a neighborhood of the zero, then

(1) U is absorbing

(2) $\exists$ neighborhood $V \subseteq U$ & V is balanced.

(3) If E is LCS, then V in (2) can be chosen to be absolutely convex

<u>Corollary</u>. Every TVS (LCS) has a neighborhood base for the origin consisting of balanced sets (absolutely convex sets).

<u>proof of theorem</u>. (1) Since scalar multiplication is continuous, U is neighborhood of zero.

$$\left[\begin{array}{l} K \times E \longrightarrow E \\ (0, x) \longmapsto 0x = 0 \in U \quad \forall x \in E \end{array} \right]$$

$\exists\ \varepsilon > 0$ s.t. $|\lambda| < \varepsilon \Rightarrow \lambda x \in U$. Hence U is absorbing.

(2) $K \times E \longrightarrow E$

$(0, 0) \longmapsto 0 \in U$,

thus $\varepsilon > 0$ & V open about 0 in E s.t.

$$|\lambda| \leq \varepsilon,\ x \in V \Rightarrow \lambda x \in U$$

$$\therefore\ \varepsilon V \subseteq U \quad (\varepsilon V \text{ is open})$$

Define $W = \bigcap_{|\mu| \geq 1} \mu U$. Then $0 \in W$

If $|\mu| \geq 1$, $|\frac{\varepsilon}{\mu}| \leq \varepsilon \Rightarrow \frac{\varepsilon}{\mu} V \subseteq U \Rightarrow \varepsilon V \subseteq \mu U$.
Thus $\varepsilon V \subseteq W$ and W is a neighborhood of zero contained in U. Now we need to show that W is balanced. Let $\omega \in W$, let $|\lambda| \leq 1$. for μ with $|\mu| \geq 1$, $|\frac{\mu}{\lambda}| \geq 1 \Rightarrow \omega \in \frac{\mu}{\lambda} U$ $\Rightarrow \lambda\omega \in \mu U \Rightarrow \lambda\omega \in W$. Thus W is balanced and Int W works for (2). [(3) reduces to (2). We may assume U is convex in (3) $\Rightarrow$ W convex.]

Let A be any collection of semi-norms on the vector space E. Then there is a TVS topology on E generated by the below process making E a LCS s.t. each $p \in A$ is continuous in this topology and it is the weakest (i.e. having the fewest open sets) TVS topology in which each semi-norm in A is continuous.

If we want all semi-norms to be continuous, then

$$W_{p,\varepsilon} = \{x \in E \mid p(x) < \varepsilon\},$$

for any $p \in A$, must be open.

E: TVS, $y \in E$

$W_{\rho,\varepsilon,y} = y + W_{\rho,\varepsilon}$ (open)

Subbase: $\mathcal{W} = \{ W_{\rho,\varepsilon,y} \mid \rho \in A, \varepsilon > 0, y \in E \}$

Base: $\mathcal{U} = \{ U \mid U$ is a finite intersection of elements of $\mathcal{W} \}$

Topology: $\mathcal{T} = \{ F \mid F$ is arbitrary union of elements of $\mathcal{U} \}$.

Must show that this is TVS topology. First think of the vector addition

$$E \times E \longrightarrow E$$
$$(x, y) \longmapsto x + y.$$

We now show that $\forall$ open set V about $x+y$, $\exists$ an open set $V_x \times V_y$ about (x, y) s.t. $V_x + V_y \subseteq V$. Note that if $U \in \mathcal{U}$, then

$$U = W^1_{\rho_1, \varepsilon_1, z_1} \cap \cdots \cap W^n_{\rho_n, \varepsilon_n, z_n}.$$

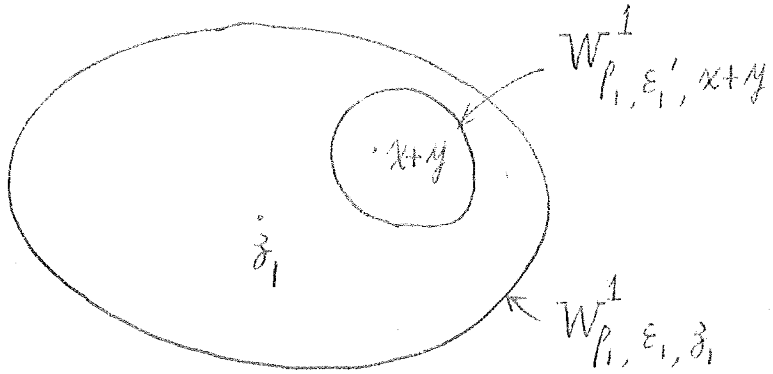
Claim: $x+y \in V \in \mathcal{U}$ & V has the form

not quite we known $\exists V' \leq V$ with $x+y \in V' \in \mathcal{U}$. but we do not know $V \in \mathcal{U}$.

$$V' = W^1_{\rho_1, \varepsilon_1', x+y} \cap \cdots \cap W^n_{\rho_n, \varepsilon_n', x+y}$$

Since V is open, $\exists U \in \mathcal{U}$ s.t. $V \supseteq U$; hence the claim.

follows at once from the following figure



Let

$$V_x = W^1_{\rho_1, \varepsilon_1'/2, x} \cap \cdots \cap W^n_{\rho_n, \varepsilon_n'/2, x}$$

$$V_y = W^1_{\rho_1, \varepsilon_1'/2, y} \cap \cdots \cap W^n_{\rho_n, \varepsilon_n'/2, y}$$

Note that $V_x \times V_y$ is open neighborhood of (x, y) in $E \times E$.

Claim: $V_x + V_y \subseteq V'$. Take $z \in W^1_{\rho_1, \varepsilon_1'/2, x}$ and $w \in W^1_{\rho_1, \varepsilon_1'/2, y}$, we have

$$\rho_1(z + w - (x+y)) \le \rho_1(z - x) + \rho_1(w - y)$$

$$< \frac{\varepsilon_1'}{2} + \frac{\varepsilon_1'}{2} = \varepsilon_1'$$

Thus, $w + z \in W^1_{\rho_1, \varepsilon_1', x+y}$. Therefore, the vector addition is continuous. The scalar multiplication is similar.

If $\mathcal{U}$ is collection of absolutely convex, open neighborhood of 0 which is a base for the neighborhoods of the origin, then

$$A = \{ \|\cdot\|_V : V \in \mathcal{U} \}$$

are continuous semi-norms on E, the topology generated is the original topology.

Suppose E has LCS topology generated by the semi-norms in A. Then

$\{x_n\} \longrightarrow \{x\}$ in this topology iff $\forall p \in A$, $p(x_n - x) \longrightarrow 0$

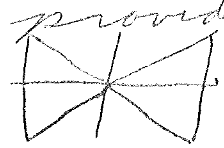
WATCH OUT SEQUENCES ARE NOT ENOUGH.

If E is generated by A
F is generated by B
$L : E \longrightarrow F$ is linear,

then L is continuous iff L is continuous at zero iff $\forall q \in B$, $\exists p \in A$, $M \in \mathbb{R}$ s.t. $q(Le) \leq M\, p(e) \quad \forall e \in E$

Jan 10, 1977

Add to the hypothesis of problem 1 that 0 is an element of the interior of the balanced set. Otherwise, a counter-example is provided by:



Theorem: Let $L : E \to F$ be a linear map between two locally convex spaces. Then L is continuous if and only if for all continuous seminorms on F, q, there exists a continuous seminorm on E, p, and an M such that, for all $e \in E$,

$$q(L(e)) \le M p(e)$$

Proof: It is left as an exercise to establish that L is continuous if and only if it is continuous at 0.
($\Rightarrow$) If L is continuous at 0 and q is a continuous seminorm on F, let $V = \{f \in F : q(f) < 1\}$.
Since L is continuous, there exists a $2V$ absolutely convex such that $L(2V) \subseteq V$.
Now $\|\cdot\|_V$ is a continuous seminorm on E, so if $x \in V$, $Lx \in V$. Hence,

$$\|x\|_V = 1 \Rightarrow x \in 2V \Rightarrow q(Lx) \le 1.$$

In general, we may consider

Case I) $\|e\|_V \neq 0$

Let $x = \frac{e}{\|e\|_V}$. $q(Lx) \leq 1$, so

$q(L(\frac{e}{\|e\|_V})) \leq 1$. Hence, $q(L(e)) \leq \|e\|_V$.

Case II) $\|e\|_V = 0$

Since $\|e\|_V = 0$, for all λ, $\|\lambda e\|_V = 0$. Then if $q(L(e)) \neq 0$, there exists a large enough λ_0 so that $q(L(\lambda_0 e)) > 1$.

Now $\|\lambda e\|_V = 0 \; \forall \lambda \Rightarrow \lambda e \in V \; \forall \lambda$ but $L(\lambda_0 e) \notin V$. This is a contradiction

Hence, $\|e\|_V = 0 \Rightarrow q(Le) = 0$.
Then the ($\Rightarrow$) part is finished, and $M = 1$.

($\Leftarrow$) V is absolutely convex in F.
$\|\cdot\|_V$ will give us M and ρ.

Let $V = \{e \in E : \rho(e) < \frac{1}{M}\}$. If $e \in V$,
$\|L(e)\|_V \leq M \rho(e) \leq \frac{M}{M} = 1$
Hence, $L(e) \in V$

Duals of Locally Convex Spaces —
HB Theorem: Let E be a locally convex space. Let M be a subspace. Let $f : M \to \mathbb{R}$ (reals), f continuous. Then f can be extended to all of E.

Proof: Since f is continuous on M, there exists a continuous seminorm ρ on M such that for $e \in E$, $|f(e)| \leq \rho(e)$

If $\rho(e) = 0$, then $f(e) = 0$

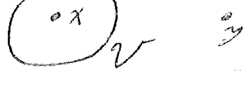
$$\begin{tikzcd}
E \arrow[ddr, "\varphi_\rho"'] & M \arrow[l] \arrow[r, "f"] \arrow[d, "\varphi_\rho"] & \mathbb{R} \\
& M_\rho \arrow[ur, "\hat{f}"] \arrow[d] & \\
& E_\rho \arrow[uur, "\tilde{f}"'] &
\end{tikzcd}$$

$E_\rho = E/\ker\rho$

$\hat{f}$ is continuous from $(M_\rho, \hat{\rho}) \to \mathbb{R}$

$\tilde{f}$ can be defined by the usual HB theorem, extending $\hat{f}$ to $\tilde{f}$ with $\tilde{f}$ continuous. This extension is $\tilde{f}\varphi_\rho$. ∎

A topological space is T_1 if for all x, y, there is an open set V such that $x \in V$, $y \notin V$



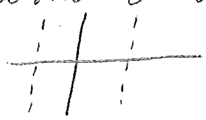
A topological space is T_2 if for all x, y, there exist open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$

If a topological space is not T_2, then a sequence may not have a unique limit.

Let $M = \text{cl.}(0)$

$M = \bigcap_{\substack{\text{open } V \\ 0 \in V}} V = \{0\}$ if and only if the space is T_1. Otherwise, M is a closed subspace of E.

Let $\rho(x, y) = |x|$. Open neighborhoods about 0 are of the form



We will only consider T_2 TVS's.

Examples

I) Λ is an index set. For each $\alpha \in \Lambda$, let $\{B_\alpha, \|\cdot\|_\alpha\}$ be a normed space. By the set $\prod_{\alpha \in \Lambda} B_\alpha$, we mean the set of all functions $\psi : \Lambda \to \bigcup_{\alpha \in \Lambda} B_\alpha$,

with the property that $\psi(\alpha) \in B_\alpha \ \forall \alpha \in \Lambda$

$\psi(\alpha)$ is called the α<u>th</u> <u>co-ordinate</u> of ψ.
The product topology on $\prod_{\alpha \in \Lambda} B_\alpha$ is the LCS generated by the seminorms
$\psi \in \prod$, $\alpha \in \Lambda$, $\rho_\alpha(\psi) = \|\psi(\alpha)\|_\alpha$

II) $\Lambda = \mathbb{N}$, $B_n = K$
Then $\omega = \prod_{k \in \mathbb{N}} k$ is the space of all scalar sequences with seminorms
$\rho_n((x_m)) = |x_n|$, $n = 1, 2, 3, \ldots$

ω is a complete metric locally convex space. No B-space topology exists on it (There are norms, but no complete norms) e.g. the metric $d((x_n),(y_n)) =$

$\sum_{k=1}^{\infty} \frac{1}{2^k} \inf (|x_k - y_k|, 1)$.

Suppose E and F are vector spaces and F can be considered as a set of linear functionals on E. We can define a LCS topology on E using the seminorms $p_f(e) = |f(e)|$, $f \in F$. This topology is called the $\sigma(E,F)$ topology on E.

If $f: E \to K$ is linear, then $|f(x)|$ is a seminorm.
Justification: $f(0) = 0$, $|f(x)| \geq 0$,
$|f(\lambda x)| = |\lambda f(x)| = |\lambda| \, |f(x)|$.
$|f(x+y)| = |f(x) + f(y)| \leq |f(x)| + |f(y)|$.
Hence, $|f(\cdot)|$ is a seminorm

Suppose X is a normed space and Y its completion. Let X^* be its dual and X^{**} its bidual.
On X, the $\sigma(X, X^*)$ topology is the weak topology on X. On Y, the $\sigma(Y, X^*)$ topology is the weak topology on Y. The weak topology on X is a subspace of the weak topology on Y.

On X^*, the $\sigma(X^*, X^{**})$ topology is the weak topology on X^*.

Now X or Y can be considered as linear functionals on X^*.

On X^*, the $\sigma(X^*, X)$ topology is the weak star topology on X^*. $\sigma(X^*, Y)$ is also called the weak star topology on X^*, although they are different if $X \neq Y$. (proof later)

On X^{**}, the $\sigma(X^{**}, X^*)$ topology is called the weak star topology on X^{**}.

1-12-77

Let E & F be vector spaces. Think of the elements of F as linear functions on E.

What is the dual of E, $\sigma(E,F)$? What are cont. lin. fcnals on E in this topology?

1. Each $f \in F$ is linear & it is cont. in this top. since $|f(x)| \le p(x)$.
   a cont. seminorm $\longrightarrow p(x) = |f(x)|$

2. Is there anything else?

No. Pf. Let g be cont. lin. on E, $\sigma(E,F)$.
$\exists$ open set U about zero s.t. $u \in U \Rightarrow |g(u)| \le 1$.

$\exists\ f_1 \cdots f_n \in F$ and $\epsilon_1 \cdots \epsilon_n > 0$ s.t.
$$W = \{e \in E \mid |f_i(e)| < \epsilon_i\ ,\ i = 1,2,\cdots,n\} \subseteq U.$$

So $e \in W \Rightarrow |g(e)| \le 1$.

Claim: g is a linear combination of $f_1 \cdots f_n$.

Lemma (p. 157): If $f_1 \cdots f_{n+1}$ lin. ind. fcnals on E $\exists\ x_1, \cdots, x_{n+1}$ in E with $f_i(x_j) = \delta_{ij}$.

We are going to show that if $f_i(x) = 0$, $i = 1, \cdots, n$ then $g(x) = 0$. This would imply, by the lemma, that g is dependent on $f_1, \cdots, f_n$.

So suppose $\exists\ x$ s.t. $f_i(x) = 0$ $i = 1, \cdots, n$.

And suppose $g(x) \neq 0$. Then there is a λ large enough so that $|g(\lambda x)| > 1$.

$f_i(\lambda x) = \lambda f_i(x) = 0 < \epsilon_i \ \forall i$. So $\lambda x \in W$
$\Rightarrow |g(\lambda x)| \leq 1$ # contradiction. Done with claim.

Cor. If X is normed, Y its completion, both duals $X^* = Y^*$ and $X \neq Y$,
then $X^*, \sigma(X^*, X) \neq X^*, \sigma(X^*, Y)$.

X LCS, X^* set of cont. lin. fcns.
Let U be nbhd of $0 \subseteq X$,
$U^\circ = \{ f \in X^* \mid |f(u)| \leq 1 \text{ for } u \in U \}$.

If U is the unit ball in the normed space X, then U° will be any ball of the B-space X^*.

Bourbaki-Alaglou Thm:
In the top. $\sigma(X^*, X)$, U° is compact.

Pf. $X^*, \sigma(X^*, X) \xrightarrow{\Phi} \prod_{x \in X} K$

$f \in X^*$, then $\Phi(f)$ is said to be the ~~element~~ element whose x^{th} coordinate is $f(x)$.

Φ is 1-1 into.
Φ is a homomorphism into

$p(f) = |f(x)|$, $\quad \prod p_x(f) = |f(x)|$

Let $x \in X$. The range of values of $|f(x)|$ as $f \in U^\circ$ is bounded (i.e. $\exists M_x$ s.t. $f \in U^\circ \Rightarrow |f(x)| \leq M_x$.)

U is a nbhd & so is absorbing.
$\exists \omega$ s.t. $\omega \geq \lambda > 0$, $\lambda x \in U$,

$|f(x)| = \frac{1}{|\lambda|} \; |f(\lambda x)| \leq \frac{1}{u}$

$|f(x)| \leq \frac{1}{u} = M_x .$

$$\begin{array}{ccc} \mathfrak{X}^* & \xrightarrow{\Phi} & \prod_{x \in \mathfrak{X}} K \\ \cup\!\mid & & \\ U^\circ & \xrightarrow{\Phi} & \prod_{x\in\mathfrak{X}} D_x \end{array} \qquad D_x = \{ \lambda \in K \mid |\lambda| \leq M_x \}.$$

So D_x is compact & the Tychonoff Product Th. says $\prod_{x\in\mathfrak{X}} D_x$ is compact, i.e. we only need to show U° is closed as a subset of this product.

$U^\circ = \bigcap_{u \in U} \{ f \in \mathfrak{X}^* \mid |f(u)| \leq 1 \}$. U°, $\sigma(\mathfrak{X}^*, \mathfrak{X})$ is closed.

Let $g \in \prod_{x\in\mathfrak{X}} D_x$ and suppose g is in the closure of U°.

g is additive: for $x, y \in \mathfrak{X}$, show $g(x+y) = g(x) + g(y)$.
Fix $\epsilon > 0$, $x, y, x+y \in \mathfrak{X}$.

$W = \{ f \in \prod_{x\in\mathfrak{X}} D_x \mid |f(x)| < \epsilon/3, \; |f(y)| < \frac{\epsilon}{3}, \; |f(x+y)| < \frac{\epsilon}{3} \}.$

Since $g \in \operatorname{cl} U^\circ$, $\exists f \in U^\circ$ with $f - g \in W$.

$f(x+y) - f(x) - f(y) = 0$ with $|g(x+y) - f(x+y)| < \frac{\epsilon}{3}$,
$|g(x) - f(x)| < \frac{\epsilon}{3}$, $|g(y) - f(y)| < \frac{\epsilon}{3} \Rightarrow$

$|g(x+y) - g(x) - g(y)| < \epsilon.$

Since this is true $\forall \epsilon > 0$, we have $g(x+y) - g(x) - g(y) = 0$

Similarly, $g(\lambda x) = \lambda g(x)$.

Let $u \in U$. We want to show $|g(u)| \leq 1$.

We will show for $\epsilon > 0$, $|g(u)| < 1 + \epsilon$.

$W = \{ f \in \prod_{x \in X} D_x \mid |f(u)| < \epsilon \}$

$\exists f \in U$ with $g - f \in W$
$g(u)$ within ϵ of $f(u)$ & $|f(u)| \leq 1$

$x \in X$, $\|x\|_u \leq 1 \Rightarrow |g(u)| \leq 1$

$\|x\|_u \geq |g(u)|$. Hence g is cont. $\in X^*$.

So $g \in U^\circ$, and U° is closed in $\prod_{x \in X} D_x$.

<u>Note</u>: If X is a reflexive B-space,

$X \xrightarrow{J} X^{**}$ J canonical injection

X is reflexive iff J is onto.

If X reflexive B-space, $X = X^{**}$,

U unit ball of X, U° unit ball of X^*;

$\underset{U}{\overset{\shortparallel}{(U^\circ)^\circ}}$ unit ball of $X^{**} = X$.

The Thm says $(U^\circ)^\circ$ is $\sigma(X^{**}, X^*)$ compact in X^{**}, U is $\sigma(X, X^*)$ compact in X.

Further will show that converse is also true.

Ex. J (in homework), $c_0, l_1, l_\infty, C([0,1])$ are each non-reflexive.
l_p, L_p $1<p<\infty$ are reflexive.

In c_0 the unit ball is not $\sigma(c_0, l_1)$ compact (weakly compact)

$$\underbrace{(1,1,\cdots,1}_{n},0,0,\cdots) = x_n$$

$\{x_n\}$ weak Cauchy seq.

For each $f\in l_1$, $|f(x_n)|$ is a C.S. of scalars

$f=(f_1, f_2, \cdots)$ $\quad \|f\| = \sum |f_n|$.

AND THE ONLY THING x_n CAN CONV TO IS THE SEQ

$(1,1,1,\ldots\ldots \quad)$
all 1's

WHICH IS NOT IN c_0

Approximation Theory

Let X be a B-space, and F be a subspace of X. Let $x \in X$ be given. We want to know which element of F "best approximates" x i.e., we want $\varphi \in F$ such that

$$\|\varphi - x\| \leq \|\psi - x\| \quad \text{for all } \psi \in F$$

Ex1 In the Hilbert Space H, if F is a closed subspace and $\{f_\alpha\}_{\alpha \in \Lambda}$ is an orthonormal basis for F. The $x \in H$ can be "best approximated" by

$$\varphi = \sum_{\alpha \in \Lambda} \langle x, f_\alpha \rangle f_\alpha$$

Ex2 The first n terms of the Fourier

Theorem If F is a finite dimensional subspace, φ always exists

Proof: Pick $f \in F$, $\|f - x\| = \lambda$. Let $B_\lambda(x) = \{ y \mid \|y - x\| \leq \lambda \}$. Then $B_\lambda(x)$ is closed and bounded. It follows that $B_\lambda(x) \cap F$ is compact. This implies that the continuous function $y \mapsto \|y - x\|$ from $K \to \mathbb{R}^+$ obtains its minimum value. i.e. there exist $\varphi \in K$ such that $\|\varphi - x\| \leq \|\psi - x\|$ for all $\psi \in F$.

Uniqueness

Def: X is strictly convex if $\|x\| = \|y\| = 1$ with $x \neq y$ implies $\| \frac{x+y}{2} \| < 1$.

Now assume that φ_1 and φ_2 are two different best approximations of x.

W.L.O.G. assume $\|\varphi_1 - x\| = \|\varphi_2 - x\| = 1$.

If X is strictly convex then

$$\left\| \frac{\varphi_1 + \varphi_2}{2} - x \right\| = \left\| \frac{(\varphi_1 - x) + (\varphi_2 - x)}{2} \right\|$$

< 1 by the definition of strictly convex.

This is a contradiction since

$\frac{\varphi_1 + \varphi_2}{2}$ is also in F and is

even better that the best.

<u>Conclusion:</u> If the Best Approximation exists in a strictly convex space, it is unique.

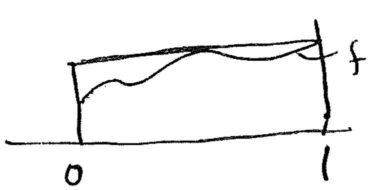
We know from example 4 that $\mathbb{R}^n, \|\ \|_\infty$ is not strictly convex. However there exists an equivalent norm, namely $\|\ \|_2$ which makes $\mathbb{R}^n$ strictly convex.

Consider $C[0,1]$, $\|\cdot\|_\infty$.

Question: Is there an equivalent norm which makes $C[0,1]$ strictly convex?

Answer: Yes $|||f||| \equiv \|f\|_\infty + \|f\|_2$

Proof: Note that $\|f\|_2 \leq \|f\|_\infty$.



(see picture ~~for~~ Use Hölder's inequality)

Hence $\|f\|_\infty \leq \|f\|_\infty + \|f\|_2 \leq 2\|f\|_\infty$

i.e., $\|f\|_\infty \leq |||f||| \leq 2\|f\|_\infty$ and

~~and $|||f|||$~~ $|||\cdot|||$ is equivalent to $\|\ \|_\infty$

Suppose $|||f||| = |||g||| = 1 \qquad f \neq g$

$$|||f+g||| = \|f+g\|_\infty + \|f+g\|_2$$

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

Check as an exercise that in a Hilbert space $\|f+g\|_2 < \|f\|_2 + \|g\|_2$ as long as $f \neq \alpha g$ for some constant α.

This condition is satisfied here (check)

$$\Rightarrow |||f+g||| < \|f\|_\infty + \|g\|_\infty + \|f\|_2 + \|g\|_2$$

$$= |||f||| + |||g||| = 2$$

$\Rightarrow |||\frac{f+g}{2}||| < 1$ i.e

$(C[0,1], |||\cdot|||)$ is strictly convex.

<u>Definition</u> A top. space is separable if it has a countable subset which is dense

<u>Theorem</u> Every Separable Normed Space is isometrically a subspace of $C[0,1]$

<u>Cor</u>: Every Separable Normed space can be given an equivalent strictly convex norm.

$\mathbb{X}$ norm space

$U^\circ \subseteq X^*$. The "functions" in $\mathbb{X}$

$\subseteq X^{**}$ are $\sigma(X^*, X)$-continuous

on $\mathbb{X}^*$. Hence they are continuous

functions on $(U,^\circ \sigma(X^*, \mathbb{X}))$.

Now U° is $\sigma(X^*, X)$-compact

$$\|x\|_{\mathbb{X}} = \|x\|_{\mathbb{X}^{**}} = \sup_{f \in U^\circ} |f(x)|$$

So $\mathbb{X}$ is isometrically a subspace of $C(U^\circ; \sigma(X^*, \mathbb{X}))$ with $\|\cdot\|_\infty$ norm

$(U^\circ, \sigma(X^*, X))$ is compact, metrizable connected and locally connected. The Hahn Mazurkiewicz Theorem then implies that it is the continuous image of $[0,1]$.

(Check Hocking - Young Topology and/or Moore's Elementary topology)

i.e

$$I \xrightarrow[\text{onto, continuous}]{\varphi} U^\circ, \sigma(X^*, X)$$

to get

$$T : C(U^\circ) \longrightarrow C(I)$$

linear, isometry and onto let

$$T(f) = f \circ g$$

Want to show if $\mathfrak{X}$ is separable normed space then U° in $\sigma(\mathfrak{X}^*, \mathfrak{X})$ topology is metrizable.

Ask: 1) When is the topology of TVS given by a norm?

2) When is the topology of TVS given by a metric?

DEF. $A \subseteq E$ is bounded in ξ LCS TOPOLOGY if $\{\rho(a) : a \in A\}$ is a bounded set of reals for each continuous semi-norm ρ.

Equivalently, A is bounded if for each nbd. of 0, U, $\exists \lambda > 0$, s.t. $A \subset \mu U$ for all $|\mu| \geq \lambda$.

To answer 1):

IF (E, ξ) is a TVS and is given by a norm, then i) (E, ξ) is a LCS.

ii) (E, ξ) has a bounded nbd. of 0.

Since every normed space is a LCS and the unit ball of a normed space is a convex bounded nbd. of 0.

Conversely, suppose the LCS (E, ξ) has U as

a bounded nbd. of 0; then $\exists$ abscov nbd. $V \subseteq U$,
and V is obviously bounded.

Let W be any nbd. of 0, then

$\exists \lambda$. s.t. $V \subseteq \lambda W$. $\Rightarrow \frac{1}{\lambda} V \subseteq W$.

Thus, $\{\varepsilon V, \varepsilon > 0\}$ form a local base.

Moreover, $\|\cdot\|_V$ gives the required norm.

To answer 2):

Suppose (E, ξ) be a metric TVS, then there
is a countable local base,

i.e. $\exists \{U_n\}$ nbds of 0, $\ni$. $\forall$ V nbd. of 0,
$\exists n$. with $U_n \subseteq V$.

eg. take $U_n = \{x : d(x, 0) < \frac{1}{n}\}$.

Conversely, we have:

<u>THM.</u> In a TVS, 1st countable $\Rightarrow$ metrizable.

<u>pf</u>: Ref. Kelley, --- etc.

For L.C.S's we have $\{U_n\}$ as a basis for nbd. of 0.

May assume $U_1 \supseteq U_2 \supseteq \cdots$. also each U_n is absconvex.

Set $d(x, y) = \sum_{n=1}^{\infty} \frac{\min(p_n(x-y), 1)}{2^n}$, where

p_n is the gauge fenal of U_n,

i) Is d a metric ?

ii) Does it give the topology ξ ?

Answer : Yes.

~~Proof~~ i) note $d(x, y) = d(x-y, 0)$.

Need to see $d(x, y) \leq d(x, z) + d(z, y)$.

i.e. $d(\xi+\eta, 0) \leq d(\xi, 0) + d(\eta, 0)$. (*)

where $\xi = x - z$, $\eta = z - y$. $\xi + \eta = x - y$.

Since $p_n(\xi+\eta) \leq p_n(\xi) + p_n(\eta)$. $\forall n$

$\Longrightarrow \min(p_n(\xi+\eta), 1) \leq \min(p_n(\xi), 1) + \min(p_n(\eta), 1)$

(*) is then obvious. The rest are easy.

For ii) Need to do :

a) For every n. $\exists \varepsilon > 0$, s.t. $\{x : d(x, 0) < \varepsilon\} \subseteq U_n$;

and

b) $\forall \varepsilon > 0$. $\exists n$. s.t. $U_n \subseteq \{x : d(x, 0) < \varepsilon\}$

a) let $\varepsilon = 1/2^n$. then $d(x,0) < 1/2^n \Rightarrow \rho_n(x) < 1$

$\Rightarrow x \in U_n$.

b) $\varepsilon > 0$. $\exists N$. $\ni \ldots \sum_{n=N}^{\infty} \frac{1}{2^n} < \varepsilon/2$.

if $x \in \frac{\varepsilon}{2} U_N$, $x \in \frac{\varepsilon}{2} U_n$, $\forall n \le N$.

Assume $\varepsilon < 1$.

$$d(x,0) = \sum_{n=1}^{\infty} \frac{\min(\rho_n(x), 1)}{2^n} \le \sum_{n=1}^{N} \frac{\rho_n(x)}{2^n} + \sum_{n=N}^{\infty} \frac{1}{2^n}$$

$$\le \frac{\varepsilon}{2} \left(\sum_{1}^{N} \frac{1}{2^n} \right) + \frac{\varepsilon}{2} < \varepsilon.$$

<u>Coro</u>. Any topology generated by a countable number of semi-norms is metrizable.

<u>Note</u>: X is LCS. X^* with $\sigma(X^*, X)$ topology. One would expect that in general, this topology is not metrizable.

<u>FACT</u>: X^*, $\sigma(X^*, X)$ is normable $\Leftrightarrow$ X is finite dimensional.

X^*, $\sigma(X^*, X)$ is metrizable $\Leftrightarrow$ X has a ctble Hamel basis.

<u>THM</u>. $\mathfrak{X}$ is separable iff there is a set $\{e_n\}$ with dense linear span

<u>pf</u>. "$\Rightarrow$" $\mathfrak{X}$ separable $\Rightarrow \exists \{X_n\}_{n\geq 1}$ ctble dense in $\mathfrak{X}$

so Cl lin span $\{X_n\} \supseteq Cl\{X_n\} = \mathfrak{X}$.

"$\Leftarrow$" If $\{e_n\}$ has dense linear span

Consider $A = \{ \sum_{n=1}^{\infty} r_n e_n$, r_n rational, only finitely many r_n's are nonzeros $\}$.

then A is countable and dense

Since $Cl(A) \supseteq$ lin span $\{e_n\}$.

and $Cl(A) = Cl\, Cl(A) \supseteq Cl$ lin. span $\{e_n\} = \mathfrak{X}$.

Q.E.D.

Now, look at $\{e_n\} \subseteq \mathfrak{X}$. with Cl lin. span $\{e_n\} = \mathfrak{X}$.

<u>Fact</u>. suppose $f \in \mathfrak{X}^*$, st. $f(e_n) = 0$. $\forall n$,

then $f \equiv 0$.

Since $f = 0$ on lin. span of $\{e_n\}$.

f cont. $\Rightarrow \ker f$ is closed.

$\Rightarrow \ker f \supset Cl$ lin. span of $\{e_n\} = \mathfrak{X}$.

DEF. A subset $B \subseteq X$ is total over X^*, if $f \in X^*$, & $f(b) \equiv 0$ $\forall b \in B \Rightarrow f \equiv 0$.

DEF A subset $B \subseteq X^*$ is total over X if $x \in X$, & $b(x) = 0$. $\forall b \in B \Rightarrow x = 0$.

NOTE: If X has ctble total subset A over X^*

Consider $\sigma(X^*, A)$ topology on X^* generated by the separating family of seminorms:

$$\{ P_a(X^*) = |X^*(a)|, \ a \in A \}.$$

then $(X^*, \sigma(X^*, A))$ is LCS Hausdorff. weaker than $X^*, \sigma(X^*, X)$.

i.e. $$X^{* \, \sigma(X^*, X)} \xrightarrow[\text{Continuous}]{1_{X^*}} X^{* \, \sigma(X^*, A)}$$

Also $$U^{\circ \, \sigma(X^*, X)} \xrightarrow{1_{X^*}} U^{\circ \, \sigma(X^*, A)}$$

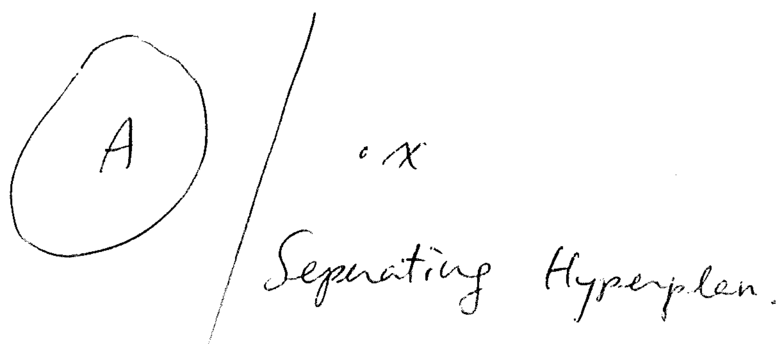
compact, T_2, metrizable. Since A is countable.

THM. If $f: U \to V$ is 1-1, onto, continuous map where U is compact and V is T_2, then f is a homeomorphism;

proof: f ~~is obviously closed map~~ maps closed sets to closed sets. Q.E.D.

Thm E; LCS, $f : E \longrightarrow K$ linear.
Then f is conti $\Longleftrightarrow$ $\operatorname{Ker} f$ is closed.

Thm E: LCS, A: convex $\subseteq E$, $x \notin \operatorname{cl} A$.
Then $\exists f \in E^*$, $\alpha > 0$ s.t.
$f(x) > \alpha > f(a)$ for $\forall a \in A$.



(E, ξ); any LCS. E^*; conti dual.

Result; The topologies $\sigma(E, E^*)$, and ξ have the same closed convex sets.

⟨Reason⟩ $\sigma(E, E^*) \subseteq \xi$
$\therefore$ $\sigma(E, E^*)$-closed $\Rightarrow$ ξ-closed.
But if $\sigma(E, E^*) \subsetneqq \xi$, then $\exists$ ξ-closed sets that are not $\sigma(E, E^*)$-closed.

But let A be convex and ξ-closed, $x \notin A$.

By Thm, $\exists f$ s.t. $f(x) > \alpha > f(a)$, $a \in A$.

$\{y;\ f(y) > \alpha\}$ $\sigma(E, E^*)$-open set misses A, contains x.

Thus A is $\sigma(E, E^*)$-closed. //

Norm-closed $\not\Rightarrow$ weak closed.

<Example> Consider c_0, $\{e_n\}$ usual basis

$$e_n = (0, 0, \cdots 0, \underset{\uparrow\, n\text{th spot.}}{1}, 0, \cdots)$$

$A = \{e_n\}$ is normed closed. (since there ... no limit point like discrete set).

But $e_n \longrightarrow 0$ in the $\sigma(c_0, l_1)$ top.

$\because$ Let $f = (f_n) \in l_1$.

Want to show $\lim_n f(e_n) = 0$

But $\lim_n f(e_n) = \lim_n f_n = 0$ since $f \in l_1$.

Let X be a B-sp. X^*, dual, X^{**}, bidual.

On X^*, $\sigma(X^*, X^{**})$ topology has the same closed convex sets as the norm.

But if $X \neq X^{**}$, there are $\sigma(X^*, X^{**})$-closed Hyperplanes which are not $\sigma(X^*, X)$-closed.

<Reason>

$\mathcal{J}$; norm top. in X^*.

$\sigma(X^*, X) \subset \sigma(X^*, X^{**}) \subset \mathcal{J}$

have same closed convex sets

Let $f \in X^{**} \setminus X$

i.e. f is $\sigma(X^*, X^{**})$-conti, but not $\sigma(X^*, X)$-conti.

Hence $\ker f$ is $\sigma(X^*, X^{**})$ closed, but not $\sigma(X^*, X)$ closed.

$\therefore$ $\ker f$ is $\sigma(X^*, X)$ dense.

<u>Thm</u> $X \subseteq X^{**}$ is $\sigma(X^{**}, X^{*})$-dense in X^{**}.

<u>Pf</u> Let $\varphi \in X^{**}$.

Let $f_1, \dots f_n \in X^*$. Assume f_i's are indep.

Want $x \in X$ s.t $f_i(x)$ is close to $\varphi(f_i)$

But $\exists\ x_j$ in X, s.t $f_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$x = \sum_{i=1}^{n} x_i \varphi(f_i) \in X.$$

$$f_j(x) = f_j\left(\sum x_i \varphi(f_i)\right) = \varphi(f_j) \qquad //$$

U : unit ball of X

U° : " of X^*

$U^{\circ\circ}$: " of X^{**}

<u>Thm</u> U is $\sigma(X^{**}, X^*)$ dense in $U^{\circ\circ}$

Pf C : closure in $\sigma(X^{**}, X^*)$ of U.

Then $U \subseteq C \subseteq U^{\circ\circ}$ is $\sigma(X^{**}, X^*)$-compact

hence closed.

Supp. $\varphi \in U^{\circ\circ} \setminus C$. Then

Since C is convex, $\sigma(X^{**}, X^*)$-LCS,

$\exists$ $\sigma(X^{**}, X^*)$ conti ft f, s.t

$f(\varphi) > \alpha > f(c)$ $\forall c \in C$ by Thm. Sep. Hyperplan.

Now $f \in X^*$ and $\|f\| = \sup_{x \in U} |f(x)| \leq \alpha$

~~Take φ s.t $\|\varphi\| \leq$~~

$\varphi \in U^{\circ\circ} \Rightarrow \|\varphi\| \leq 1$.

$\therefore$ $\alpha < |\varphi(f)| \leq \|\varphi\| \|f\| \leq \alpha$ # contradiction

$\therefore$ $\varphi \notin U^{\circ\circ} \setminus C$

$\therefore$ U is $\sigma(X^{**}, X^*)$-dense in $U^{\circ\circ}$.

__Cor__ X is reflexive iff Unit ball is $\sigma(X, X^*)$-compact.

Pf. $\Rightarrow$ done

$\Leftarrow$ Supp. U is $\sigma(X, X^*)$-compact. then

U is $\sigma(X, X^*)$-closed.

$U \subseteq U^{\circ\circ} \subseteq X^{**}$.

Since U is $\sigma(X^{**}, X)$ closed,

$\mathrm{cl}_{\sigma(X^{**}, X)} U = U^{\circ\circ} \Rightarrow U = U^{\circ\circ} \Rightarrow X = X^{**}$.

//

<u>Thm</u> If X is a B-sp. then
X is reflexive iff X^* is reflexive.

<u>Remark</u> Reflexive normed spaces are B-sps.

<u>Pf</u> ($\Longrightarrow$) X : reflexive. then

$J_2 : X \longrightarrow X^{**}$ onto, $(J_2 x)(x^*) = x^*(x)$

Let $f \in X^{***}$. Want $f \in X^*$.

Since $X \subseteq X^{**}$, $f|_X = x^*$.

Now $J_3 : X^* \longrightarrow X^{***}$, $(J_3 x^*)(x^{**}) = x^{**}(x^*)$

$\therefore$ $x^{**} = J_2 x$ for some x.

$(J_2 x)(x^*) = x^*(x) = f(J_2 x) = f(x^{**})$.

so $J_3 x^* = f$ and X^* is reflexive.

($\Longleftarrow$) Supp. X is not reflexive, then $X \subsetneq X^{**}$.

X is normed closed in X^{**}.

Let $\varphi \in X^{**} \setminus X$ $f \in X^{***}$, $f(\varphi) = 1$, $f(x) = 0$ $\forall x \in X$.

And extending f by H.B, $f \neq 0$.

Supp. $x^* \in X^*$ s.t $J_3 x^* = f$

Then for $x \in X$, $f(x) = 0$ so

$$(J_3 x^*)(x) = x(x^*) = 0.$$

so x^* is zero.

Contradiction since

$$0 = J_3 x^* = f \neq 0.$$

so X^* is not reflexive //

Thus the double sequence

$$X^* \subseteq X^{***} \subseteq X^{*****} \subseteq \cdots$$

$$X \subseteq X^{**} \subseteq X^{****} \subseteq \cdots\cdots$$

either contains a reflexive space (in which case everything is reflexive and all the even $*$'s are X & odd $*$'s are X^*) or it contains a non-reflexive space (in which case everything is nonreflexive and each of the inclusions is proper),

POINT SET TOPOLOGY (FILTERS & NETS)

LET Γ BE AN UNCOUNTABLE SET

$X = \prod_{\alpha \in \Gamma} \mathbb{R}_\alpha$ UNCOUNTABLE PRODUCTS OF REALS
(EVERY WEAK TOPOLOGY IS A SUBSET)

Θ, WHOSE α^{th} COORDINATE IS 1, $\alpha \in \Gamma$

ϕ_F, $F \subseteq \Gamma$ AND F IS FINITE

$$\phi_F(\alpha) = \begin{cases} 1 & \alpha \in F \\ 0 & \alpha \notin F \end{cases}$$

PROP: IF $A = \{\phi_F : F \text{ IS FINITE} \subseteq \Gamma\}$, THEN $\Theta \in \mathrm{cl}\, A$

PF: LET U BE OPEN ABOUT Θ. INSIDE U THERE IS A BASIC OPEN SET, V, WHERE $\alpha_1, \alpha_2, \cdots, \alpha_n \in \Gamma$, $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$

$V = \{\psi \in X \mid |\psi(\alpha_i) - 1| < \epsilon_i,\ i = 1, 2, \cdots, n\}$

$F = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. NOTE $\phi_F \in V$.

THERE IS NO SEQUENCE IN A THAT CONVERGES TO Θ.
SUPPOSE $\phi_{F_1}, \phi_{F_2}, \cdots$ IS A SEQUENCE CONVERGING TO Θ:

$Z = \bigcup_{i=1}^{\infty} F_i \longleftarrow$ COUNTABLE SUBSET OF Γ

$\alpha \in \Gamma / Z$

$V = \{\psi \in X \mid |\psi(\alpha) - 1| < 1/2\}$

$\Theta \in V$ OPEN $\phi_{F_n} \notin V$ $\phi_{F_n}(\alpha) = 0$

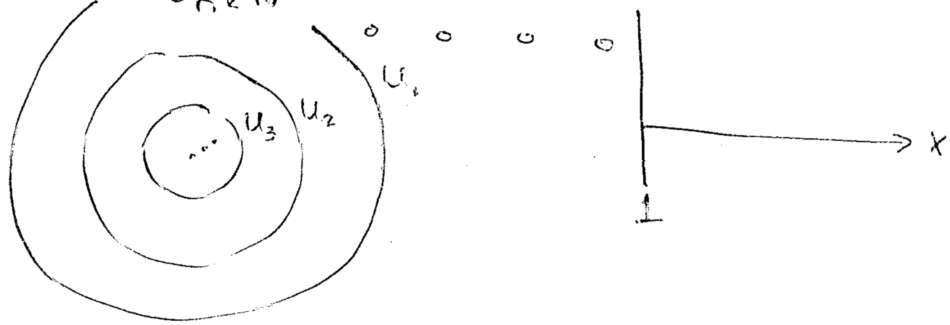
$f : A \cup \Theta \longrightarrow \{0, 1\}$

$f(A) = 0$ & $f(\Theta) = 1$ f IS NOT CONTINUOUS BUT f IS SEQUENTIALLY CONTINUOUS. i.e. $x_n \longrightarrow x$ THEN $f(x_n) \longrightarrow f(x)$. SEQUENTIAL CONTINUITY IS NOT ENOUGH.

(EXAMPLE IN WEAK TOPOLOGY THAT SHOWS WE CAN'T DEPEND ON SEQUENCES)

(ON NEXT PAGE)

EX: $\{x_n\}_{n \in N}$



IN EXAMPLE ABOVE WE DON'T HAVE SET UP AS ILLUSTRATED IN PICTURE.

$\mathcal{U}_x = \{ U_{\text{OPEN}} \mid x \in U \}$

$\mathcal{N}_x = \{ N \mid x \in \text{INT } N \}$ ← NHBDS

(ORDER THE STRUCTURE WE GET ON $\mathcal{U}_x$ & $\mathcal{N}_x$)

$\mathcal{B}_x = \{ B_{\text{OPEN}} \in \mathcal{U}_x \in \mathcal{U} \}$ BASE FOR TOPOLOGY AT x.

$U \le V \iff U \supseteq V$

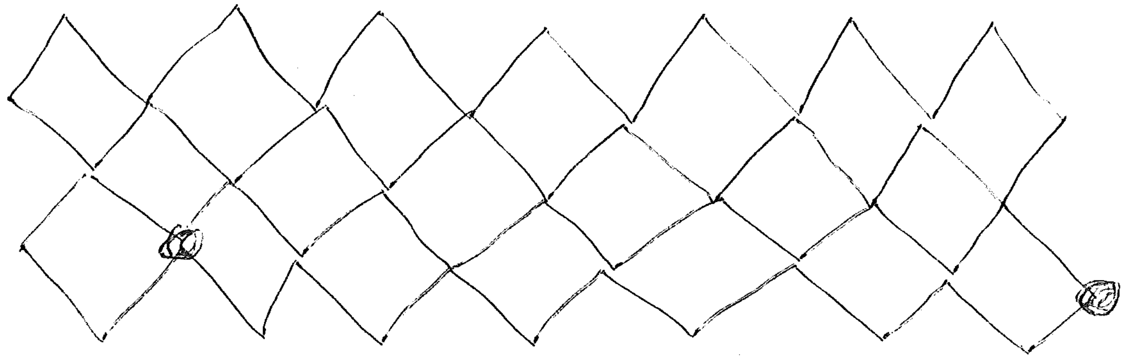
TAKE ANY TWO OPEN SETS, THE INTERSECTION IS OPEN ($\cap$ TWO NHBD'S IS A NHBD)

DEF: A PARTIALLY ORDERED SET $(D, \le)$ IS A <u>DIRECTED</u> <u>SET</u> IF $d_1, d_2 \in D$, $\exists\, d_* \in D \ni d_1 \le d_*$ & $d_2 \le d_*$

EX: $\mathcal{U}_x, N_x, \mathcal{B}_x$ ARE DIRECTED SETS

EX: INTEGERS ARE DIRECTED SETS

EX: PICTURE BELOW IS A DIRECTED SET



DEF: A <u>NET</u> IS A FUNCTION $f: D \to X$. $\alpha \in D$ $\{x_\alpha\}_{\alpha \in D}$

D ↑ DIRECTED SET

NET $\{x_\alpha\}_{\alpha \in D} \longrightarrow x \iff \forall$ OPEN $U, x \in U\ \exists N \ni \alpha \geq N$ THEN $x_\alpha \in U$.

x IS CLUSTER PT $\{x_\alpha\} \longleftrightarrow \forall$ OPEN $U, x \in U\ \forall N\ \exists \alpha \geq N\ x_\alpha \in U$.

PROP: $x \in cl\, A \iff \{x_\alpha\}_{\alpha \in D}$ NET $\subset A$ WITH $\{x_\alpha\} \to x$

PROP: f CONTINUOUS AT $x \iff (\{x_\alpha\} \to x \Rightarrow f(x_\alpha) \to f(x))$

PROP: A IS COMPACT, IF EVER NET HAS A CLUSTER ~~CLUSNET~~ THAT ~~CONVERGES TO A~~ POINT IN A (NOT TRUE FOR SEQUENCES)

DEF: A SUBNET OF $\{x_\alpha\}_{\alpha \in D}$ IS ANY $\{x_\alpha\}_{\alpha \in B}$, $B \subseteq D$

S.T. B SATISFIES $\forall \alpha \in D\ \exists \beta \in B$ with $\alpha \leq \beta$ [COFINAL]

CONSIDER: A, $x \in cl\, A$, $D = \mathcal{U}_x$, $U \leq V$, $\alpha \in D$

$\alpha = U$ OPEN

$x_\alpha = U \cap A$

$\{x_\alpha\} \to x$ LET U BE OPEN, $U = \alpha$

$\beta \geq \alpha$, $\beta = V$, $V \subset U$, $x_\beta \in V \cap A \subseteq U \cap A$

RECALL THEOREM: IF $\{x_n\} \subseteq$ NORMED SPACE, X, $x_n \to x$

THEN $\{x_n\}$ IS BDD.

PF: $\epsilon = 1$, PICK $N \ni n \geq N \Rightarrow \|x_n - x\| < 1$

$B = \text{MAX}\{\|x_1\|, \|x_2\|, \cdots \|x_n\|, 1 + \|x\|\}$ QED

NOT TRUE FOR NETS. CONVERGING NETS DO NOT HAVE TO BE BOUNDED, EVEN IN NORM SPACES.

FILTERS

CONSIDER $\{x_n \mid n \geq N\} = A_n$ (TAIL END OF SEQ)

WE KNOW $\{x_n\}_{n \in N} \longrightarrow x \iff \forall U$ OPEN $\exists N \ni n \geq N \Rightarrow x_n \in U$.

$\iff \exists_N\ A_n \subseteq U$

$\iff \exists\ N, W$, WITH $A_n \subseteq W \subseteq U$

$\iff \mathcal{F} = \{W \subset X \mid \exists N \ni A_n \subseteq W\}$

DEF: $\mathcal{F} \subseteq P(X)$ POWER SET OF X ∋

① $\emptyset \notin \mathcal{F}$

② $U \in \mathcal{F},\ V \supseteq U \Rightarrow V \in \mathcal{F}$

③ $U_1, U_2 \in \mathcal{F} \Rightarrow U_1 \cap U_2 \in \mathcal{F}$

THEN $\mathcal{F}$ IS A FILTER ON X.

EX: N_x = NHBDS OF x IS A FILTER

EX: B_x IS NOT A FILTER BUT A FILTER BASIS

i.e: ① $\emptyset \notin B_x$

② $U, V \in B_x \Rightarrow \exists W \in B_x$ WHERE $W \subset U \cap V$

IF ONE TAKES THE SET GENERATED BY B_x, ONE GETS A FILTER.

i.e. $\mathcal{F}(B_x) = \{W \mid \exists\ U \in B_x\ \&\ W \supseteq U\}$ i.e. THREW IN SUPERSETS

DEF: $\mathcal{F} \to x \iff \mathcal{F} \supseteq N_x$

REMARKS: SAME TYPES OF THINGS AS WITH NETS, $\{x_\alpha\}_{\alpha \in D}$

$A_\alpha = \{\beta \in D \mid \beta \geq \alpha\}$ $B = \{A_\alpha \mid \alpha \in D\}$

$\mathcal{F} = \mathcal{F}(B)$

$\{x_\alpha\} \to x \iff \mathcal{F} \to x$

WE CAN PARTIALLY ORDER THE FILTERS ON X: i.e.

$\mathcal{F} \geq \mathcal{G} \iff \mathcal{F} \supseteq \mathcal{G}$.

THEOREM: $\forall\ \mathcal{F}$ FILTERS ON X, $\exists\ \mathcal{U}$, ULTRAFILTER, $\mathcal{U} \supseteq \mathcal{F}$

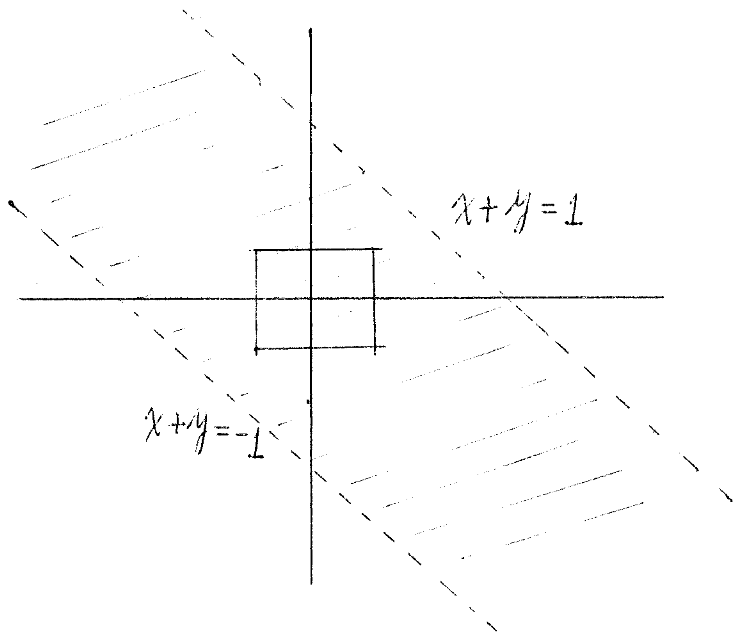
(IF $\mathcal{G}$ FILTER $\mathcal{G} \supseteq \mathcal{U} \Rightarrow \mathcal{U} = \mathcal{G}$)

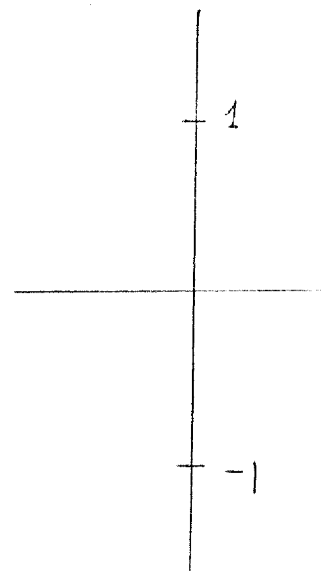
COMPACT $\iff$ ULTRAFILTER CONVERGES

THEOREM: IF $\mathcal{F}$ IS A FILTER ON X, $\mathcal{F}$ IS AN ULTRAFILTER $\iff \forall\ A \subseteq X$ EITHER $A \in \mathcal{F}$ OR ~~M~~ $X \setminus A \in \mathcal{F}$.

Notes of Jan 24

$\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$





A generating family of semi-norms

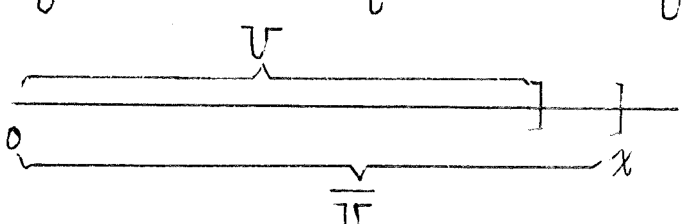
$$(E,\xi) \xrightarrow[1-1]{T} \prod_{p\in A} E_p \quad . \ T \text{ homeomorphism}$$

pick $p_1,\dots,p_n$. $U_i \subseteq E_{p_i}$ $\quad U_i = \{e\in E \mid p_i(e)<1\}$

$$U_1\times\cdots\times U_n \qquad W = \bigcap U_i \subset E$$

$$T(W) = V, \quad V = \{x\in \prod E \mid p_i(x_{p_i}) < 1\},\ i=1,2,\dots,n \}$$

$\|\ \|_U \ge \|\ \|_{\bar U}$ since $\|x\|_U > 1 = \|x\|_{\bar U}$



$\|\ \|_U = \|\ \|_{\bar U}$ ~~iff U is bounded~~. if U is a neighborhood

<u>Defn</u>. $\{X_\beta\}_{\beta\in B}$ is a sub net ~~cofinal subset~~ of $\{X_\alpha\}_{\alpha\in D}$ if $B\subseteq D$ and $\forall\, d\in D\ \exists\, b\in B$ with $d\le b$. (B is a cofinal subset of D)

<u>Theorem</u> A is compact $\Leftrightarrow \forall$ net $\{X_\alpha\}$ have cluster point in A.

<u>Defn</u>. $\{X_\alpha\}_{\alpha\in D}$ is a Cauchy net in TVS E if $\forall\, U$ open about $0\ \exists\, N\in D,\ \alpha,\beta\ge N \Rightarrow X_\alpha - X_\beta \in U$.

<u>Defn</u> ~~Theorem~~. TVS E is complete ~~compact~~ if every Cauchy net converges.

Look $\sigma(X, X^*)$ on normed space X.

$\{X_\alpha\}_{\alpha\in D} \longrightarrow X$ $\sigma(X,X^*)$ iff $\forall\, f\in X^*$ $\{f(X_\alpha)\}_{\alpha\in D} \longrightarrow f(X)$ in K

$\sigma(X^*, X)$: $\{f_\alpha\}\to f \Leftrightarrow \forall\, x\in X$ $f_\alpha(X)\longrightarrow f(X)$.

<u>Theorem</u>. If $\dim X < +\infty$, then $X, \sigma(X,X^*) \equiv$ norm topology and is complete.

Suppose $\dim X = +\infty$. Let $\{f_\alpha\}_{\alpha\in\Lambda}$ be a Hamel basis for X^* s.t. $\|f_\alpha\| = 1$.

$X \quad \sigma(X, X^*)$

if $f \in X^*$, then $f = \alpha_1 f_1 + \cdots + \alpha_n f_n$, $f_i \in \{f_\alpha\}$.

$|f(x)| \le |\alpha_1| |f_1(x)| + \cdots + |\alpha_n| |f_n(x)|$. thus

$X \quad \sigma(X, X^*) \longrightarrow \prod_{\alpha \in \Lambda} K_\alpha$ is homeomorphism into

$x \longmapsto \tilde{x} \quad f_\alpha(x) = \alpha$ th co-ordinate of $\tilde{x}$.

$D = \{F \subseteq \Lambda \mid F \text{ finite}\}$

$F \le G \iff F \subseteq G$ (directed set)

Let x_F be s.t. $f_\alpha \in F$, $f_\alpha(x) = 0$

If # of elements in $F = n$, then $\|x_F\| = n$.

$F = (f_1, \cdots, f_n, \cancel{f_{n+1}})$ add f_{n+1}

$\exists\, x_j$, $j = 1, \cdots, n+1 \ni f_i(x_j) = \delta_{ij}$

Let $x_F = n \dfrac{x_{n+1}}{\|x_{n+1}\|}$

$\{x_F\}_{F \in D} \longrightarrow 0$ in $\sigma(X, X^*)$

pick $f_1, \cdots, f_n$. Let $V = W(f_1, \xi_1) \cap \cdots \cap W(f_n, \xi_n)$ (open)

If $G \ge F = \{f_1, \cdots, f_n\}$, then $x_G \in V$. $\{x_F\}_{F \in G}$ is not bounded

$\{f_n^*\}_{n\in I} \cup \{f_\alpha\}_{\alpha\in \Lambda\setminus I}$

Let $f_1^*, \cdots, f_n^*, \cdots \subseteq \Lambda$, create $\{y_F\}_{F\in D}$ s.t.

$F = \{f_{n_1}^*, \cdots, f_{n_m}^*, g_1, \cdots, g_k\} \subseteq \Lambda$ where $f_{n_i}^*(y_F) = n_i$, $g_j(y_F) = 0$

for example,
$$\begin{array}{cccc} f_1^* & f_2^* & f_3^* & g \\ X_1 & X_2 & X_3 & X_4 \end{array}$$
$$y_F = X_1 + 2X_2 + 3X_3$$

claim $\{y_F\}_{F\in D}$ is a Cauchy net. Suppose $\{y_F\}_{F\in D} \to X \in \mathfrak{X}, \sigma(\mathfrak{X}, \mathfrak{X}^*)$

$$\|X\| = \sup_{\substack{f\in\mathfrak{X}^* \\ \|f\|\le 1}} |f(X)| \ge n \text{ for all } n \quad ※$$

since
$$\begin{array}{ccc} f_n^*(y_F) & \longrightarrow & f_n^*(X) \\ \text{eventually: } \| & & \| \\ n & \longrightarrow & n \end{array}$$

<u>Theorem</u> $\mathfrak{X}$ B-space, $\mathfrak{X}$ $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ complete $\Leftrightarrow$ $\dim \mathfrak{X} < +\infty$.

U = unit ball of $\mathfrak{X}$

U is $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ complete $\Leftrightarrow$ U is $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ compact

$\Leftrightarrow$ $\mathfrak{X}$ is reflexive

WHAT ABOUT WEAK SEQUENCE ?

$\mathfrak{X}$ is weakly sequentially complete iff every weak Cauchy sequence converges to a element of $\mathfrak{X}$.

$\{X_n\} \subseteq X$ is a weak Cauchy sequence $\Leftrightarrow \forall f \in X^* \; \{f(x_n)\}$ is a Cauchy sequence in K.

Example. C_0 is not weakly sequentially complete

Let $X^{(n)} = (\underbrace{1, 1, \cdots, 1}_{n}, 0, \cdots)$, then $\{X^{(n)}\}$ is a weak Cauchy sequence. (For $f \in l_1$, $\exists N \ni n, m \geq N \Rightarrow$

$|f(X_n - X_m)| < \varepsilon$)

But $\{X^{(n)}\}$ does not converge to an element of C_0, since

$C_0 \subseteq l_\infty \quad \sigma(l_\infty, l_1)$ topology

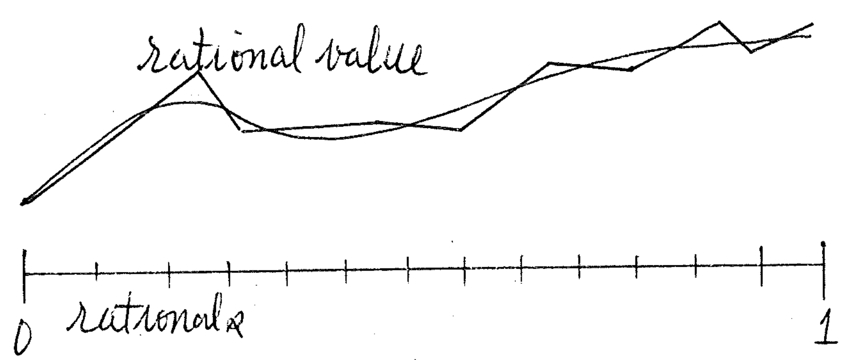
$\sigma(l_\infty, l_1)|C_0 = \sigma(C_0, l_1)$

$\{X^{(n)}\}$ is weak* Cauchy sequence in l_∞

$\{X^{(n)}\} \longrightarrow \theta = (1, 1, \cdots, 1) \notin C_0$

Theorem. $\{f_n\}$ is weak Cauchy sequence. Then $\{f_n\}$ is norm bounded

#6 of HW. $C[0,1]$ is separable



Theorem - Let X be a Banach Space. Let $B \subseteq X^*$ be $\sigma(X^*, X)$ bounded. Then B is norm bounded.

Proof: It follows by definition that B is bounded in $\sigma(X^*, X)$ if and only if for every $x \in X$, $\sup_{f \in B} |f(x)| < \infty$.
By the Principle of Uniform Boundedness, there exists an M such that $\|f\| \leq M$ for all $f \in B$. Hence, B is norm bounded.

Note that the above may be false if X is only a norm space and not a Banach space.

Cor: If X is a normed space and $B \subseteq X$ is $\sigma(X, X^*)$ bounded, then B is norm bounded.
Proof: Consider $X \subseteq X^{**}$.
The $\sigma(X^{**}, X^*)$ topology, restricted to X, is the $\sigma(X, X^*)$ topology on X.
Then a set that is weak bounded in X is weak star bounded in X^{**}.
Since X^* is a Banach space, the result follows from the theorem above.

Def: $\{x_n\} \subseteq X$ is a $\sigma(X, X^*)$ Cauchy sequence if and only if for all $f \in X^*$, $\{f(x_n)\}$ is a Cauchy sequence of scalars.

Cor: Let X be a Banach space. If $\{x_n\} \subseteq X$ is a $\sigma(X, X^*)$ Cauchy sequence, then $\{x_n\}$ is norm bounded

Proposition: If $\{x_n\} \subseteq \mathbb{X}$ is a sequence such that $\{x_n\} \longrightarrow x$ in norm, then $\{x_n\} \longrightarrow x$ in $\sigma(\mathbb{X}, \mathbb{X}^*)$.

Proof: Let $\{x_n\} \subseteq \mathbb{X}$ be a sequence such that $\|x_n - x\| \longrightarrow 0$.

We want to show that for all $f \in \mathbb{X}^*$, $|f(x_n - x)| \to 0$.

Now $\|f\| \, \|x_n - x\| \longrightarrow 0$, so $|f(x_n - x)| \to 0$ ∎

Let $\{u_n\} \subset l_2$ be the usual basis for l_2 - that is, $\{u_n\}$ has 1 in the nth slot and 0 elsewhere. $\{u_n\}$ does not converge to anything in norm, but $\{u_n\} \longrightarrow 0$ weakly.

Claim: Let $\{x_n\} \subset l_2$. Then $\{x_n\} \longrightarrow x$ in norm if and only if $\{x_n\} \longrightarrow x$ weakly <u>and</u> $\|x_n\| \longrightarrow \|x\|$.

Proof: ($\Longrightarrow$) This part of the problem follows immediately from the continuity of $\|\cdot\|$ and the fact that the weak topology is weaker than the norm topology.

($\Longleftarrow$) If $\|x\| = 0$, we are done.

If $\|x\| \neq 0$, we may assume without loss of generality that $\|x\| = 1$.

We may also assume $\|x_n\| = 1$ for all n

(Let $\lambda_n = \|x_n\|$. For n sufficiently large, $\|x_n\| \neq 0$, so we can ignore the terms such that $\|x_n\| = 0$.

Let $y_n = \frac{x_n}{\lambda_n}$. Now $\lim_n \lambda_n = 1$, so $\lim_n \frac{1}{\lambda_n} = 1$. Since $x_n \to x$ and $\frac{1}{\lambda_n} \to 1$, $\frac{x_n}{\lambda_n} \longrightarrow x$. Hence, $y_n \longrightarrow x$ and $\|y_n\| \underset{n}{\equiv} 1$)

Let $\{x_n\} \to x$ weakly, where $\|x_n\| \equiv \|x\| = 1$.
We want to show that $\{x_n\} \to x$ in norm.

Let $\varepsilon > 0$ be given. Let $x = x^1, x^2, x^3, \ldots$ and
let $x_n = x_n^1, x_n^2, x_n^3, \ldots$.
There is an N such that $\left[\sum_{i \geq N} |x^i|^2\right]^{\frac{1}{2}} < \varepsilon$,
since $\{x^1, x^2, \ldots\} \in l_2$. [This N is good only for x]

$\exists \delta > 0$ such that if $|y^i - x^i| < \delta$, $i = 1, 2, \ldots, N$,
then $\|(y_1, \ldots, y_n) - (x_1, \ldots, x_N)\|_{l_2} < \varepsilon$
since $f(y_1, \ldots, y_n) = \left[\sum_{i=1}^{n} (y_i - x_i)^2\right]^{\frac{1}{2}}$ is a
continuous function.

Let f_i be in l_2^*, where $1 \leq i \leq N$ and f_i has a 1
in the ith slot and 0 elsewhere.
Then $W_x = \{y \in l_2 : |f_i(y - x)| < \delta$ for $i = 1, \ldots, N\}$ is
a neighborhood of x. ($\{y \in l_2 : |f_i(y - x)| < \delta\}$ is
a neighborhood of x for $i = 1, \ldots, N$. The intersection of
a finite number of neighborhoods is still a neighborhood.)

There is a M such that $n \geq M \Rightarrow x_n \in W_x$.
Now $\|(x_n^i)_{i=1}^N - (x^i)_{i=1}^N\| < \varepsilon$, so
$$\|(x_n^i)_{i=1}^N\| > \sqrt{1-\varepsilon^2} - \varepsilon.$$
$$\left[(a^2 + \varepsilon^2)^{\frac{1}{2}} > \|x\| = 1 \Rightarrow a^2 + \varepsilon^2 > 1 \Rightarrow a > \sqrt{1 - \varepsilon^2} \right]$$

Let b denote $\|(x_n^i)_{N+1}^\infty\|$ and let c denote $\|(x_n^i)_1^N\|$.
Now $(x_n^i)_{N+1}^\infty$ and $(x_n^i)_1^N$ are orthogonal, so by
the Pythagorean theorem, $b^2 + c^2 \equiv 1$.
Hence,
$$\sqrt{1 - (1 - \varepsilon^2 - 2\varepsilon\sqrt{1-\varepsilon^2} + \varepsilon^2)} > b.$$

Then if $n \geq M$,

$\|(x_n^i)_{i=1}^{\infty} - (x^i)_{i=1}^{\infty}\| \leq$ (by the triangle inequality)

$\|(x_n^i)_{i=1}^{N} - (x^i)_{i=1}^{N}\| + \|(x_n^i)_{i=N+1}^{\infty} - (x^i)_{i=N+1}^{\infty}\| \leq$
(triangle inequality again)

$\|(x_n^i)_{i=1}^{N} - (x^i)_{i=1}^{N}\| + \|(x_n^i)_{i=N+1}^{\infty}\| + \|(x^i)_{i=N+1}^{\infty}\| <$
(making substitutions from the previous page)

$\varepsilon + \sqrt{1 - (1 - \varepsilon^2 - 2\varepsilon\sqrt{1-\varepsilon^2} + \varepsilon^2)} + \varepsilon.$

Now the above goes to zero as ε goes to zero.

Then $\lim_{n\to\infty} \|(x_n^i)_{i=1}^{\infty} - (x^i)_{i=1}^{\infty}\| = 0$, so $x_n \to x$ in norm, completing the proof.

The claim also holds for l_p $1 \leq p < \infty$ – in fact, the proof is easier in l_1.

Theorem: Let $\{x_n\} \subseteq l_1$. Then $\{x_n\} \to x$ in $\sigma(l_1, l_\infty)$ iff $\{x_n\} \to x$ in norm topology.

Jan 31

Midterm Exam:

1. $e_n = (0, \cdots, 0, 1, 0, \cdots)$
$\uparrow$ n^{th} place

a) Show $e_n \to 0$, $\sigma(l_1, c_0)$.
Let $(\xi) = (\xi_n) \in c_0$.

$|\langle \xi, e_n \rangle| = |\xi_n| \to 0$ since $\xi \in c_0$.

b) $e_n \not\to 0$, $\sigma(l_1, l_\infty)$.

$\theta = (1, 1, \cdots)$. Then $\langle \theta, e_n \rangle = 1 \not\to 0$.

2. Let X, $\|\cdot\|$ be strictly convex.
S $\|f\| = 1$, $f \in X^*$, $x \neq y$, $\|x\| = \|y\| = 1 = f(x) = f(y)$.

Then $\|\frac{x+y}{2}\| < 1$ & $f(\frac{x+y}{2}) = \frac{1}{2}(f(x) + f(y)) = 1$

$\Rightarrow \|f\| > 1$ #

3. S $X \xrightarrow{T} Y$ norm cont. Show T is wk-wk cont.

For $y^* \in Y^*$, $|\langle y^*, \cdot \rangle|$ semi-norm,

$|\langle y^*, T(\cdot) \rangle| = |\langle T^* y^*, \cdot \rangle|$

since T is norm cont. $T^* y^* \in X^*$
and so is a cont. seminorm with $\sigma(X, X^*)$ top. on X.

4. $\not\exists$ $\sigma(X^*, X^{**})$ + $\sigma(X^*, X)$
agree on $U^0 \subseteq X^*$.

U^0 is $\sigma(X^*, X)$ compact
$\Rightarrow U^0$ is $\sigma(X^*, X^{**})$ compact
$\Rightarrow X^*$ is reflexive
$\Rightarrow X$ is reflexive. done.

Note: This implies $\sigma(X^*, X^{**}) = \sigma(X^*, X)$
on X^*.

But consider $\mathbb{N}$ with top. from $\mathbb{R}$,
$P = \{ \{0\} \cup \{\frac{1}{n}\} : n \geq 2 \}$ with top. from $\mathbb{R}$.

$$\begin{array}{cccc} 1 & 2 & 3 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \\ 0 & \frac{1}{2} & \frac{1}{3} & \cdots \end{array}$$

Then P is compact while
$\mathbb{N}$ is not.
yet the topologies agree for
any finite sequence

Consider also l_f = finitely many non-zero seq.
with sup norm.

c_0 its completion.

$l_f^* = c_0^* = l_1$.

U = unit ball of l_1.

U is $\sigma(l_1, l_f)$ compact + $\sigma(l_1, c_0)$ compact,

on U ~~so~~ two top. agree,

but $l_1, \sigma(l_1, l_f) \neq l_1, \sigma(l_1, c_0)$.

5. $\ni$ $\mathfrak{X}, \sigma(\mathfrak{X}, \mathfrak{X}^*) \xrightarrow{T} Y, \ \sigma(Y, Y^*)$ norm cont.,

$U_Y = \{ y \in Y \mid \|y\| \leq 1 \}$.

<u>Claim:</u> $T^{-1}(U_Y)$ is convex balanced absorbing + norm closed.

~~Pf of claim~~: $\ni$ $x, y \in T^{-1}(U_Y)$, $\lambda \ni |\lambda| \leq 1$, $s,t \geq 0$, $s+t=1$.

Then [redacted] $\|T(sx+ty)\| = s\|Tx\| + t\|Ty\| \leq 1$
$\Rightarrow sx+ty \in T^{-1}(U_Y)$.

$\|T(\lambda x)\| = |\lambda| \cdot \|Tx\| \leq 1 \Rightarrow \lambda x \in T^{-1}(U_Y)$.

$z \in \mathfrak{X}$, $Tz \in Y$, $\|T(z)\| < +\infty$.

$\forall \epsilon \ni \epsilon\|Tz\| < 1$ we have $\epsilon z \in T^{-1}(U_Y)$
since $\|T(\epsilon z)\| = |\epsilon| \cdot \|Tz\| < 1$.

So U_Y - norm closed + convex
U_Y - wk convex
$T^{-1}(U_Y)$ - wk closed + convex [T is wk cont]
$T^{-1}(U_Y)$ - norm closed

$\Rightarrow$ result by Baire category th.

~~Back to class notes~~:

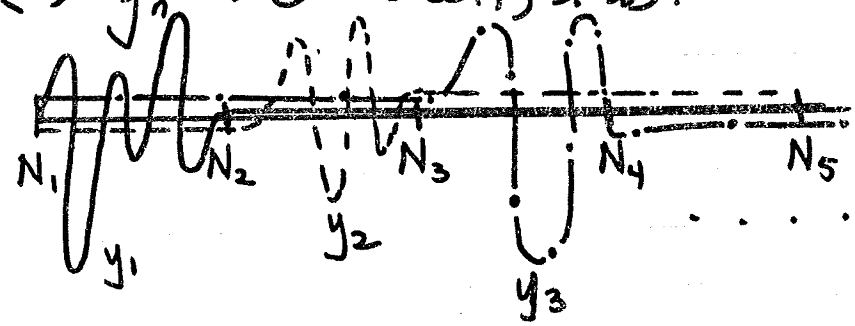
<u>Thm.</u> $\{x_n\} \subseteq l_1$ then $\{x_n\} \to x$ in norm
$\Leftrightarrow \{x_n\} \to x \quad \sigma(l_1, l_\infty)$.

Pf (⇒) easy since wk. top. is weaker.

<u>Lemma:</u> Suppose $0 = N_1 < N_2 < N_3 < \cdots$ is an increasing sequence of integers, $\{e_n\}$ is the usual basis for l_1.

Suppose $y_n = \sum_{i>N_n}^{N_{n+1}} \alpha_i^n e_i$. Then $y_n \to 0$ in norm

$\Leftrightarrow y_n \to 0 \quad \sigma(l_1, l_\infty)$.

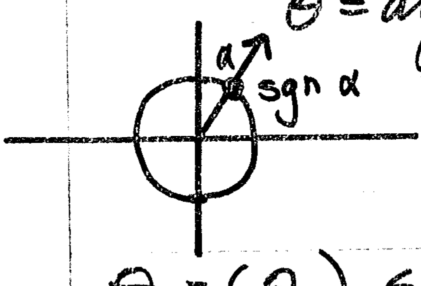


<u>Pf of lemma</u>. (⇒) Already.

(⇐) Suppose $y_n \to 0$, $\sigma(l_1, l_\infty)$ but $y_n \not\to 0$ in norm.

$\exists\ \epsilon > 0$, subseq. $\{y_{n_j}\} \ni \|y_{n_j}\| \geq \epsilon$.

Suppose $\alpha \in \mathbb{C}$, then $\operatorname{sgn} \alpha = e^{i\theta} \ni |\alpha| \operatorname{sgn} \alpha = \alpha$.
$\theta = \arg \alpha$.



If $\alpha \in \mathbb{R}$, $\operatorname{sgn} \alpha = \begin{cases} 1 & \alpha \geq 0 \\ -1 & \alpha < 0 \end{cases}$

$|\alpha| = \overline{\operatorname{sgn} \alpha} \cdot |\alpha| \cdot \operatorname{sgn} \alpha = \alpha\, \overline{\operatorname{sgn} \alpha}$.

$\theta = (\theta_k) \in l_\infty$,

$\theta_k = \overline{\operatorname{sgn} \alpha_k^{n_j}}$ for $N_{n_j} < k \leq N_{n_j+1}$, $j = 1, 2, \cdots$

$\theta_k = 0$ otherwise.

$$\langle \theta, y_{n_j} \rangle = \sum_{N_{n_j}+1}^{N_{n_{j+1}}} \overline{\operatorname{sgn} \alpha_i^{n_j}}\, \alpha_i^{n_j} = \sum_{N_{n_j}+1}^{N_{n_{j+1}}} |\alpha_i^{n_j}|$$

$= \|y_{n_j}\| \geq \epsilon$ # contradiction ~~since~~ to

y_{n_j} ~~θ_n~~ $\to 0$ wkly. ($\theta \in \ell_\infty$ if $\|\theta\|_\infty = 1$).

Back to pf. of thm. ($\Leftarrow$).

We may assume $x = 0$ since $x_n \to x \Leftrightarrow x_n - x \to 0$ in either top.

We may assume $\{x_n\} \to 0$ $\sigma(\ell_1, \ell_\infty)$ + $\{x_n\} \not\to 0$ in norm.

In fact, we may assume $\|x_n\| = 1$ for all n.

$\|x_n\| \geq \epsilon$ for a subseq. (throw everything else away).

$\exists M \ni \epsilon \leq \|x_n\| \leq M$. (wk. c.s. are norm bdd).

$\lambda_n = \frac{1}{\|x_n\|} \Rightarrow \epsilon \leq \lambda_n \leq M$, $\lambda_n \in [\epsilon, M]$

hence $\exists$ a subseq. of λ_n which converges to λ^* (pass to that subseq.)

$\lambda_n x_n \to 0 \Leftrightarrow x_n \to 0$ in either top. since $\lambda_n \to \lambda^*$

$\lambda_n^{-1} \to \lambda^{*-1}$!

Inductively define

$\left.\begin{array}{l} N_1 = 0 < N_2 < \cdots \\ \& \; n_1 < n_2 < n_3 < \cdots \end{array}\right\}$ *integers*

$\ni$ (1) $\sum_{N_i+1}^{N_{i+1}} |x_{n_i}^k| \geq \frac{3}{4}$ *norm of* X_{n_i} *between* N_j+1 *and* N_{j+1}. *If I can do this, I can obtain a contradiction.*

[We have, as before $\theta_k = \overline{sgn\, x_{k_j}^k}$,
$\theta_k = 0$ *otherwise.*

$\langle \theta_j, x_{n_j} \rangle = \sum_{i=N_j+1}^{N_{j+1}} \overline{sgn\, x_{n_j}^k}\; x_{n_j}^k$ *+ other terms.]*

$\|$ *other terms* $\| \leq \|\theta\| \cdot \| x_{n_j}$ off of $N_{n_j}+1$ to $N_{n_{j+1}} \| < 1 \cdot \frac{1}{4}$

So $|\langle \theta, x_{n_j} \rangle| \leq \frac{1}{4} + \sum_{N_{n_j}+1}^{N_{n_{j+1}}} |x_{n_j}^k|$

$|\langle \theta, x_{n_j} \rangle| \geq \sum_{k=N_j+1}^{N_{j+1}} |x_{n_j}^k| - \frac{1}{4}$

$\geq \frac{3}{4} - \frac{1}{4} \geq \frac{1}{2}$ # $x_{n_j} \to 0$ $\sigma(l_1, l_\infty)$.

Back to pf.

$N_1 = 0$, $n_1 = 1$, $\|x_1\| = \sum_{k=1}^{\infty} |x_1^k|$. $\exists N_2$

$\ni \sum_{k=N_2+1}^{\infty} |x_{n_i}^k| < \frac{1}{4}$. *Since* $\{x_n\} \to 0$ *weakly,*

$\exists n_2 \ni m > n_2 \Rightarrow \sum_{k=1}^{N_2} |x_m^k| < \frac{1}{8}$.

Pick $N_3 > N_2 \ni \sum_{k=N_3+1}^{\infty} |x_{n_2}^k| < \frac{1}{8}$. *By induction, done.*

Let Ω be a set, Σ a σ-algebra defined on Ω, and μ a positive countably additive measure defined on Σ.

Define

$$L_1(\mu) \stackrel{\text{def}}{=} \{f : \Omega \to K \ni f \text{ is measurable}$$

$$\text{and } \|f\|_1 < \infty\} \quad \text{where } \|f\|_1 = \int_\Omega |f|\, d\mu .$$

$$L_\infty(\mu) \stackrel{\text{def}}{=} \{f : \Omega \to K \ni f \text{ is measurable}$$

$$\text{and } \|f\|_\infty < \infty\} \quad \text{where } \|f\|_\infty = \operatorname*{ess\,sup}_{t\in\Omega} |f(t)| =$$

$$= \inf\{M : \mu\{t : f(t) > M\} = 0\}$$

= the infimum of $\sup g(t)$ as g ranges over all functions which are equal to f almost everywhere μ.

Note that for each

$$g \in L_\infty(\mu) \qquad T_g(f) = \int_\Omega f g \, d\mu$$

defines a linear functional on $L_1(\mu)$

In fact with this identification

$$L_1(\mu)^* = L_\infty(\mu)$$

i.e., $L_\infty(\mu)$ is the dual of $L_1(\mu)$

<u>Examples</u>

(1) If $\Omega = [0,1]$, Σ = Borel σ-algebra

and μ = Lebesgue measure.

$$L_1(\mu) = L.$$

(2) If $\Omega = \mathbb{N}$, $\Sigma = \mathcal{P}(\mathbb{N})$

and μ = counting measure

$$L_1(\mu) = \ell_1$$

Note that:

(1) $\{f_n\} \subseteq L_1(\mu)$ is <u>weak cauchy</u> iff

$T_g(f_n)$ is a cauchy sequence of

scalars for each $g \in L_\infty(\mu)$.

iff $\lim_{n\to\infty} T_g(f_n)$ exists $\forall g \in L_\infty(\mu)$

(2) $\{f_n\} \subseteq L_1(\mu)$ <u>converges weakly</u> to f

iff $\lim_{n\to\infty} T_g(f_n) = T_g(f)$ ~~exists~~ for all $g \in L_\infty(\mu)$

<u>Theorem 1</u> (1) $\{f_n\} \subseteq L_1(\mu)$ is a cauchy sequence $\Longleftrightarrow$ $\{f_n\}$ is norm bounded and $\forall E \in \Sigma$, $\lim_{n\to\infty} \int_E f_n \, d\mu$ exists

(2) $\{f_n\} \subseteq L_1(\mu)$ converges weakly to f $\Longleftrightarrow$ $\{f_n\}$ is bounded and $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$ for all $E \in \Sigma$.

(3) L_1 is weakly sequentially complete

Def: A space is weakly sequentially complete if every weak cauchy sequence has a weak limit

Remark: Every reflexive space is weakly sequentially complete since $\{f\}$ wk cauchy sequence $\Rightarrow$ norm bounded $\Rightarrow \{f_n\} \subseteq$ wk compact set $\Rightarrow f_n \xrightarrow{wk} f_0$ for some f_0.

Proof of (1).

($\Rightarrow$) We already know $\{f_n\}$ is norm bounded. Now $E \in \Sigma \Rightarrow \chi_E \in L_\infty$

so $\lim_{n\to\infty} \int_E f_n \, d\mu = \lim_{n\to\infty} \int f_n \chi_E \, d\mu$

$= \int_E f \, d\mu$ exists

($\Leftarrow$) Suppose s is a simple function

$$s = \sum_1^n \alpha_i \chi_{E_i} \in L_\infty$$

$$\int f_n s \, d\mu = \sum_{i=1}^N \alpha_i \int_{E_i} f_n \, d\mu$$

Hence $\int f_n s \, d\mu$ has a limit as $n \to \infty$

Now we need a lemma.

<u>Lemma:</u> Simple functions are dense in $L_\infty(\mu)$ $[\forall g \in L_\infty(\mu) \; \forall \epsilon > 0, \exists s \ni \|g - s\|_\infty < \epsilon]$.

Proof: Let g be given consider the ball of radius $\|g\|_\infty$ in K

There exists a partition of this set into a finite collection of measurable sets $\{A_i\}$ such that $t, s \in A \Rightarrow |s - t| < \epsilon$

$\sum g(a_i) \chi_{g^{-1}(A_i)} \equiv s \quad a_i \in A_i$

off a set of measure zero

Hence $\|s - g\|_\infty < \varepsilon$ ∎

To finish (1) need to show $g \in L_\infty$

$\Rightarrow \lim\limits_{n \to \infty} \int f_n g \, d\mu$ exists

$\forall \varepsilon > 0 \ \exists s$ simple function with

$\|g - s\|_\infty < \varepsilon$ since f_n is bounded

$\exists M$ such that $\|f_m\| < M$ for all m

$$\left| \int_\Omega f_n h \, d\mu - \int_\Omega f_n s \, d\mu \right| \le$$

$$\|f_n\|_1 \, \|g - s\| \le M\varepsilon$$

$\exists M$, $m, n \ge N \Rightarrow$

$$\left| \int f_n s \, d\mu - \int f_n s \, d\mu \right| < \varepsilon$$

Thus $\int f_n g \, d\mu$ is C.S.

$m, n \ge N$

$$\left|\int f_n g - \int f_m g\right| \le \left|\int f_n(g-s)\,d\mu + \int (f_n - f_m) s\,d\mu + \int f_m (s-g)\,d\mu\right|$$

$$\le \left|\int f_n (g-s)\,d\mu\right| + \left|\int (f_n - f_m) s\,d\mu\right| + \left|\int f_m(s-g)\,d\mu\right| \le 2M\varepsilon + \varepsilon$$

done with 1.

(2) $\Rightarrow$ easy since wk bounded $\Rightarrow$ norm bounded & $\chi_E \in L_\infty$

Suppose $\{f_n\}$ is bdd & $\lim_n \int_E f_n \,d\mu = \int_E f\,d\mu \quad \forall E \in \Sigma$

want to show

$$\lim_{n \to \infty} \int f_n g\,d\mu = \int fg\,d\mu \quad \forall g \in L_\infty$$

Well as before it works for g a simple function in L_∞

$h \in L_\infty$ $\quad$ s simple

$$\int fh\,d\mu = \int fs\,d\mu + \int f(h-s)\,d\mu$$

want $\quad$ know $\quad$ both these can be made small

$$\int f_n h\,d\mu = \int f_n s\,d\mu + \int f_n(h_n - s)\,d\mu$$

(or you can proceede as in (1) above)

To show (3)

Proof: (3) If $\{f_n\}$ is wk C.S $\Rightarrow$ $\exists f$ such that $\{f_n\} \to f$ wk. Let $\{f_n\}$ be a wk C.S. We may assume (Ω, Σ, μ) is σ-finite

supp $f_n = \{x \in \Omega \mid f(x) \neq 0\}$ - σ-finite measure

restrict $\Omega = \bigcup_{n=1}^{\infty} \text{supp}(f_n)$

$\lambda(E) = \lim_{n\to\infty} \int_E f_n\,d\mu$ countable additive measure finite signed measure $\lambda_n \ll \mu$

need to show countably additivity will do next time. Now assume it

[count additive seems to require Vitali - Hahn - Saks Thm]

$\exists f \in L(\mu)$ s.t. $\lambda(E) = \int_E f \, d\mu$

Thus $\lim \int_E f_n \, d\mu = \int f \, d\mu = \lambda(E)$

for $E \in \Sigma$ hence

$\{f_n\} \to f$ wk by (2).

Let C be a convex set in Vector sp.

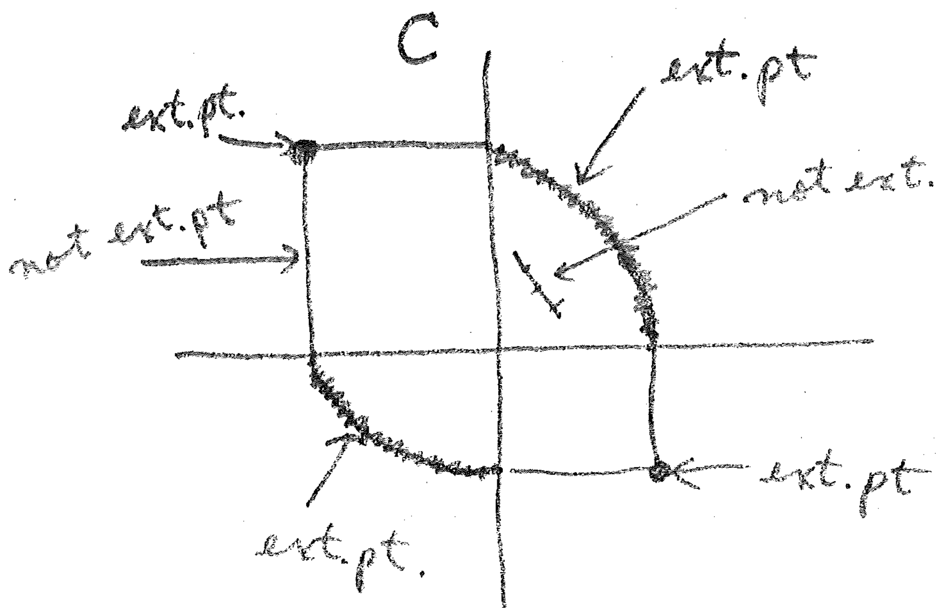
<u>Defn</u> $x \in C$ is said to be an <u>extreme pt.</u>
iff any of equivalent conditions hold

(1) $y, z \in C$ & $\frac{y+z}{2} = x \Longrightarrow y = x = z$
or
(2) x is not in the "interior" of any line segment $\subseteq C$

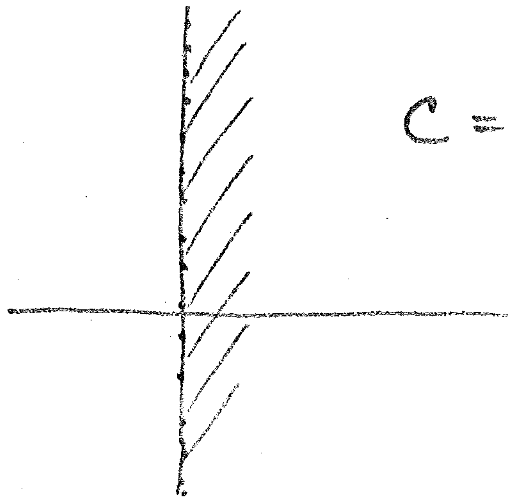
$[y, z] = \{sy + (1-s)z, \ 0 \le s \le 1\}$

"interior" of $[y, z] = \{sy + (1-s)z, \ 0 < s < 1\}$

<u>These are equivalent</u>



C : convex hull of its extreme pt.



$C = \{(x,y) ; x > 0\}$

C has not ext. pt.

$\overline{C}$ 〃 .

The unit ball of a strictly convex norm sp has the unit sphere as its set of extreme pts.

Pf U : unit ball. S : Unit sphere $= \{x : \|x\| = 1\}$

Let $x \in S$. Want to show x is ext. pt. of U.

Supp. $y, z \in U$. and $\frac{y+z}{2} = x$.

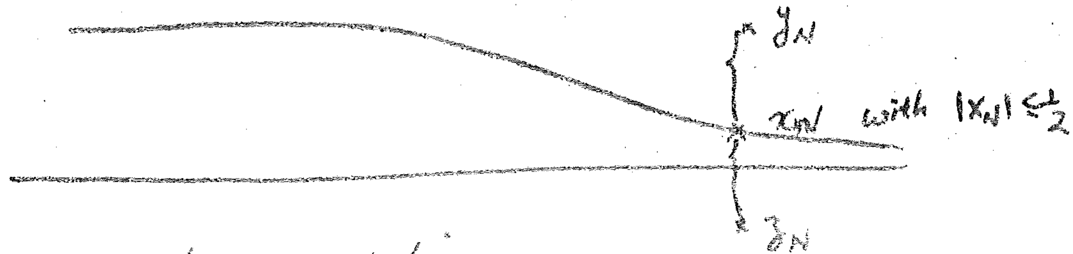
If $y \neq z$, then $\|\frac{y+z}{2}\| < 1$.

Thus $x = y = z$. //

Since every seperable sp. can be renormed to be strict convex, it has lots of ext. pts.

The unit ball of c_0 or L_1 has no ext. pts. T>

$(x_n) \in c_0$, $\|(x_n)\| = 1$



Supp. $x_n \to 0$. Then define

$$y_n = \begin{cases} x_n & \text{if } n \neq N \\ 1 & n = N \end{cases} \qquad \therefore \|(y_n)\| = \|(z_n)\| = 1$$

$$z_n = \begin{cases} x_n & \text{if } n \neq N \\ x_N - (1 - x_N) & \text{if } n = N \end{cases}$$

But $\frac{(y_n) + (z_n)}{2} = (x_n)$ and $(y_n) \neq (x_n) \neq (z_n)$ //

$f \in L_1$, $\|f\| = 1$

pick $s \in [0,1]$ s.t. $\int_0^s |f| \, d\mu = \frac{1}{2}$

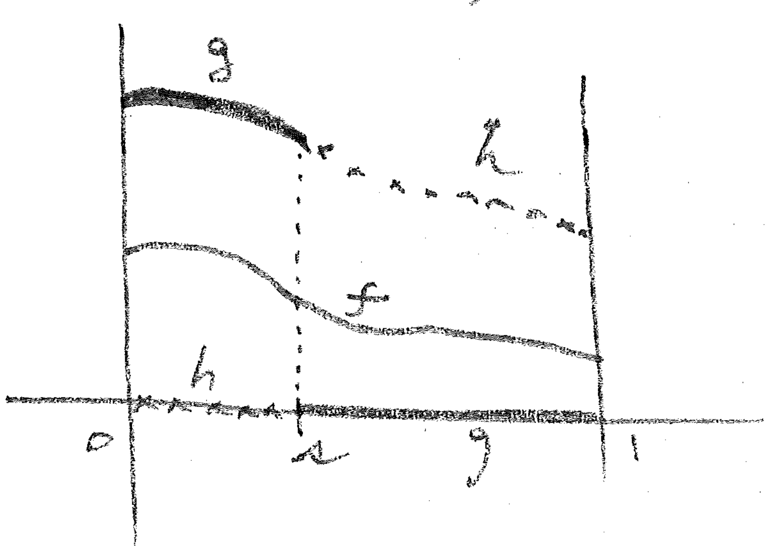
Define
$$g(t) = \begin{cases} 2f(t) & t < s \\ 0 & \text{o.w.} \end{cases}$$

$$h(t) = \begin{cases} 0 & t < s \\ 2f(t) & \text{o.w.} \end{cases}$$

$\therefore \frac{g+h}{2} = f$ a.e

$g \neq h$.

$\|g\| = \|h\| = 1$.



$\|g\| = \|h\| = 1$

Theorem (Krein Milman) If C is a compact, convex set in Real LCS E, then C is closed convex hull of its ext. pts.

Cor The unit ball of a reflexive sp. or a dual sp. is the closed convex hull of it extreme pt. ($\because$ unit ball is w^*-compact)

e.g. $\{\pm e_n\}$: ext. pts. of l_1.

Lemma Every compact convex set has an extreme pt. in LCS.

proof of Thm given the lemma.

Let C be a compact convex set

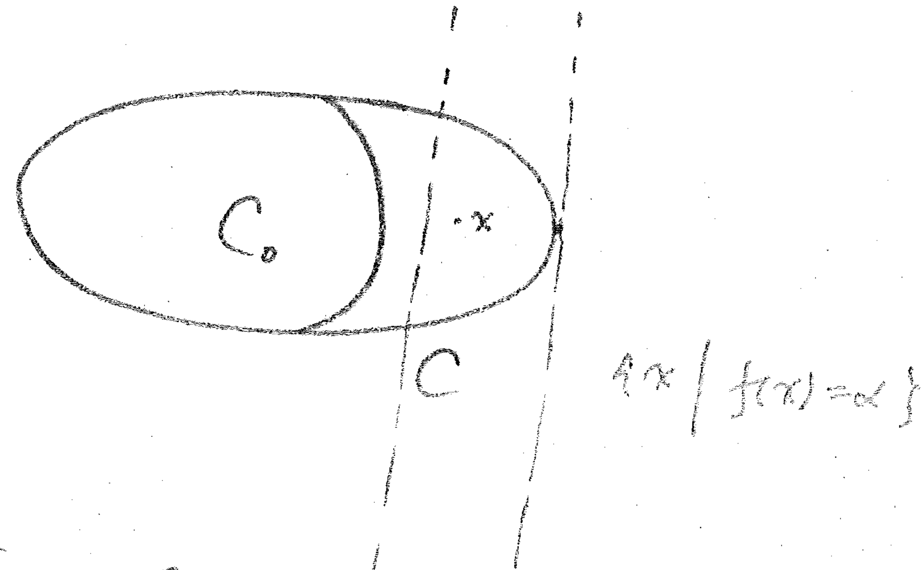
Let C_0 be a closed convex hull of its extreme pts.

If $C = C_0$, then done.

O.W. $\exists\, x \in C \setminus C_0$.

Since E is LCS & H.B Thm,

$\exists\, f \in E^*$ s.t $f(x) > \alpha > f(c) \quad c \in C_0$



Since C is compact, $\exists\, y \in C$, s.t. $f(y) = \sup\limits_{c \in C} f(c)$

Since f is conti. on compact set,
it obtains its sup.

$K = C \cap \{x : f(x) = f(y)\}$

K is convex, compact

By lemma, must have an extreme pt. $k \in K$.

<u>claim</u> k is ext. pt. of C.

Supp. not. $\exists\, x \neq z \in C$ s.t $\frac{x+z}{2} = k$

$f(k) = \frac{1}{2}\left(f(x) + f(z)\right) \leq f(k)$

$\Rightarrow f(x) = f(z) = f(k) = f(y)$

$\Rightarrow$ $x \neq z$, $x, z \in K$ ⨳

But this is a contradiction since $f(k) > \alpha > \underset{c \in C_0}{f(c)}$

and k ext. pt $\Rightarrow k \in C_0$ $\therefore$ $C = C_0$

FEB 7, 1977

PROBLEM SET TWO, NUMBER TWO:

X IS NORMAL

EVERY CAUCHY NET CONVERGES $\Leftrightarrow$ EVERY CAUCHY SEQUENCE CONVERGES

PF: ($\Longrightarrow$) ANY CAUCHY SEQ $\{x_n\}_{n\in N}$ IS A CAUCHY NET

($\Longleftarrow$) LET $\{x_\alpha\}_{\alpha\in D}$ BE A CAUCHY NET. LET $\epsilon = 1/n$.

$B_n = \{x \mid \|x\| < 1/n\}$

$\exists\, \alpha_n \in D \ni \alpha_n \geq \alpha_{n-1}, \dots, \alpha_1$ AND $\beta, \alpha \geq \alpha_n \Rightarrow \|x_\alpha - x_\beta\| < 1/n$

$\{x_{\alpha_n}\}$ IS A CAUCHY SEQ.

PICK $\epsilon > 0$. PICK $m \ni 1/m < \epsilon$

IF $i, j \geq m$ $\|x_{\alpha_i} - x_{\alpha_j}\| < 1/m < \epsilon$ $\forall \alpha_i, \alpha_j \geq \alpha_m$.

THUS $\{x_{\alpha_n}\}_{n\in N} \longrightarrow x$ BY HYP.

WANT TO SHOW $\{x_\alpha\} \longrightarrow x$:

LET $\epsilon > 0$. LET $M \ni 1/m < \epsilon/2$

IF $\beta \geq \alpha_m$

REMARKS ON NO. FIVE

$(X/M)^* = M^+ \subseteq X^*$ IS REFLEXIVE

HENCE TEXT $\Rightarrow$ M^+ IS REF $\Rightarrow (X/M)^*$ REF. $\Longrightarrow$ X/M IS REF.

NOTE J NOT REFLEXIVE

J ISOMETRIC J^{**}

BACK TO NOTES

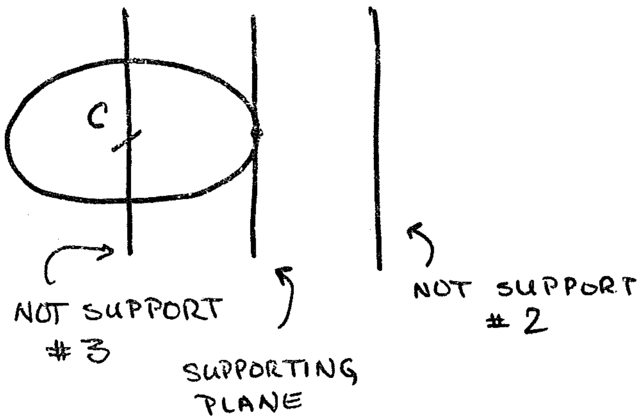
DEF: H IS SUPPORTING PLANE TO THE CONVEX SET, C IF:

① H IS A TRANSLATED OF A CLOSED VECTOR SPACE

② $H \cap C \neq \emptyset$

③ IF $x, y \in C$ & $[x,y] = \{sx + (1-s)y : 0 \leq s \leq 1\}$ AND $\exists$ AN ~~INTEGER~~ INTERIOR POINT OF $[x,y]$ IN H, THEN $[x,y] \subseteq H$

PICTURE:



SUPPOSE $C \subseteq E$ LCS

① THEN H IS A TRANS. OF A CLOSED V.S. $\Leftrightarrow$ THERE IS A COLLECTION, $A \subseteq E^*$ $\ni$ $H = \bigcap_{b \in A} b^{-1}(\alpha_b)$, $\alpha_b \in \mathbb{R}$.

PROOF OF: C, CONVEX COMPACT IN LCS HAS EXTREME POINT

$M = \{$SUPPORTING PLANES OF $C\}$. PARTIAL ORDER M BY $H < G \Leftrightarrow G \subseteq H$.

LET $\{H_\alpha\}_{\alpha \in \Lambda}$ BE A CHAIN IN M. $H_\alpha \supseteq H_\beta \supseteq \cdots$

$H = \bigcap_{\alpha \in \Lambda} H_\alpha$, TRANS. OF CLOSED V.S.

CLAIM H IS SUPPORT OF C. DONE ①

② $H_\alpha \cap C \neq \emptyset$ CLOSED & $\bigcap_{i=1}^{n} (H_{\alpha_i} \cap C) \neq \emptyset$

COLLECTION OF CLOSED SETS WITH FINITE INTERSECTION PROPERTY (FIP) IN COMPACT SET, C, SO $H \cap C =$
$= [\bigcap H_\alpha] \cap C = \bigcap_\alpha (H_\alpha \cap C)$

③ IF $x, y \in C$ & $[x,y] = \{sx + (1-s)y \mid 0 \leq s \leq 1\}$ AND THERE IS AN INTERIOR POINT OF $[x,y]$ IN H THEN $[x,y] \subseteq H$

THUS M (BY ZORN'S LEMMA) CONTAINS A MAXIMAL ELEMENT,
$H \in M \ni \forall G \in M$, $G \supseteq H \Rightarrow G = H$.

CLAIM $C \cap H$ CONSIST OF ONE POINT. SUPPOSE $x \neq y$, $x, y \in C \cap H$. LET $g \in E^*$ WITH $g(x) \neq g(y)$. LET $\alpha = \sup_{z \in C \cap H} g(z)$ (A COMPACT SET)

LET $G = H \cap g^{-1}(\alpha)$

$G \cap C \neq \emptyset$ $G \subseteq H$. LET $\xi, \eta \in C \ni G$ CONTAINS AN INTERIOR POINT. $\alpha = g(z) = s\,g(\xi) + t\,g(\eta) \leq \alpha$

AND $= \alpha \Leftrightarrow g(\xi) = g(\eta) = \alpha$

$\Rightarrow [\xi, \eta] \subset G$.

BUT $G \subsetneqq H$ ✻ BY MAXIMALITY OF H. $\therefore$ H CONSIST OF ONE PT., e, e IS EXTREME POINT OF C BY ③ SINCE $H = \{e\}$ IS SUPPORTING PLANE OF C. THUS COMPLETING THE KM THEOREM FOR REAL V.S.

$C[0,1]$ HAS A BASIS $\{b_n\}$

RECALL: $\{b_n\}$ IS A BASIS FOR $X \Leftrightarrow \forall x \in X \ \exists! \ (\alpha_n) \ni$ $x = \sum_{i=1}^{\infty} \alpha_i b_i \Leftrightarrow \{b_n\}$ LIN. DEP. WITH DENSE LINEAR SPAN & $\exists M \ni \| \sum_1^p \alpha_i b_i \| \leq M \| \sum_1^{p+q} \alpha_i b_i \| = ✯ \ \forall p, q, \ \& \ \{\alpha_i\}$

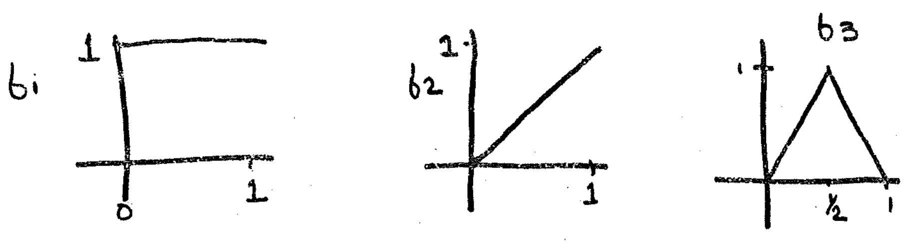
IF M IS THE SMALLEST NUMBER SATISFYING ✯, M IS CALLED THE BASIS CONSTANT OF $\{b_n\}$ (BIGGEST PROJECTION)

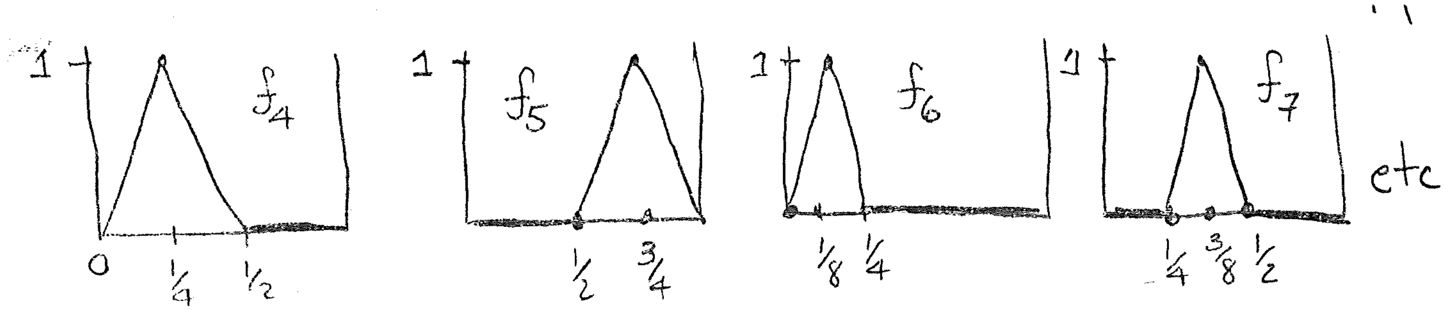
$M = \sup_{n=1}^{\infty} \|P_n\|$ WHERE $P_n : X \to X$

$$x = \sum_{i=1}^{\infty} \alpha_i b_i \mapsto \sum_{i=1}^{n} \alpha_i b_i$$

THEOREM: EVERY BANACH SPACE HAS A SUBSPACE WITH A BASIS.

CONSTRUCTION OF b_n





IN GENERAL $n \geq 2$ IS ONE ON THE N^{th} number in the list: $0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \ldots$ linearly drops to zero on both sides at the nearest no.'s already in the list & zero everywhere else

Since $\delta_t \in (C[0,1])^* \ \forall t \in [0,1]$ where $\delta_t(g) = g(t)$

since $|g(t)| \leq \|g\|_\infty \Rightarrow \|\delta_t\| \leq 1$.

If $g = \sum_{n=1}^{\infty} \alpha_n f_n$ then $g(t) = \delta_t(g) = \delta_t(\sum_1 \alpha_n f_n) = \sum_1 \alpha_n f_n(t)$

Thus letting $t = 0, 1, \frac{1}{2}, \ldots$ we obtain

$$g(0) = \sum_1 \alpha_n f_n(0) = \alpha_1 \quad [\text{since } f_n(0) = 0 \quad n \geq 1]$$

$$g(1) = \sum_1 \alpha_n f_n(1) = \alpha_1 + \alpha_2 \Rightarrow \alpha_2 = g(1) - g(0)$$

$$g(\tfrac{1}{2}) = \alpha_1 + \tfrac{1}{2}\alpha_2 + \alpha_3 \Rightarrow \alpha_3 = g(\tfrac{1}{2}) - g(0) - \tfrac{1}{2}(g(1) - g(0))$$

$$\alpha_3 = g(\tfrac{1}{2}) - \tfrac{1}{2}g(0) - \tfrac{1}{2}g(1)$$

$$g(\tfrac{1}{4}) = \alpha_1 + \tfrac{1}{4}\alpha_2 + \tfrac{1}{2}\alpha_3 + \alpha_4 \Rightarrow \alpha_4 = g(\tfrac{1}{4}) - \alpha_1 - \tfrac{1}{4}\alpha_2 - \tfrac{1}{2}\alpha_3$$

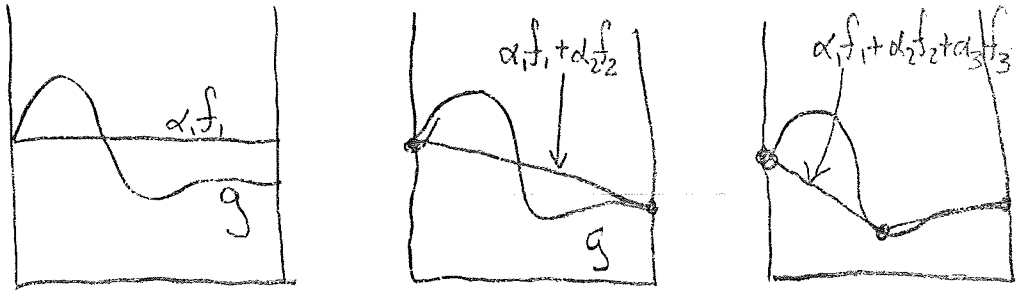
$\vdots$ etc

So that $\{\alpha_n\}$ are unique ie.

if $g = \sum_{n=1}^{\infty} \alpha_n f_n$ & $g = \sum_{n=1}^{\infty} \beta_n f_n$ then $\alpha_n = \beta_n \quad n = 1, 2, \ldots$

TO COMPLETE THE PROOF THAT $\{f_n\}$ is a basis we need only show each $g \in C[0,1]$ the seq of α_n obtained above have the property that $\sum_{i=1}^{N} \alpha_n f_n \longrightarrow g$ in norm.

pictures:



we know that $\sum_{i=1}^{N} \alpha_i f_i$ agrees with g at the first N pts on the list $0, 1, \frac{1}{2}, \frac{1}{4}, \ldots$ and by induction we could show that $\sum_{i=1}^{N} \alpha_i f_i$ just connects there points by straight lines. It should intutively clear that $\sum_{i=1}^{N} \alpha_i f_i$ converge uniformly to $g(x)$ [for proof use the uniform continuity of $g(x)$]

If it is easy to see $\|\sum_{n=1}^{N} \alpha_n f_n\|_\infty \leq \|g\|$ so that each of the projections $P_N : \sum_{n=1}^{\infty} \alpha_n f_n \longmapsto \sum_{n=1}^{N} \alpha_n f_n$ have norm one. Thus the basis constant for $\{f_n\}$ is one and the basis is a normalized $[\|f_n\| = 1]$ monotone [constant $= 1$] basis for $C[0,1]$.

Def: Let X be a Banach space. A sequence $\{x_n\} \subseteq X$ is a *basic sequence* if and only if $\{x_n\}$ is a basis for the closed linear span of $\{x_n\}$.

Example – Let $\{e_n\}$ be the usual basis for l_2. Define x_n to be e_{2n-1}. Then $\{x_n\}$ is a basic sequence.

Def: Let $\{x_n\}$ be a basic sequence. Let $0 = N_1 < N_2 < \dots$ be an increasing sequence of integers. Let $\{\beta_n\}$ be a sequence of scalars such that for all m, $\sum_{i=N_m+1}^{N_{m+1}} \beta_i x_i \neq 0$. Then

the sequence $\{b_m\}$ is called a *block basic sequence*, where $b_m = \sum_{i=N_m+1}^{N_{m+1}} \beta_i x_i$.

Lemma: A block basic sequence is a basic sequence.

Proof: The sequence $\{x_n\}$ is independent, and $b_n \neq 0$. Hence, the sequence $\{b_n\}$ is linearly independent.

It suffices to show that there is a K such that for all integers p, q and sequences of scalars $\{\alpha_n\}$,

(i) $$\left\| \sum_{n=1}^{p} \alpha_n b_n \right\| \leq K \left\| \sum_{n=1}^{p+q} \alpha_n b_n \right\|.$$

Let K_x be the basis constant. Then

$$\left\| \sum_{n=1}^{p} \alpha_n b_n \right\| \leq \left\| \sum_{n=1}^{N_{p+1}} a_n^* \beta_n x_n \right\| \leq$$

$K_x \| \sum_{n=1}^{N_{p+q+1}} \alpha_n^* \beta_n x_n \|$, where

$$\alpha_n^* = \begin{cases} \alpha_k & \text{for } N_k+1 \le n \le N_{k+1} \\ 0 & \text{otherwise} \end{cases}.$$

But $\| \sum_{n=1}^{N_{p+q+1}} \alpha_n^* \beta_n x_n \| = \| \sum_{n=1}^{p+q} \alpha_n b_n \|$.

Hence, the basis constant becomes the K in line (i), and

$$\| \sum_{n=1}^{p} \alpha_n b_n \| \le K_x \| \sum_{n=1}^{p+q} \alpha_n b_n \|.$$

Let X be an infinite dimensional B space. We want to show there is a basic sequence in X. It suffices to assume X is separable, and thus a subset of $C[0,1]$, because every infinite dimensional B space has separable infinite dimensional B subspace. (Let $\{x_n\}$ be any infinite independent set in X. Then closed linear span of $\{x_n\}$ is separable.)

Lemma: If $\{f_n\}$ is a basis for $C[0,1]$ and X an infinite dimensional ~~Banach space~~ Banach space, then for all $\varepsilon > 0$, there is a $\{b_j\}$ block basic sequence of $\{f_n\}$ with $\|b_j\| = 1$, and there exists a $\{z_n\}$ such that $\sum_{n=1}^{\infty} \|z_n\| < \varepsilon$, and $\{b_j + z_j\} \subseteq X$.

Proof: We will choose the N_i's, $\{b_i\}$, and $\{z_i\}$ by induction, such that

$0 = N_1 < N_2 < \cdots$,

$\|b_i\| = 1$ where b_i is a block basic sequence with respect to $\{f_n\}$ and $\{N_i\}$,

$\|z_i\| < \varepsilon/2^i$, and

$\{b_n + z_n\} \in \mathbb{X}$.

Let y_1 be an element of $\mathbb{X}$ such that $\|y_1\| = 1$. Since $\{f_n\}$ is a basis for $C[0,1]$ and $\mathbb{X}$ is assumed to be a subspace of $C[0,1]$, there is a sequence of scalars $\{\alpha_i^1\}$ such that $y_1 = \sum_{i=1}^{\infty} \alpha_i^1 f_i$.

Let $\delta = \frac{\varepsilon}{\varepsilon + 3}$. Since $\sum_{i=1}^{\infty} \alpha_i f_i$ converges, there is a N_2 such that $\|\sum_{i=N_2+1}^{\infty} \alpha_i f_i\| < \delta$.

Let $b_1 = \dfrac{\sum_{i=1}^{N_2} \alpha_i^1 f_i}{\|\sum_{i=1}^{N_2} \alpha_i^1 f_i\|}$ and $z_1 = \dfrac{\sum_{i=N_2+1}^{\infty} \alpha_i f_i}{\|\sum_{i=1}^{N_2} \alpha_i f_i\|}$.

Now $\|b_1\| = 1$ and $\|z_1\| \leq \frac{\delta}{1-\delta}$. But $\frac{\delta}{1-\delta} = \frac{\varepsilon}{3} < \frac{\varepsilon}{2}$, and $(b_1 + z_1) \in \mathbb{X}$.

Induction step: Choose any $y_2 \in \mathbb{X}$ such that $\sum_{i=1}^{\infty} \alpha_i^2 f_i = y_2$, and $\alpha_i = 0$ for $i = 1, 2, \ldots, N_2$.

Repeating this process will produce the sequences $\{b_j\}$ and $\{z_j\}$ ∎

We will show that if ε is small enough, $\{b_j + z_j\}$ is a basic sequence.

If ε is small, $\{b_j + z_j\}$ is independent. It will suffice to show there is a K such that $\|\sum_{i=1}^{p}\alpha_i y_i\| \leq K\|\sum_{i=1}^{p+q}\alpha_i y_i\|$ for any p, q, $\{\alpha_i\}$, where $y_i = b_i + z_i$.

We know that $\|\sum_{i=1}^{p+q}\alpha_i y_i\| =$

$\|\sum_{i=1}^{p+q}\alpha_i b_i + \sum_{i=1}^{p+q}\alpha_i z_i\| \geq$

$\|\sum_{i=1}^{p+q}\alpha_i b_i\| - \|\sum_{i=1}^{p+q}\alpha_i z_i\|$.

It is clear that $\|\sum_{i=1}^{p+q}\alpha_i z_i\| \leq \sup_{1\leq i\leq p+q}|\alpha_i|\sum\|z_i\|$.

However, $\sup_{1\leq i\leq p+q}|\alpha_i| \leq 2\|\sum_{i=1}^{p+q}\alpha_i b_i\|$.

(Justification: Let j be such that $|\alpha_j| = \sup_{1\leq i\leq p+q}|\alpha_i|$. Now
$$\alpha_j = \sum_{i=1}^{j}\alpha_i b_i - \sum_{i=1}^{j-1}\alpha_i b_i, \text{ so}$$

$$|\alpha_j| = \|\sum_{i=1}^{j}\alpha_i b_i - \sum_{i=1}^{j-1}\alpha_i b_i\|.$$

Then $|\alpha_j| \leq \|\sum_{i=1}^{j}\alpha_i b_i\| - \|\sum_{i=1}^{j-1}\alpha_i b_i\|$. Since the basis constant for $\{f_n\}$ is 1, the basis constant for the block basic sequence is also 1. Thus, $\|\sum_{i=1}^{j}\alpha_i b_i\| \leq \|\sum_{i=1}^{p+q}\alpha_i b_i\|$ and $\|\sum_{i=1}^{j-1}\alpha_i b_i\| \leq \|\sum_{i=1}^{p+q}\alpha_i b_i\|$. Hence, $|\alpha_j| \leq 2\|\sum_{i=1}^{p+q}\alpha_i b_i\|$.)

Let ε be less than $\frac{1}{2}$. Then

$\|\sum_{i=1}^{p+q} \alpha_i z_i\| < 2\varepsilon \|\sum_{i=1}^{p+q} \alpha_i b_i\|$, so

$\|\sum_{i=1}^{p+q} \alpha_i b_i\| - \|\sum_{i=1}^{p+q} \alpha_i z_i\| \geq (1-2\varepsilon) \|\sum_{i=1}^{p+q} \alpha_i b_i\|$

$\geq (1-2\varepsilon) \|\sum_{i=1}^{p} \alpha_i b_i\|$. Therefore,

$\|\sum_{i=1}^{p+q} \alpha_i y_i\| \geq (1-2\varepsilon) \|\sum_{i=1}^{p} \alpha_i b_i\|$.

Now $\|\sum_{i=1}^{p} \alpha_i y_i\| \leq \|\sum_{i=1}^{p} \alpha_i b_i\| + \|\sum_{i=1}^{p} \alpha_i z_i\|$

$\leq (1+2\varepsilon) \|\sum_{i=1}^{p} \alpha_i b_i\|$.

Therefore, $\|\sum_{i=1}^{p} \alpha_i y_i\| \leq \frac{1+2\varepsilon}{1-2\varepsilon} \|\sum_{i=1}^{p+q} \alpha_i y_i\|$, establishing a constant K ∎

This shows that $\{b_n\}$ and $\{y_n\}$ are equivalent basic sequences, in the sense that:

(1) $\sum \alpha_i b_i$ converges if and only if $\sum \alpha_i y_i$ converges

(2) $T: b_i \rightarrow y_i$, extended linearly, is an isomorphism.

2/11/77

Def. A basis $\{x_n\}$ for X is shrinking if $\forall f \in X^*$, $\lim_{n\to\infty} [\![f]\!]_n = 0$,

where $[\![f]\!]_n = \| f|\, \text{cl lin span} \{x_n, x_{n+1}, \cdots\}\|$.

Examples:

1. If $e_n = (0, \cdots, 0, \underset{n\text{th}}{1}, 0, \cdots)$, then $\{e_n\}$ is a shrinking basis for c_0.

Pf. Let $f \in l_1 = c_0^*$, $f = (f_1, f_2, \cdots)$.

$[\![f]\!]_N = \| f|\, \text{cl lin span}\{x_N, x_{N+1}, \cdots\}\|$

$= \sup\limits_{\substack{x \in c_0 \\ \|x\|=1}} |f(x)|$

$x = (x_1, x_2, \cdots)$ with $x_1 = x_2 = \cdots = x_{N-1} = 0$.

So $[\![f]\!]_N = \sum_{i=N}^{\infty} |f_i| \to 0$.

2. $\{e_n\}$ is not a shrinking basis for l_1 because $\theta = (1, 1, \cdots) \in l_\infty$,

$[\![\theta]\!]_N \geq |\theta[e_N]| = 1$. $\therefore \lim_{N\to\infty} [\![\theta]\!]_N \neq 0$.

<u>Lemma</u>. Let $\{x_n\}$ be a basis for $\mathfrak{X}$ and let $\{f_n\}$ be the coefficient fcnals,
$f_n(\sum_{n=1}^{\infty} \alpha_i x_i) = \alpha_n$. Then $\{f_n\}$ is a basic sequence and

(1) If $\{x_n\}$ is normalized, so is $\{f_n\}$;
(2) If $\{x_n\}$ is monotone, so is $\{f_n\}$
(3) $\{x_n\}$ shrinking $\Leftrightarrow$ $\{f_n\}$ has dense linear span in $\mathfrak{X}^*$.

Pf. It is clear that $\{f_n\}$ is independent.

S $\{\alpha_n\}$ is a seq. We want to find $K \ni$
$\|\sum_1^p \alpha_i f_i\| \leq K \cdot \|\sum_1^{p+q} \alpha_i f_i\|$.

We can assume, W.L.O.G., $\|\sum_1^p \alpha_i f_i\| = 1$.
Then $\exists\, x = \sum_{i=1}^{\infty} \beta_i x_i \ni \|x\| = 1$ and
$|(\sum_1^p \alpha_i f_i)(x)| > 1-\delta$.
We have $|(\sum_1^p \alpha_i f_i)(x)| = |(\sum_1^p \alpha_i \beta_i)| = |(\sum_1^p \alpha_i f_i)(\sum_1^p \beta_i x_i)|$.

Let Q = basis constant for $\mathfrak{X}$, $K = \frac{Q}{1-\delta}$, $\delta > 0$.
$1-\delta < \|\sum_1^p \beta_i x_i\| \leq Q\|x\| = Q$.

$\|\sum_1^{p+q} \alpha_i f_i\| \geq |(\sum_1^{p+q} \alpha_i f_i)\left(\frac{\sum_1^p \beta_i x_i}{Q}\right)| \geq \frac{1-\delta}{Q}$.

This also proves (2).

$\mathcal{S}$ $\|x_n\| = 1 \quad \forall x_n$.
$\|f_j\| = \sup_{\|\sum_1^\infty \alpha_i x_i\| = 1} |\alpha_j|$.

So if $\|f_j\| > 1$, $\exists$ a seq. $\{\alpha_i\}$ with

$|\alpha_j| > 1$ and $\|\sum \alpha_i x_i\| = 1$.

$\mathcal{S}$ $\{x_n\}$ is monotone.
We have both $\|\sum_{i=1}^{j} \alpha_i x_i\| \leq 1$, $\|\sum_{i=1}^{j-1} \alpha_i x_i\| \leq 1$,
& suppose $|\alpha_j| \leq 2$.

We have shown for $\{x_n\}$ normalized,
$1 \leq \|f_n\| \leq 2K$. (Check later for 1).

(3) ($\Rightarrow$) $\mathcal{S}$ $\{x_n\}$ is shrinking.
We wish to show $\{f_i\}$ has dense linear span in X^*.

Let $F \in X^*$, $\beta_i = F(x_i)$.

Then $\|F - \sum_1^{N-1} \beta_i f_i\| = [\![F]\!]_N \to 0$ by def. of shrinking

because $F - \sum_1^{N-1} \beta_i f_i = F$ on cl lin span $\{x_N, x_{N+1}, \ldots\}$.

($\Leftarrow$) Now $\mathcal{S}$ $\{f_n\}$ is a basis for X^*.

Then let $F \in X^*$, $F = \sum_{i=1}^{\infty} \beta_i f_i$, $\beta_i = F(x_i)$.

So $\lim_{N \to \infty} \| \sum_{i=N}^{\infty} \beta_i f_i \| = [F]_N \to 0$.

So $\{x_n\}$ is shrinking. ✓

Def. $\{x_n\}$, a basis for X, is boundedly complete if $\{\alpha_n\}$ is a sequence of scalars $\ni$ $\{\| \sum_{i=1}^{N} \alpha_i x_i \|, N = 1, 2, \cdots\}$ is bounded $\Rightarrow \exists y \in X \ni y = \sum_{i=1}^{\infty} \alpha_i x_i$.

Examples

1) Consider c_0 with basis $\{e_n\}$. It is not bdly complete, for consider $\alpha_n \equiv 1$. Then $\| \sum_1^n \alpha_n x_n \| = 1 \ \forall n$. But $(\alpha_n) \notin c_0$.

2) Consider l_1 with basis $\{e_n\}$. It is bdly complete. For if $\{\alpha_n\}$ is a seq. with $\sup_N \| \sum_1^N \alpha_n e_n \| \leq M < +\infty$,

$\sum_1^N |\alpha_n| = \| \sum_1^N \alpha_n e_n \| \leq M$, and

$\sum_1^{\infty} |\alpha_i| = \lim_{N \to \infty} \sum_1^N |\alpha_i| = \sup \sum_1^N |\alpha_i| = \sup \| \sum_1^N \alpha_i x_i \| \leq M$

$\Rightarrow \| (\alpha_n) \|_1 < +\infty$.

Prop. If $\{x_n\}$ is a basis, and $\{f_n\}$ is the seq. of coefficient fncs, then

1. $\{x_n\}$ is shrinking $\Leftrightarrow$ $\{f_n\}$ is bddly complete.
2. $\{x_n\}$ is bddly complete $\Leftrightarrow$ $\{f_n\}$ is a shrinking basis for its cl lin span.

Pf. (1)($\Rightarrow$). $\supset$ $\{x_n\}$ is shrinking.
Let $\{\beta_n\}$ be a seq. $\ni$

$$\|\sum_1^n \beta_n f_n\| \leq M < +\infty.$$

We want $y \in X^* \ni y = \sum_1^\infty \beta_n f_n$.

Let $y_N = \sum_1^N \beta_i f_i$. Then $\|y_N\| \leq M$, so

$\{y_N\} \subseteq \{x^* \mid \|x\|^* \leq M\}$ which is $\sigma(X^*, X)$

seq. compact (it is compact & metrizable, X sep.)

So $\exists$ subseq. $\{y_{N_j}\}$ that converges to some $y \in X^*$ in $\sigma(X^*, X)$.

$y(x_n) = \lim y_{N_j}(x_n) = \beta_n$.

$y = y - \sum_{i=1}^N \beta_i x_i$ on lin span $\{x_{N+1}, x_{N+2}, \ldots\}$

$\|y - \sum_{i=1}^N \beta_i x_i\| = [\![y]\!]_{N+1} \to 0$ by shrinking hypothesis.

Thus, $y = \sum_1^\infty \beta_i x_i$.

Last class we finished ($\Rightarrow$).

($\Leftarrow$) i.e. $\{f_n\}$ bddly complete basic seq. $\Rightarrow$ $\{x_n\}$ shrinking.

Let $F \in X^*$. $\beta_i = F(x_i)$

<u>claim</u> $\sum_1^N \beta_i f_i \longrightarrow F$ in $\sigma(X^*, X)$ top as $N \to \infty$

<pf> Let $x \in X$. $x = \sum_1^\infty \alpha_i x_i$ Then

$$F\left(\sum_1^\infty \alpha_i x_i\right) \qquad \lim\left(\sum_1^N \beta_i f_i\right)\left(\sum_1^\infty \alpha_i x_i\right)$$

$$= \lim_N F\left(\sum_1^N \alpha_i x_i\right) \qquad = \lim_N \sum_1^N \beta_i \alpha_i \quad (\because f_i : \text{coeff. ft})$$

$$= \lim_N \sum_1^N \alpha_i \beta_i \quad = $$

$$\therefore F\left(\sum_1^\infty \alpha_i x_i\right) = \lim_N \left(\sum_1^N \beta_i f_i\right)\left(\sum_1^\infty \alpha_i x_i\right)$$

Done w/claim ∎

Since $\sum_1^N \beta_i f_i \longrightarrow F$ $\sigma(X^*, X)$.

and X : B-sp. $\exists M < +\infty$ s.t. $\|\sum_1^N \beta_i f_i\| \le M$

By bddly complete, $\exists y \in X^*$ s.t

$$y = \sum_1^\infty \beta_i f_i .$$

$F_1 - y$ is zero at each x_n since $y(x_n) = \beta_n$.

$F_1 - y = 0$ on lin sp $\{x_n\}$

$F_1 - y$ conti $\Rightarrow$ $F - y = 0$ on cl. lin. sp $\{x_n\} = X$.

i.e $F_1 = y$ and thus $\{f_i\}$ have dense lin span.

q.e.d of (1).

Supp X is B-sp with shrinking monotone basis $\{x_n\}$

We know that X^* has bddly complete monotone basis $\{f_n\}$.

<u>Claim</u> $X^{**} = \{ (\xi_i) \text{ s.t } \|(\xi_i)\|_{X^{**}} = \lim_N \| \sum_1^N \xi_i x_i \|_X \text{ exists} < \infty \}$

Supp. $\{\xi_i\}$ is a seq. of scalars s.t.

$\lim_N \| \sum_1^N \xi_i x_i \|_X = A < +\infty$ exists

define $F_1 : X^* \longrightarrow K$ scalars.

by $g \in X^*$ $\quad g = \sum_1^\infty \beta_i f_i \quad F(g) = \lim_N \sum_1^N \xi_i \beta_i$

$= \lim_N g(\sum_1^N \xi_i x_i)$.

$$\left|\sum_N^M \beta_i \xi_i\right| \le \underbrace{\left\|\sum_N^M \beta_i f_i\right\|_{\mathbb{X}^*}}_{\substack{\downarrow \\ 0 \\ \text{as } M,N\to\infty}} \cdot \underbrace{\left\|\sum_1^M \xi_i x_i\right\|_{\mathbb{X}}}_{\uparrow \text{ bdd. by } A} \qquad (*)$$

and by monotone, $\|\sum_1^N \xi_i x_i\| \le \|\sum_1^{N+k} \xi_i x_i\|$

$\therefore$ $(*) \longrightarrow 0$.

Thus $\sum_1^\infty \beta_i \xi_i$ exists and $< +\infty$.

$\therefore$ $F(g)$ is well defined.

F is linear. (easy)

<u>Claim $\|F\| = A$.</u>

pf. Let $g \in \mathbb{X}^*$. $\|g\| = 1$.

$g = \sum_1^\infty \beta_i f_i$. $\quad |\sum_1^N \xi_i \beta_i| \le \|g\|_{\mathbb{X}^*} \|\sum_1^N \xi_i x_i\|_{\mathbb{X}} \le 1 \cdot A$

$\therefore$ $|F(g)| \le A \Rightarrow \|F\| \le A$.

Since limit is A, $\forall \varepsilon > 0$, $\exists N$ s.t. $\|\sum_1^N \xi_i x_i\|_{\mathbb{X}} > A - \varepsilon$

$\exists g \in \mathbb{X}^*$, $\|g\| = 1$. $g = \sum_1^\infty \beta_i x_i$ $\quad$ H.B Thm

$A - \varepsilon < g(\sum_1^N \xi_i x_i)$

$\therefore\ A-\varepsilon < g\left(\sum_1^N \gamma_i x_i\right) = \sum_1^N \gamma_i \beta_i = \left(\sum_1^N \beta_i f_i\right)\left(\sum_1^N \gamma_i x_i\right)$

$= \left(\sum_1^N \beta_i f_i\right)\left(\sum_1^P \gamma_i x_i\right)$ for $P \geq N$

$\downarrow$

$F\left(\sum_1^N \beta_i f_i\right)$ as $P \to \infty$.

$\therefore\ F\left(\sum_1^N \beta_i f_i\right) > A - \varepsilon$

and $\left\| \sum_1^N \beta_i f_i \right\| \leq \left\| \sum_1^\infty \beta_i f_i \right\| = \|g\| = 1$

$\uparrow$ monotone of $\{f_n\}$

$\therefore\ \|F\| > A - \varepsilon$

$\therefore\ \|F\| \geq A$.

$\therefore\ \|F\| = A$ Done w/ claim $\ast\ast$.

Let $F \in X^{**}$. Let $\gamma_i = F(f_i)$

Consider

$\left\| \sum_1^N \gamma_i x_i \right\|_X$ monotone increasing in N.

We want to show $\left\| \sum_1^N \gamma_i x_i \right\|_X \leq \|F\|$

Let $\varepsilon > 0$ be given.

$\exists\ g \in X^*$ s.t. $\|g\| = 1$ $g\left(\sum_1^N \gamma_i x_i\right) > \left\| \sum_1^N \gamma_i x_i \right\|$

$g = \sum_1^\infty \beta_i f_i$, $\left\| \sum_1^N \beta_i f_i \right\| \leq 1$ monotone

$$\left| g\left(\sum_1^N \xi_i x_i\right)\right| = \left| \left(c \sum_1^N \beta_i f_i\right)\left(\sum_1^N \xi_i x_i\right)\right| = \left| \sum_1^N \beta_i \xi_i \right|$$

$$= \left| F\left(\sum_1^N \beta_i f_i\right)\right| \qquad F(f_i) = \xi_i$$

$$\therefore \quad \|F\| > \left\| \sum_1^N \xi_i x_i \right\| - \varepsilon$$

$\therefore$ $\|F\|$ is upper bdd. to $\left\{ \left\| \sum_1^N \xi_i x_i \right\|_X \right\}_N$

thus $\lim_{N\to\infty} \left\| \sum_1^N \xi_i x_i \right\|$ exists

done by 1st half.

Let $J = \left\{ (\xi_i) \in c_0 \;\middle|\; \|(\xi_i)\| = \sup_{(p_i)} \left\{ \left[\sum_1^{n-1} (\xi_{p_i} - \xi_{p_{i+1}})^2 + (\xi_{p_n} - \xi_{p_1})^2 \right]^{\frac{1}{2}} \right.\right.$

$p_1 < p_2 < \cdots$ increasing seq. of integers

$n = 3, 4, 5, \ldots \}$

The $\{e_n\}$ is monotone shrinking basis.

We can show J is not reflexive.

J^{**} has one-more dimension.

CLAIM: $e_n = (0,0,\dots,0,\underset{n\text{TH}}{1},0,\dots)$ IS A SCHRINKING MONOTONE BASIS FOR J.

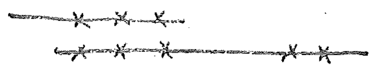
ASSUME CLAIM

$$J^{**} = \left\{(\xi_i) \,\middle|\, \lim_{n\to\infty} \left\|\sum_{1}^{n} \xi_i e_i\right\|_J < \infty\right\} \text{ AND } \|\xi\|_J = \lim_{n\to\infty} \left\|\sum_{i=1}^{n} \xi_i e_i\right\|_J$$

① J IS NOT REFLEXIVE

$\theta = (\xi_i)$ ALL $\xi_i = 1$ THEN $\left\|\sum_{i=1}^{N} \xi_i e_i\right\|_J = \|(\underbrace{1,1,1,\dots}_{n},0,0,\dots)\|_J$

AND THE $\|\theta\| = \sqrt{2}$



② $(\xi_i) \in J^{**} \Rightarrow \lim_{i\to\infty} \xi_i$ EXISTS.

SUPPOSE NOT. THEN $\exists\, \epsilon > 0 \ni \forall N\ \exists\, i,j \ni |\xi_i - \xi_j| \ge \epsilon$

SO THERE IS AN INCREASING SEQUENCE $\{n_k\} \ni$

$|\xi_{n_{2k}} - \xi_{n_{2k-1}}| > \epsilon$ LET M BE GIVEN CHOOSE $N \ni \sqrt{N}\cdot\epsilon > M$

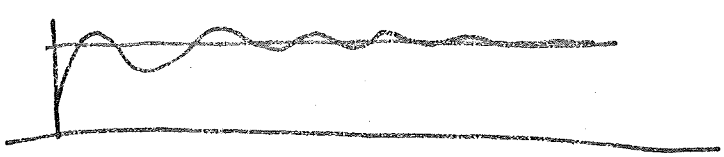
$P_i = n_i \quad i = 1,2,\dots,2N \quad K > n_{2N}$

$\|(\xi_i)\|_{J^{**}} \ge \left\|\sum_{i=1}^{K} \xi_i e_i\right\|_J \ge (N\epsilon^2)^{1/2} \ge \sqrt{N}\epsilon \ge M$

CONTRADICTION

③ IF $(\xi_i) \in J^{**}$ AND $f = \lim_{i\to\infty} \xi_i$

$(\xi_i) - f(\theta) \in J$



KNOW $(\eta_i) \in c_0$. JUST SHOW HAS FINITE NORM

IF $P_1 < \dots < P_N$ IS A CHOICE OF INTEGERS

$\eta_{P_i} - \eta_{P_{i+1}} = \xi_{P_i} - \xi_{P_{i+1}}$

AND $\exists\, N \ni N \ge P_n \Rightarrow \left\|\sum_{i=1}^{N} \xi_i e_i\right\|_J \ge \left[\sum (\eta_{P_i} - \eta_{P_{i+1}})^2 + (\eta_{P_n} - \eta_{P_1})^2\right]^{1/2}$

$\therefore \|(\eta_i)\|_J \le \|(\xi_i)\|_{J^{**}}$

THEREFORE J IS CODIM ONE IN J^{**}

J AND J^{**} ARE ISOMETRIC

TAKE $x_i \in J^{**}$ AND $T : J^{**} \to J$

$T(x_i) = (-\ell, x_1 - \ell, x_2 - \ell, \cdots)$ WHERE $\ell = \lim_{i \to \infty} x_i$

$S : J \to J^{**}$

$(\xi_i) \mapsto (\xi_2 - \xi_1, \xi_3 - \xi_1, \xi_4 - \xi_1, \cdots)$

S AND T ARE LINEAR (EASY)

IDENTITY ON $J = TS : J \to J$

IDENTITY ON $J^{**} = ST : J^{**} \to J^{**}$

TO COMPUTE NORM SUFFICES TO SHOW $\|S\|, \|T\| \leq 1$

~~SUPPOSE~~ $\cancel{p_1 < \cdots < p_n \Rightarrow \left[\sum (\lambda p_i - \lambda p_{i+1})^2 + (\lambda p_n - \lambda p_1)^2\right]^{1/2} > 1 - \epsilon}$

SUPPOSE $\lambda_i = (-\ell, x_1 - \ell, x_2 - \ell, \cdots) = T(x_i)$

SUPPOSE $\|x_i\|_{J^{**}} = 1$

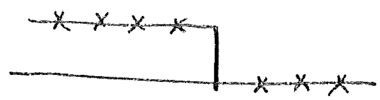
$\exists N \ni \|\sum_1^N x_i e_i\| > 1 - \epsilon$

$$P_1 < P_2 < \cdots P_{N+1}\Big|_N < P_n\Big|_N \ni \left[\sum_1^{n-1} (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_n} - x_{p_1})^2\right]^{1/2} > 1 - \epsilon$$

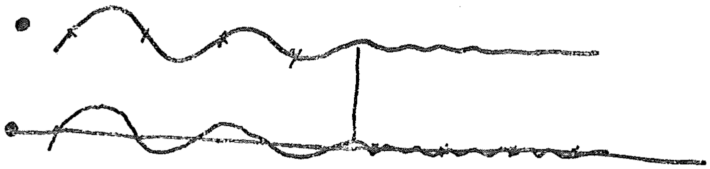
$$\sum_1^{n-2} (\lambda_{p_i+1} - \lambda_{p_{i+1}+1})^2 + \left(\lambda_{p_{n-1}+1} + \begin{Bmatrix} \lambda_{p_n+1} \\ \lambda_1 \end{Bmatrix}\right)^2 + \begin{Bmatrix} (\lambda_{p_n+1} - \lambda_{1+1})^2 \\ (\lambda_1 - \lambda_{p_1+1})^2 \end{Bmatrix}^2 \Bigg]^{1/2}$$

$\|T(x_i)\| > 1 - \epsilon \Rightarrow \|T(x_i)\| \geq 1$

$\therefore \|S\| \leq 1$



OTHER DIRECTION INTUITIVELY



$\therefore$ NON REFLEXIVE. B SPACE CODIM IN SECOND DUAL AND ISOMETRIC

IF X IS B-SPACE AND $X \xrightarrow{T} X^{**}$ IS ISOMETRY ONTO, CAN'T NECESSARILY SAY X IS REFLEXIVE.

$\therefore X = X^{**} \not\Rightarrow$ REFLEXIVE

NEED $X \hookrightarrow X^{**}$ BY CANONICAL INJECTION MAP.

CLAIM $e_n = (0, \cdots 0, 1, 0 \cdots)$ IS SCHRINKING

SUPPOSE NOT.

$\exists\ F \in J^*$ WITH $[\![F]\!]_K \searrow 1$

WE CAN ASSUME $F \in J^*$ WITH $[\![F]\!]_N > 1$ FOR ALL N, AND

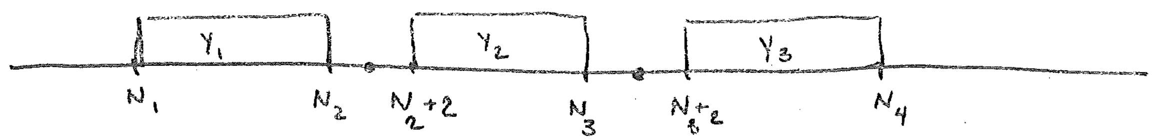
$\lim [\![F_N]\!]_N \longrightarrow \alpha > 1$

PICK $\epsilon > 0$ SMALL, $N_1 \ni 1+\epsilon > [\![F]\!]_{N_1} > 1$

FIND $x_1 \in$ cl. lin. span $[e_{N_1}, e_{N_2}, \cdots] \ni \|x_1\| = 1$ & $F(x_1) > 1$.

FIND $N_2 \ni \|P_{N_2}\| \le 1$ BUT $F(P_{N_2}(x)) > 1$

$y_1 = P_{N_2}(x_1)$ DO IT TWICE MORE



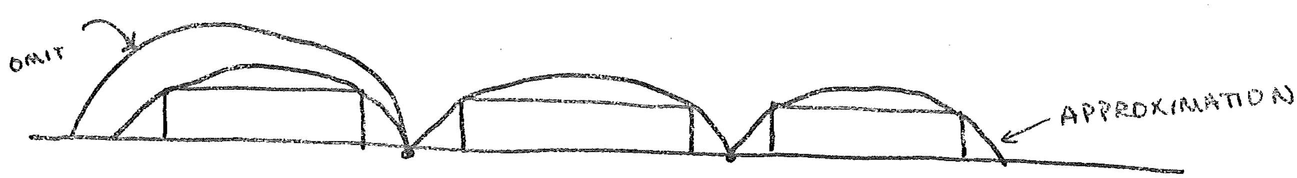
KNOW $\|y_i\| \le 1$ $\quad i = 1, 2, \cdots$

KNOW $1+\epsilon > F(y_i) > 1$ $\quad i = 1, 2, 3, \cdots$

$F(y_1+y_2+y_3) > 3$

$\|y_1+y_2+y_3\| < \frac{3}{1+\epsilon} \Longrightarrow [\![F]\!]_{N_1} > 1+\epsilon$ A CONTRADICTION SINCE

$\| \frac{1+\epsilon}{3} + y_1+y_2+y_3\| \le 1$ IF $\omega = y_1+y_2+y_3$ $\quad \|F(\omega)\| > 1+\epsilon$



IF APPROXIMATION HITS THE ZEROS THEN IT GIVES AN ESTIMATION OF y_1, y_2, AND y_3. THEREFORE MAKE APPROX. HIT ZERO

KNOW $\|y_i\| \le 1$ $\quad i = 1, 2, 3$

KNOW $1+\epsilon > F(y_i) > 1$

PICTURE ILLUSTRATES $\|y_1+y_2+y_3\| < \sqrt{6} < 3$.

Let X be a vector space over $\mathbb{C}$, the complex numbers. Since $\mathbb{R}$ is a subspace of $\mathbb{R}$, X is also a vector space over $\mathbb{R}$.

Let $\tilde{X}$ be the real TVS corresponding to the complex TVS X. Then $I_X : X \to \tilde{X}$ is a homeomorphism of topologies. That is, whether a TVS is complex or real has nothing to do with the topologies.

There are Banach spaces T such that T is not $\tilde{X}$, corresponding to any complex Banach space X — for example, the Banach space T given on page 95.

Let $g : X \longrightarrow \mathbb{C}$ be a linear functional. Then for $x \in X$, $g(x) = Re(g(x)) + i\, Im(g(x))$, and the functionals $Re(g(x))$, $Im(g(x))$ are real linear functionals on $\tilde{X}$.

That is,

$$Re(g(x+y)) = Re(g(x) + g(y)) = Re(g(x)) + Re(g(y)) \quad \text{and,}$$

if $\lambda \in \mathbb{R}$, $Re(g(\lambda x)) = Re(\lambda g(x)) = \lambda Re(g(x))$.

The functional $Im\, g$ is similar.

If g is continuous, then both $Re\, g$ and $Im\, g$ are continuous.

Let $K = \ker g$. Then K is of codimension 2 in $\tilde{X}$ and of codimension 1 in X.

Then $\exists\, y \in X$ such that

$\{K + \lambda y\} = X$, where $\lambda \in \mathbb{C}$ or

$\{K + \xi y + \eta [iy]\} = X$, where $\xi, \eta \in \mathbb{R}$

Then codimension $K(\subseteq \tilde{X})$ is ≤ 2.

Let $y = k + \lambda i y$, where $\lambda \in \mathbb{R}$.
Then $(1-\lambda i)y = k$, which implies $y \in K$.
This is a contradiction.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\operatorname{Re} g"] \arrow[dr, "\varphi"'] & & \mathbb{R} \\ & \tilde{X}/K = \mathbb{R}^2 \arrow[ur, "\widehat{\operatorname{Re} g}"'] & \end{tikzcd}$$

There exists a $\widehat{\operatorname{Re} g}$ such that $\operatorname{Re} g = \widehat{\operatorname{Re} g} \circ \varphi$.
Both $\widehat{\operatorname{Re} g}$ and φ are continuous, so $\operatorname{Re} g$ is continuous.

Suppose $f: \tilde{X} \to \mathbb{R}$ is the real part of a complex linear functional $g: X \to \mathbb{R}$ - that is, $f = \operatorname{Re} g$.
Consider $f(ix)$. Then $f(ix) =$
$\operatorname{Re}(g(ix)) = \operatorname{Re}(i\, g(x)) =$
$\operatorname{Re}[\, i[\operatorname{Re} g(x) + i \operatorname{Im} g(x)]\,] =$
$-\operatorname{Im} g(x)$. Then if $f = \operatorname{Re} g$,
then $\operatorname{Im} g(x) = -f(ix)$.

Theorem: Every $f: \tilde{X} \to \mathbb{R}$, f ~~a~~ linear functional, is the real part of some complex linear functional g, $g: X \to \mathbb{C}$

Proof: Define $g(x) = f(x) - i f(ix)$.
First we will establish that g is additive.

$$g(x+y) = f(x+y) - i f(i(x+y)) =$$
$$f(x) + f(y) - i f(ix) - i f(iy) =$$
$$\{f(x) - i f(ix)\} + \{f(y) - i f(iy)\} =$$
$$g(x) + g(y).$$

Now we establish that g is multiplicative. Let $\lambda = \xi + i\eta$, where ξ, η are reals.

$$g(\lambda x) = g(\xi x + i\eta x) =$$
$$f(\xi x + i\eta x) - i f(i\xi x - \eta x) =$$
$$(\xi + i\eta)(f(x) - i f(ix)) =$$
$$(\lambda)(g(x)).$$

Further, if f is continuous, then so is g, since g is a continuous function of f. ∎

Suppose $f \neq 0$. Let $H = \ker f$. Then $iH = \{ih : h \in H\}$ is the kernel of $f(ix)$. So then $\ker g = \{H\} \cap \{iH\}$.

Complex Hahn-Banach Theorem: Let M be a closed complex subspace of a complex B space $\mathfrak{X}$, and C an open convex set such that $M \cap C = \Phi$. Then there exists a closed complex hyperplane K with $K \cap C = \Phi$ and $K \supseteq M$.

Proof: Use the real Hahn-Banach Theorem to find a closed real hyperplane H with $H \cap C = \Phi$ and $H \supseteq M$.

Now $iH \supseteq iM$, but since M is a complex subspace, $iM = M$. Also, $H \cap C = \Phi$. If we let $K = H \cap iH$, we are done. ∎

Analytic version of ~~proof~~ Thm: If f is linear, $f: M \longrightarrow \mathbb{C}$, where M is a subspace of a complex space X, and $\rho: X \to \mathbb{R}^+$ is a seminorm such that for all $x \in M$, $|f(x)| \le \rho(x)$, then there exists a $\hat{f}$, $\hat{f}: X \longrightarrow \mathbb{C}$, linear, with $\hat{f}|_M = f$ and $|\hat{f}(x)| \le \rho(x)$ $\forall x \in X$.

proof: Let $N = \ker f$, $N \subseteq M$.

Let $z \in M$ be such that $\{f(z) = 1\}$, which implies $\rho(z) \ge 1$.

Let $C = \{w \in X : \rho(z - w) < 1\}$

Now $N \cap C = \emptyset$, since otherwise there exists a $n \in N$ such that $\rho(n - z) < 1$, which implies ~~that~~ $|f(z-n)| \le \rho(z-n) < 1$. But $|f(z-n)| = |f(z) - f(n)| = |f(z)|$, since $n \in N$. Then $|f(z)| < 1$, which contradicts the ~~definition~~ of z.

We can use the geometric H.B. Theorem to obtain a closed $K \supseteq N$ with $K \cap C = \emptyset$.

Define $\hat{f}(k + \lambda z) = \lambda$ $\quad k \in K$

~~(Proof to be completed next time)~~

(1) $\hat{f}$ extends f on M since $f(m + \lambda z) = \lambda$ for $m \in M \subseteq K$

(2) $\hat{f}$ is continuous since K is closed

(3) finally $|\hat{f}(x)| \le \rho(x)$ $\quad x \in X$

for if not $\exists\, x$ with $\rho(x) < 1$ and $|\hat{f}(x)| = 1$

we may assume $\hat{f}(x) = 1$ [multiply by right $e^{i\theta}$].

thus $x = z + k$ $\quad k \in X$

and $z - (-k) = x$

$\Rightarrow \rho(z - (-k)) < 1$ so $-k \in C \cap K = \emptyset$

Answer to H.W #3.

prblem 1) . Supp $f \in l_\infty^*$.

Define $\mu_f : \mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{R}$ s.t

$S \subseteq \mathbb{N}$, $\chi_S \in l_\infty$, $\mu_f(S) = f(\chi_S)$

(1) μ_f is finitly additive

Let $A_1 \cdots A_n$ disj $\subseteq \mathbb{N}$.

$$\therefore \chi_{\bigcup_1^n A_i} = \sum_1^n \chi_{A_i}$$

$$\mu_f\left(\bigcup_1^n A_i\right) = f\left(\chi_{\bigcup_1^n A_i}\right) = f\left(\sum_1^n \chi_{A_i}\right) = \sum_1^n f(\chi_{A_i})$$
$$= \sum_1^n \mu_f(A_i)$$

(2) $\|\mu\| = \sup\{\sum_1^n |\mu(A_i)| \quad A_1 \cdots A_n \text{ disj}\}$, then

$\|\mu\| \le \|f\|$ ~~ie~~ (here $\mu = \mu_f$)

i.e μ has bounded variation.

Again, let $A_1 \cdots A_n$ be disj

Define $\alpha_1 \cdots \alpha_n$ by $\alpha_i = \begin{cases} 1 & \text{if } \mu(A_i) \geq 0 \\ -1 & \mu(A_i) < 0 \end{cases}$.

$$\sum_1^n |\mu(A_i)| = \sum_1^n \alpha_i \mu(A_i) = \sum_1^n \alpha_i f(\chi_{A_i})$$

$$= f\left(\sum_1^n \alpha_i \chi_{A_i}\right) = \left| f\left(\sum_1^n \alpha_i \chi_{A_i}\right)\right| \le \|f\| \, \left\|\sum_1^n \alpha_i \chi_{A_i}\right\| = \|f\|$$

If $s \in l_\infty$ and s is simple,

$$s = \sum_1^n \alpha_i \chi_{A_i}, \quad A_1 \cdots A_n \text{ disj}$$

$$\int s \, d\mu = \sum_1^n \alpha_i \mu(A_i) = \sum_1^n \alpha_i f(\chi_{A_i})$$

$$= f\left(\sum_1^n \alpha_i \chi_{A_i}\right) = f(s)$$

$s \in l_\infty$ $\quad$ s simple

$$\int s \, d\mu = f(s)$$

Let μ finitely add. b.v. $\mathcal{P}(\mathbb{N}) \to \mathbb{R}$

Define f_n on the simple functions in l_∞

$$f(s) = f\left(\sum_1^n \alpha_i \chi_{A_i}\right) = \sum_1^n \alpha_i \mu(A_i) = \int s \, d\mu$$

f linear: $\quad f(\lambda s) = \int \lambda s \, d\mu = \lambda \int s \, d\mu.$

$$f(s+t) = \int (s+t) \, d\mu = \int s \, d\mu + \int t \, d\mu = f(s) + f(t)$$

Show $\|f\| \le \|\mu\|$

s : simple, $\|s\| \le 1$.

$$s = \sum_{i}^{n} \alpha_i \chi_{A_i}, \quad A_1, \cdots A_n \text{ disj} \quad |\alpha_i| \le 1.$$

$$|f(s)| = |f(\sum_{i}^{n} \alpha_i \chi_{A_i})| \le \sum_{i}^{n} |\alpha_i| |f(\chi_{A_i})|$$

$$\le \sum_{i}^{n} 1 \cdot |\mu(A_i)| \le \|\mu\|$$

$$\therefore \|f\| \le \|\mu\|$$

<Remark>

$$\varphi : C \longrightarrow \mathbb{R}$$

$\varphi(\xi_n) = \lim \xi_n \in C^*$, $\varphi(C_0) = 0$.

Since $C \subseteq l_\infty$, H.B. ext. φ to f on l_∞.

$f(C_0) = 0$, $f(\theta) = 1$, $\theta = (1, 1, \cdots)$

μ_f : finitely additive b.v. measure on $P(\mathbb{N})$.

If A is finite $\subseteq \mathbb{N}$, $\chi_A \in C_0$, $\mu(A) = f(\chi_A) = 0$

If A s.t $\mathbb{N} - A$ is finite, $\mu(A) = \mu(\mathbb{N} - A) + \mu(A)$

$= \mu(\mathbb{N}) = f(\theta) = 1$

Wierd !?

Problem 3)

U_X : unit ball

$\{x_n\} \subseteq T(U_X) \subseteq l_1$

$\therefore \exists y_n \in U_X$ s.t $Ty_n = x_n$

$\therefore \exists \{y_{n_k}\}$ wkly conv. subseq of $\{y_n\}$

Since T is conti, T is wkly conti.

So $\{x_{n_k}\}$ converges wkly. ($x_{n_k} = Ty_{n_k}$)

Since in l_1, wkly convergence of $\{x_{n_k}\}$ is equiv to norm convergence,

$\therefore$ T is compact.

Problem 4) $\varphi : l_1 \xrightarrow{\text{onto}} l_2$, $X = \ker\varphi$.

If we show $(I-P)l_1 \simeq l_2$, then

Infinite dim. subsp. of $l_1 \simeq l_2$.

$\therefore l_2 \xrightarrow[\subseteq]{} l_1$

$\therefore l_2$: finite dim since unit ball in l_2 is compact by #3.

$\#$

Let $Y = \ker \rho$

$$\begin{array}{ccccc} X \times Y & \xrightarrow{\simeq} & l_1 & \longrightarrow & l_2 \\ \downarrow \varphi & & & & \\ l_2 & & & & \end{array}$$

problem 6) Y is $N+1$ dim.

Write down $N+1$ indep. vectors

$$\begin{array}{llllll} \alpha_1^{(1)}, & \alpha_2^{(1)}, & \cdots & \alpha_n^{(1)} & \alpha_{n+1}^{(1)} & \alpha_{n+2}^{(1)} \quad \cdots \\ \alpha_1^{(2)} & \alpha_2^{(2)} & \cdots & \alpha_n^{(2)} & \alpha_{n+1}^{(2)} & \alpha_{n+2}^{(2)} \quad \cdots \\ \vdots & & & & & \\ \alpha_1^{(n+1)} & \alpha_2^{(n+1)} & \cdots & \alpha_n^{(n+1)} & \alpha_{n+1}^{(n+1)} & \alpha_{n+2}^{(n+1)} \end{array}$$

Gauss Elimination.

problem 7) Y : infinite dim. $\subseteq l_1 \qquad 1 = N_1 < N_2 < \ldots$

b_i, z_i s.t

$b_i = \sum_{N_i+1}^{N_{i+1}} \alpha_i e_i \qquad e_i$: usual basis

$\|b_i\| = 1$

$\|z_i\| < \varepsilon/2^i \quad , \quad z_i = \sum_{N_{i+1}+1}^{\infty} \gamma_i e_i$

$b_i + z_i \in Y$

$\underline{X}$: B-sp.

$A : \underline{X} \longrightarrow \underline{X}$

Define $\sigma(A)$ – spectrum

Supp. $P : \underline{X} \longrightarrow \underline{X}$ is conti proj. i.e $P^2 = P$
and supp. $P(A) = AP$

Let $R = P(\underline{X})$ range, N : ker P.

$$R \supset PA(\underline{X}) = A(P\underline{X}) = A(R)$$

$$P(A(N)) = A(P(N)) = A(0) = 0$$

$$A(N) \subseteq N$$

Rewrite $\underline{X} = R \oplus N$

$$R \oplus N \xrightarrow{A} R \oplus N$$

$$\begin{bmatrix} A/R & 0 \\ 0 & A/N \end{bmatrix}$$

$\underline{\underline{Qn}}$ $PA \overset{?}{=} AP$

$\underline{\text{Question}}$ is what projections commute with A.
(hard)

So. $\underline{\text{Question}}$ what operator commute with A.

Some that do

$I, A, A^2, A^3, \ldots$

If it exists, A^{-1} will commute with A.

$A^{-1}, A^{-2}, \ldots$

Polynomials in A commute with A.

$B(X)$ has norm $\|\cdot\|$ which is complete.

s.t. (1) $\|S+T\| \leq \|S\| + \|T\|$

(2) $\|\alpha T\| = |\alpha| \|T\|$

(3) $\|T\| \geq 0$ and $= 0 \iff T = 0$

(4) $\|\cdot\|$ is complete

(5) $\|ST\| \leq \|S\| \|T\|$

multiplication. composition.

(6) $\exists I$ s.t. $\forall T$, $IT = TI$.

<u>Defn</u> A vector sp. and also a ring (algebra) is a normed algebra if it satisfies (1)–(3) and (5)

Banach algebra if (1) — (5)

Banach algebra with I if (1) — (6).

Def. A collection $\mathcal{A}$ is a B-algebra if it is an algebra and

(1) it is a B-space with norm $\|\cdot\|$,

(2) $\|AB\| \leq \|A\| \cdot \|B\|$ for $A, B \in \mathcal{A}$.

$\mathcal{A}$ is a B-algebra with identity if there is an $I \in \mathcal{A} \ni \forall A \in \mathcal{A}$, $IA = AI = A$.

Examples

1. $\mathcal{B}(\mathfrak{X}) = \{T : \mathfrak{X} \to \mathfrak{X}$ bdd. linear with operator norm$\}$
is a B-algebra with identity - multiplication is composition.

2. $K(\mathfrak{X}) = \{T : \mathfrak{X} \to \mathfrak{X} : T$ linear + compact$\}$

Then $K(\mathfrak{X}) \leq B(\mathfrak{X})$ is a B-alg. but does not have 1 if $\dim \mathfrak{X} = +\infty$.

3. $F(\mathfrak{X}) = \{\lambda I + K : K \in K(\mathfrak{X})\}$

4. If $\mathfrak{X}$ is compact, T_2, then $C(\mathfrak{X})$ is a B-alg. with 1 where $(fg)(t) = f(t)\, g(t)$,
$\|f\| = \sup_{t \in \mathfrak{X}} |f(t)|$.

It is commutative.

5. $L_1(\mathbb{R})$, where $fg = f * g =$ convolution

Question: If $A \in \mathcal{A}$, what is
$C(A) = \{B \in \mathcal{A} : AB = BA\}$?

We noted:

1) if $B \in C(A)$, $\lambda \in \mathbb{C}$, then $\lambda B \in C(A)$ since $\lambda BA = \lambda AB = A\lambda B$,

2) $B, C \in C(A) \Rightarrow B + C \in C(A)$ since $A(B+C) = AB + AC = BA + CA = (B+C)A$

3) $B, C \in C(A) \Rightarrow BC \in C(A)$ since
$$A(BC) = (AB)C = (BA)C = B(AC) = B(CA) = (BC)A.$$

[This implies:

~~Cor.~~ Any polynomial in A commutes with A, that is, something of the form
$$\lambda_1 I + \lambda_2 A + \lambda_3 A^2 + \cdots + \lambda_n A^{n-1}, \quad \lambda_i \in \mathbb{C}\ \forall i]$$

4) $C(A)$ is closed, that is, if $\{B_n\} \subseteq C(A)$ & $B_n \to B$ in norm, then $B \in C(A)$.

Pf of 4). It suffices to show $AB = \lim_{n\to\infty} AB_n$ & $BA = \lim_{n\to\infty} B_n A$, for that $\Rightarrow$

(since $B_n \in C(A)\ \forall n$), $\lim_{n\to\infty} AB_n = \lim_{n\to\infty} B_n A$ & so $AB = BA$ & $B \in C(A)$.

So consider $\|AB - AB_n\| = \|A(B-B_n)\| \le \|A\| \cdot \|B - B_n\|$. $\|A\|$ is fixed & $\|B - B_n\| \to 0 \Rightarrow \|AB - AB_n\| \to 0$.
If $A_n \to A$ & $B_n \to B$, then $A_n B_n \to AB$,

for $\|AB - A_n B_n\| \le \|AB - AB_n + AB_n - A_n B_n\|$
$$\le \|A\| \cdot \|B - B_n\| + \|A - A_n\| \cdot \|B_n\|.$$

Since $\|B - B_n\| \to 0$, $\|B_n\|$ is bdd. Also

$\|A\|$ is fixed, $\|B-B_n\| \to 0$ & $\|A-A_n\| \to 0$.
So we have $\|AB - A_nB_n\| \to 0$.

Properties 1) – 4) show that $C(A)$ is a B-alg. with 1.

However $C(A)$ is not in general commutative. For consider $A = I$, so $C(I) = \mathcal{A}$. Then if $\mathcal{A}$ is not commutative, neither is $C(I)$.

Consider $P(A) = \{B \in \mathcal{A} : B$ is a norm limit of polynomials in $A\}$.

Then $P(A) \subseteq C(A)$, and $P(A)$ is commutative.

[We see, by checking mult., $p(A)\,q(A) = q(A)\,p(A)$.

$\ni$ $p_n(A) \to B$, $q_n(A) \to C$.

Then $BC = \lim_{n\to\infty} p_n(A)\,q_n(A) = \lim_{n\to\infty} q_n(A)\,p_n(A) = CB$

by previous argument.]

<u>Change of Pace</u>: $\ni$ $f : \Omega \to \mathfrak{X}$, $\Omega \overset{\text{open}}{\subseteq} \mathbb{C}$, $\mathfrak{X}$ a B-space.

<u>Def</u>. f is <u>analytic</u> at $\lambda_0 \in \mathbb{C}$ if $\exists \{x_n\} \subseteq \mathfrak{X}$ $\ni$ $f(\xi) = \sum_{i=0}^{\infty} (\xi - \lambda_0)^i x_i$ (convergence in norm)

in some open set about λ_0 in Ω.

~~Thm.~~ $\cancel{\mathcal{S}}$ $\{x_i\} \subseteq \mathfrak{X}$, $\lambda_0, \xi \in \mathbb{C}$,

$f(\xi) \sim \sum_{i=0}^{\infty} (\xi - \lambda_0)^i x_i$.

Then $\exists R$ (called radius of convergence) with $0 \le R \le \infty$,

[$R \equiv \sup\{r \ge 0 : \{x_i r^i\}$ is norm bdd$\}$]

$\ni$ (1) $|\xi - \lambda_0| < R \Rightarrow \sum_{i=0}^{\infty} (\xi - \lambda_0)^i x_i$ converges absolutely in norm, ($\Leftrightarrow \sum_{i=0}^{\infty} |\xi - \lambda_0|^i \|x_i\| < +\infty$)

(2) for $0 \le r < R$, $f_n(\xi) = \sum_{i=0}^{n} (\xi - \lambda_0)^i x_i$ converges to $f(\xi) = \sum_{i=0}^{\infty} (\xi - \lambda_0)^i x_i$ uniformly on the set $\{\xi : |\xi - \lambda_0| \le r\} \equiv D_r(\lambda_0)$,

$$\lim_{n\to\infty} \sup_{\xi \in D_r(\lambda_0)} \| f_n(\xi) - f(\xi) \| = 0.$$

(3) ~~If~~ $|\xi - \lambda_0| > R$, then $\sum_{i=0}^{\infty} (\xi - \lambda_0) x_i$ diverges.

~~Pf of (1).~~ Let $|\xi - \lambda_0| < r < R$.

Then $\exists M \ni \|x_i\| r^i \le M \; \forall i$.

So $\|x_i\| \cdot |\xi - \lambda_0|^i \le \|x_i\| r^i \left(\frac{|\xi - \lambda_0|^i}{r} \right) \le M \left(\frac{|\xi - \lambda_0|^i}{r} \right)$

with $0 \le \frac{|\xi - \lambda_0|}{r} < 1$.

So $\sum_{i=0}^{\infty} |z-\lambda_0|^i \|x_i\| \le \sum_{i=0}^{\infty} M \left(\frac{z-\lambda_0}{r}\right)^i < +\infty$

since it is a geometric series with ratio < 1.

Pf. of 2). S. $r<R$. $\exists\, \epsilon>0 \ni r<r+\epsilon<R$.

Then $|z-\lambda_0| \le r < r+\epsilon \Rightarrow$

$\|\sum_{N}^{\infty}(z-\lambda_0)^i x_i\| \le \sum_{N}^{\infty} M \left(\frac{r}{r+\epsilon}\right)^i \to 0$ uniformly on N.

Pf of 3). Since $\| |z-\lambda_0|^i x_i\|$ is unbdd, ~~the~~ result follows.

Next time we with show

$$R = \frac{1}{\limsup_{i\to\infty} \sqrt[i]{\|x_i\|}}.$$

Let $\{x_n\}$ be contained in A, and let $f(\rho) = \sum_{n=0}^{\infty} (\rho - \lambda_0)^n x_n$. If the radius of convergence $(= \sup_n \{r>0 : \{\|x_n\| r^n\}\})$ is bounded, we will show

$R = 1/\limsup_n \|x_n\|^{1/n}$

where R is the radius of convergence.

Let $\limsup_{n\to\infty} \|x_n\|^{1/n} = S$.

Suppose $r > S$. Then there is an N such that if $n \geq N$, then $\|x_n\|^{1/n} < r$, or, equivalently $\|x_n\| < r^n$.

So if $|\rho - \lambda_0| < \frac{1}{r}$, $\|x_n\| |\rho - \lambda_0|^n < r^n (\frac{1}{r^n}) = 1$.

If $S < \frac{1}{r}$, $\|x_n\| S^n < r^n (\frac{1}{r^n}) = 1$, so $R \geq S$.

But $S < \frac{1}{r}$, which implies $1/S > r$, so

$\frac{1}{S} > r > S$. Then if $\frac{1}{S} > S$, $R \geq S$.

Therefore, $\frac{1}{S} \leq R$.

If $r < S$, then for all N, there exists an n such that $\|x_n\|^{1/n} > r$ (or $\|x_n\| > r^n$) infinitely often.

Let s be such that $1/S < \frac{1}{r} < s$. Then $\|x_n\| s^n > r^n (\frac{1}{r^n}) = 1$.

Then since $(sr) > 1$ and $\|x_n\| s^n > (sr)^n$, $\|x_n\| s^n$ cannot be bounded, for $\lim_{n\to\infty} (sr)^n = +\infty$.

Then if $s > 1/S$, $\|x_n\| s^n$ is not bounded. Therefore, $R \leq 1/s$ ∎

Let $\mathcal{A}$ be a Banach algebra with I. For $A \in \mathcal{A}$, define $\sigma(A)$ = spectrum of A = $\{\lambda \in \mathbb{C} : (\lambda - A)^{-1}$ does not exist$\}$.

Theorem: If $A \in \mathcal{A}$, a Banach algebra with I, and $\limsup \|A^n\|^{1/n} < \lambda$, then $(\lambda - A)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n \in \mathcal{A}$.

Proof: Now $\sum_{n=0}^{\infty} \lambda^{-n-1} A^n = \frac{1}{\lambda}\left(\sum_{n=0}^{\infty} \lambda^{-n} A^n\right)$, which converges whenever $\limsup_{n \to \infty} \|A^n / \lambda^n\|^{1/n} < 1$ – that is, whenever $\limsup_{n \to \infty} \|A_n\|^{1/n} < \lambda$.

Look at $(\lambda - A)\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{-n} A^n =$

$$\lim_{M \to \infty} \left(I - \frac{A}{\lambda}\right)\left(\sum_{n=0}^{M} \lambda^{-n} A^n\right) =$$

$$\lim_{M \to \infty} \left(\sum_{n=0}^{M} \lambda^{-n} A^n - \sum_{n=0}^{M} \lambda^{-n+1} A^{n+1}\right) =$$

$$\lim_{M \to \infty} \left(\sum_{n=0}^{M} \lambda^{-n} A^n - \sum_{n=1}^{\infty} \lambda^{-n} A^n\right) =$$

$$\lim_{M \to \infty} \left(I - \left(\frac{A}{\lambda}\right)^M\right).$$

But $\|A^M\| < \lambda^M$, since $\limsup \|A_n\|^{1/n} < r < \lambda$, and $\lambda^M > r^M$. Therefore, $\|(\frac{A}{\lambda})^M\| < |\frac{r}{\lambda}|^M$, and since $|\frac{r}{\lambda}| < 1$, $|\frac{r}{\lambda}|^M \to 0$.

Hence, $\lim_M \|(\frac{A}{\lambda})\|^M = 0$, and therefore

$$\lim_{M \to \infty} \left(I - \left(\frac{A}{\lambda}\right)^M\right) = I \quad ∎$$

Cor: $\sigma(A)$ is contained in the disk of radius ρ_A, where $\rho_A = \limsup_{n\to\infty} \|A^n\|^{1/n}$.

Denote the disk of radius A by D_{ρ_A}.
$\mathbb{C} \setminus D_{\rho_A} : \lambda \longrightarrow (\lambda - A)^{-1} \in \mathcal{A}$ is analytic - can be written as a power series.

Lemma: Suppose $\lambda, \mu \in \mathbb{C} \setminus \sigma(A)$. Then
$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}$.

Remark - Think of $\mathfrak{X}$ = Banach space $\mathcal{A}$.
Then for each $A \in \mathcal{A}$, we have an operator
$T_A : \mathfrak{X} \to \mathfrak{X}$ by $x \mapsto A\cdot x$.

Proof: Let $x \in \mathcal{A}$. There is a $u \in \mathcal{A}$ with
$u = (\lambda - A)^{-1}x$ (or $(\lambda - A)u = x$).
Then $(\mu - A)u = [(\mu - \lambda) + (\lambda - A)]u =$
$(\mu - \lambda)u + (\lambda - A)u$.
But $(\lambda - A)u = x$, so
$(\mu - A)u = (\mu - \lambda)u + x$

Then multiplying by $(\mu - A)^{-1}$, we have
$u = (\mu - A)^{-1}x + (\mu - \lambda)(\mu - A)^{-1}u$.

Then $(\lambda - A)^{-1}x = [(\mu - A)^{-1} + (\mu - \lambda)(\mu - A^{-1})(\lambda - A^{-1})]x$
for every $x \in \mathfrak{X}$.
Take $x = 1$, and the lemma will follow.

Note: $T_A x = T_B x \ \forall x \in \mathfrak{X} \Rightarrow A = B$ (T is defined above) since if we take $x = 1$,
$T_A x = T_A 1 = A\cdot 1 = A$ &
$T_B x = T_B 1 = B \cdot 1 = B$

Lemma: $(\lambda - A)^{-1} = \sum_{n=1}^{\infty} (\mu - \lambda)^{n-1} (\mu - A)^{-n}$

Proof: $(\lambda - A)^{-1} = (\mu - A)^{-1} + (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1} =$

$(\mu - A)^{-1} + (\mu - \lambda)[(\mu - A)^{-1} + (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}]$

$\times [\mu - A]^{-1} =$ (repeating the preceding N times)

$(\lambda - A)^{-1} - \sum_{n=1}^{N} (\mu - \lambda)^{n-1} (\mu - A)^{-n} =$

$(\mu - \lambda)^{N} (\lambda - A)^{-1} (\mu - A)^{-N-1}$.

Consider $\| (\mu - \lambda)^N (\lambda - A)^{-1} (\mu - A)^{-N-1} \| \leq$

$|\mu - \lambda|^N \| (\lambda I - A)^{-1} \| \; \| (\mu - A)^{-1} \|$ but

$|\mu - \lambda| \, \| \mu - A^{-1} \| \leq r < 1$, so

$|\mu - \lambda|^{N+1} \| \mu - A^{-1} \|^{N+1} \leq r^{N+1} < 1$.

Then the above goes to zero as $N \to +\infty$. Therefore, $\frac{1}{|\mu - \lambda|}$ times it goes to zero, completing the proof ∎

Let $I = [0,1]$, and let $\varphi : I \to X$ describe a curve in X. Then we define $\int_0^1 \varphi(t)\,dt$ to be

$$\lim_{|\Delta t| \to 0} \sum_{i=1}^{n} \varphi(t_i)\,\Delta t_i$$

in the norm topology, where for some partition of I, $(u_0, u_1, \ldots, u_n)$, $t_i \in [u_{i-1}, u_i)$, $\Delta t_i = |u_i - u_{i-1}|$, and $|\Delta t| = \sup_i \Delta t_i$.

It is clear that $\int_0^1 (\cdot)\,dt$ is linear, and defined for continuous φ by the same argument that establishes the existence of the Rieman integral for continuous φ.

Let $\varphi_n(t)$ be a sequence of integrable functions converging uniformly in norm to some $\varphi(t)$ on I — that is, $\sup_{t \in I} \| \varphi_n(t) - \varphi(t) \| \longrightarrow 0$.
Then $\int_0^1 \varphi_n(t)\,dt \longrightarrow \int_0^1 \varphi(t)\,dt$.

To prove the above, substitute "$\| \cdot \|$" for absolute value in the real variable argument and everything goes through.

Let C be a simple closed curve in $\mathbb{C}$, piecewise smooth. Let φ represent C ($\varphi : [0,1] \to C$) and let $\varphi(0) = \varphi(1)$. If φ is restricted to $(0,1)$, φ is 1-1 and φ is piecewise differentiable.

If $f : \mathbb{C} \to \mathbb{C}$, we define the counter-clockwise contour integral

$\oint_C f(z)\,dz$ to equal

$\int_0^1 f(\varphi(t))\varphi'(t)dt.$

Suppose $f: \mathbb{C} \to \mathfrak{X}$, and C is a simple closed smooth curve in $\mathbb{C}$. Then $\oint_c f(z)dz = \int_0^1 f(\varphi(t))\varphi'(t)dt$ where φ represents C.

Note that $f(\varphi(t))$ is in $\mathfrak{X}$, and $\varphi'(t)$ is a complex value.

If f and g are integrable, $f, g: \mathbb{C} \to \mathfrak{X}$, then

(1) $\oint_c (f+g)\, dz = \oint_c f\, dz + \oint_c g\, dz$ and

(2) $\oint_c \alpha f\, dz = \alpha \oint_c f\, dz$, where $\alpha \in \mathbb{C}$.

Both of the above follow from the definition of $\oint_c f(z)dz$ and the linearity of $\int_0^1 (\cdot)\, dt$.

If $f_n(z) \longrightarrow f(z)$ uniformly in C, then $\oint_c f_n(z)dz \longrightarrow \oint_c f(z)\, dz$, which also follows from definitions.

Let $x \in \mathfrak{X}$. Pick $g: \mathbb{C} \to \mathbb{C}$, where g is continuous. Then

$$\oint_c g(z)\, x\, dz = \lim_{|\Delta t_i| \to 0} \sum_{i=1}^{n} \{g(\varphi(t_i))x\}\varphi'(t_i)\Delta t_i =$$

$$x \lim_{|\Delta t_i| \to 0} \sum_{i=1}^{n} g(\varphi(t_i))\, \varphi'(t_i)\Delta t_i =$$

$x \oint_c g(z)\, dz.$ Notice that

$\oint_c g(z)dz$ is merely a scalar, in $\mathbb{C}$.

Let $A \in \mathcal{A}$, a Banach algebra with 1. Let C be a circle about the origin with radius $r > \rho_A$, where $\rho_A = \limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}}$.

(1) $\frac{1}{2\pi i} \oint_C Z^n (Z-A)^{-1} dZ = A^n$, where the integration is counter-clockwise

Proof: Let $|Z| > \rho_A$. Then we know that $(Z-A)^{-1} = \sum_{k=0}^{\infty} Z^{-k-1} A^k$. Therefore,

$\oint_C Z^n (Z-A)^{-1} dZ =$

$\lim_{N\to\infty} \oint_C Z^n \sum_{k=0}^{N} (Z^{-k-1}) A^k dZ =$

$\lim_{N\to\infty} [A \oint_C Z^{n-1} dZ + A^2 \oint_C Z^{n-2} dZ + \dots$

$+ A^n \oint \frac{dZ}{Z} + \dots + A^N \oint \frac{dZ}{Z^{N+1-n}}]$, by the linearity of the integral.

From complex variable theory, we know that $\oint_C Z^n dZ = \begin{Bmatrix} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{Bmatrix}$ for any circle about the origin C.

Then $\oint_C Z^n (Z-A)^{-1} dZ = \lim_{N\to\infty} 2\pi i A^n =$

$2\pi i A^n$. Hence,

$\frac{1}{2\pi i} \oint_C Z^n (Z-A)^{-1} dZ = A^n$. ∎

(2) If $p(t)$ is a polynomial – that is, $p(t) = \sum_{i=0}^{m} \lambda_i t^i$ – then
$$P(A) = \frac{1}{2\pi i} \oint_c p(z)(z-A)^{-1}\,dz.$$

Proof: By linearity,

$$\frac{1}{2\pi i} \oint_c p(z)(z-A)^{-1}\,dz =$$

$$\frac{1}{2\pi i} \sum_{k=0}^{m} \oint_c \lambda_k z^k (z-A)^{-1}\,dz =$$

$$\sum_{k=0}^{m} \lambda_k (A^k) = p(A) \quad \blacksquare$$

$A \in \mathcal{A}$ A B-ALGEBRA WITH $\mathbb{1}$.

IF $\rho_A = \limsup \|A^n\|^{\frac{1}{n}}$

$r > \rho_A$, C = CIRCLE CENTERED AT O AND RADIUS r

$$A^n = \frac{1}{2\pi i} \oint_C z^n (z-A)^{-1} dz$$

IF $p(t)$ = POLYNOMIAL

$$p(A) = \frac{1}{2\pi i} \oint_C p(z)(z-A)^{-1} dz$$

WE CAN SHOW:

IF $f(z): \mathbb{C} \to \mathbb{C}$ IS ANALYTIC FOR DISC CENTERED AT O WITH RADIUS $r+\varepsilon$ (SOME $\varepsilon > 0$)

THEN $f(A) = \oint_C f(z)(z-A)^{-1} dz$

$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n$ WITH RADIUS OF CONVERGENCE $> \rho_A$

☆ → $f(A) = \lim_{N\to\infty} \sum_1^N a_n A^n$ EXIST

$$= \lim_{N\to\infty} \frac{1}{2\pi i} \oint_C f(z)(z-A)^{-1} dz$$

$$\text{(UNIFORMLY)} = \frac{1}{2\pi i} \oint_C f(z)(z-A)^{-1} dz$$

SUPPOSE $f(z), g(z)$ ARE $\ni$ ☆ IS TRUE AND $h(z) = f(z)g(z)$

~~AND~~ $h(A) = \frac{1}{2\pi i} \oint_C f(z)(z-A)^{-1} dz$

CONSIDER $A^n = \frac{1}{2\pi i} \oint_C z^n (z-A)^{-1} dz$

THEOREM: LET $M = \max \{ |\lambda| \mid \lambda \in \sigma(A) \}$

THEN $M = \limsup \|A^n\|^{\frac{1}{n}} = \rho_A$

COR: $\sigma(A) \neq 0$ (IF SO, NO λ & MAX $= -\infty$)

(ALREADY KNOW $m \leq \limsup \|A^n\|^{1/n}$

LET $r = m + \epsilon$ FOR $\epsilon > 0$

$\|A^n\| \leq \left\| \frac{1}{2\pi i} \oint_C z^n (z-A)^{-1} dz \right\|$

LET $M = \max \|(z-A)^{-1}\| < +\infty$

CONT. ON COMPACT

$\|A^n\| \leq \frac{1}{2\pi} \oint_C r^n M \, dz$ WHERE r^n = LENGTH OF z^n

$\leq \frac{r^n M}{2\pi} \cdot 2\pi r$

$= M r^{n+1}$

$\|A^n\|^{\frac{1}{n}} \leq M^{\frac{1}{n}} r^{\frac{n+1}{n}}$

$\therefore \limsup_{n \to \infty} \|A^n\|^{\frac{1}{n}} \leq r = m + \epsilon$

THUS TRUE FOR $\epsilon > 0$ $\therefore$ DONE

THEOREM: SUPPOSE $\mathcal{A}$ IS B-ALGEBRA WITH 1

SUPPOSE $\mathcal{A}$ IS A DIVISION RING (i.e. $A \in \mathcal{A}$, $A \neq 0$ $\exists$ $\Longrightarrow$ A^{-1} EXIST $\in \mathcal{A}$)

THEN $\mathcal{A}$ IS COMPLEX NOS.

PF: $A \in \mathcal{A}$ WE KNOW $\sigma(A) \neq \phi$

$\exists \lambda \in \mathbb{C} \ni \lambda I - A$ HAS NO INVERSE IN $\mathcal{A}$

$\therefore \lambda I - A = 0$ OR $A = \lambda I$

$\mathcal{A} \to \mathbb{C}$

$A \mapsto \lambda$

$0 \mapsto 0$

$I \mapsto 1$

1-1 :
ONTO : $\lambda I \to \lambda$

LINEAR: $A = \lambda I$ $B = \beta I$,

$A + B = \lambda I + \beta I = (\lambda + \beta) I$

ETC.

THEOREM NOT TRUE FOR REALS

SUPPOSE $A \in \mathcal{A}$

CLAIM: $\sigma(A^n) = [\sigma(A)]^n = \{\lambda^n \mid \lambda \in \sigma(A)\}$

(RELATION BETWEEN SPECTRUM OF A AND SPEC. OF A^n)

CONSIDER $\lambda^n - A^n$;

$$\lambda^n - A^n = (\lambda - A)B = B(\lambda - A)$$

WHERE $B = \lambda^{n-1} + \lambda^{n-2}A + \lambda^{n-3}A^2 + \cdots + A^{n-1}$

IF $\lambda \in \sigma(A)$ THEN $(\lambda - A)^{-1}$ DOES NOT EXIST

$\mathcal{A} \xrightarrow{\lambda - A} \mathcal{A}$ OPERATOR

$x \longmapsto (\lambda - A)x$

SINCE $(\lambda - A)^{-1}$ DOES NOT EXIST THEN

(1) EITHER $\ker(\lambda - A) \neq \{0\}$

OR (2) $\text{RANGE}(\lambda - A) \neq A$ (BY OPEN MAPPING THEOREM)

SUPPOSE (1):

$\exists x \neq 0 \quad x \in A \ni (\lambda - A)x = 0$

$$(\lambda^n - A^n)x = B(\lambda - A)x = B(0) = 0$$

$\ker(\lambda^n - A^n) \neq 0$

SUPPOSE (2): $\exists y \in A \ni \forall x \in A$

$(\lambda - A)x \neq y$ ★

SUPPOSE $\exists z$, $(\lambda^n - A^n)z = y$

$\Rightarrow y = (\lambda^n - A^n)z = (\lambda - A)Bz$ so $x = Bz$,

CONTRADING ★.

THIS SHOWS $\lambda \in \sigma(A) \Rightarrow \lambda^n \in \sigma(A^n)$

$\Rightarrow \sigma(A^n) \supseteq [\sigma(A)]^n$

LET $\mu \in \sigma(A^n)$ SHOW: SOME POWER OF POINT IN SPECTRUM OF A.

CONSIDER $z^n - \mu$, A COMPLEX POLYNOMIAL

LET $\lambda_1, \lambda_2, \cdots, \lambda_n$ BE COMPLEX ROOTS OF $z^n - \mu$

THE FOLLOWING IS TRUE: $\lambda_i^n = \mu$

$$(z^n - \mu) = (z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n)$$

$$A^n - \mu = (A - \lambda_1)(A - \lambda_2)\cdots(A - \lambda_n)$$

IF ALL λ_i DO NOT BELONG TO $\sigma(A)$ THEN $(A - \lambda_i)^{-1}$ EXIST

FOR $i = 1, 2, \cdots, n$

$\Rightarrow B(A^n - \mu) = (A^n - \mu)(B) = I$

$B = (A - \lambda_1)^{-1}(A - \lambda_2)^{-1}\cdots(A - \lambda_n)^{-1}$

SO $B = (A^n - \mu)^{-1}$, A CONTRADICTION

$\therefore$ SOME $\lambda_i \in \sigma(A)$

$\therefore \sigma(A^n) = [\sigma(A)]^n$

LET $m = \rho_A = \lim\sup_n \|A^n\|^{1/n}$

THIS IMPLIES $\max\{|\lambda| \mid \lambda \in \sigma(A^n)\} = m^n$

ALSO KNOW $m^n \le \|A^n\|$

$\therefore m \le \|A^n\|^{1/n}$

$m \le \lim\inf \|A^n\|^{1/n} \le \lim\sup \|A^n\|^{1/n} = m$

$\therefore \lim \|A^n\|^{\frac{1}{n}}$ EXIST

ALSO TRUE: $\inf \|A^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$

CONSIDER:

$$\|A^{2n}\|^{\frac{1}{2n}} \le (\|A^n\|\|A^n\|)^{\frac{1}{2n}} = \|A^n\|^{\frac{1}{n}}$$

THUS: $\|A\| \ge \|A^2\|^{1/2} \ge \|A^4\|^{1/4} \ge \|A^{2n}\|^{\frac{1}{2n}} \ge \cdots$

LIMIT EXIST, SO MUST BE INF

$\ni$ $\mathcal{A}$ is a B-alg. w/ 1, $A \in \mathcal{A}$, $\sigma(A)$ - spectrum of $A \subseteq \mathbb{C}$.

$\rho_A = \inf_n \|A^n\|^{\frac{1}{n}} = \max \{|\lambda| : \lambda \in \sigma(A)\}$.

$H_A = \{f : U_f \to \mathbb{C} \mid f$ is analytic on U_f & $U_f \supseteq \sigma(A)\}$

<u>Lemma.</u> U open $\supseteq K$ compact, then $\exists\, C =$ disjoint union of $C_1, \cdots, C_n$, where C_i is a simple closed curve in $U \setminus K$, "containing" K inside $\bigcup_i \operatorname{int} C_i$.

<u>Thm</u> (w/o proof). $\forall f \in H_A$ & C in lemma, then $f(A) = \frac{1}{2\pi i} \oint_C f(z)(z-A)^{-1}dz$, $f(A) \in \mathcal{A}$. also, if $f, g \in H_A$, $h(z) = f(z)g(z) \in H_A$, then

$$h(A) = f(A)g(A) = \frac{1}{2\pi i} \oint_C f(z)g(z)(z-A)^{-1}dz.$$

<u>Applications.</u>

1) If $\sigma(A) \cap \{t \in \mathbb{R}, t \leq 0\} = \emptyset$, $\exists\, A^{-\frac{1}{2}} \in \mathcal{A}$ & $A^{\frac{1}{2}} A^{\frac{1}{2}} = A$

2) <u>Spectral Mapping Th</u>: $f \in H(A)$, then $\sigma(f(A)) = f(\sigma(A))$

<u>Pf.</u> ($\subseteq$) $\ni$ $\mu \in \sigma(f(A))$ but $f(\lambda) \neq \mu$ $\forall \lambda \in \sigma(A)$.
$f(z) - \mu \neq 0$ on $\sigma(A)$ & so there is U open $\supseteq \sigma(A)$ w/ $f(z) - \mu \neq 0$ on U.
Then $g(z) = \frac{1}{f(z)-\mu} \in H_A$. So $g \in \mathcal{A}$.

$$g(A)(f(A) - \mu) = (f(A) - \mu)g(A)$$
$$= \frac{1}{2\pi i} \oint_C \frac{f(z)-\mu}{f(z)-\mu}(z-A)^{-1}dz = 1 \ \#$$

since $\Rightarrow g(A) = [f(A) - \mu]^{-1}$ but $\mu \in \sigma(f(A))$ & so can't have an inverse. $\therefore\ \sigma(f(A)) \subseteq f(\sigma(A))$.

($\supseteq$) Let $\lambda \in \sigma(A)$. Let $\mu = f(\lambda)$.

Define $g(z) = \begin{cases} \dfrac{f(z)-\mu}{z-\lambda} & z \neq \lambda \\ f'(\lambda) & z = \lambda \end{cases}$.

g is analytic wherever f is.

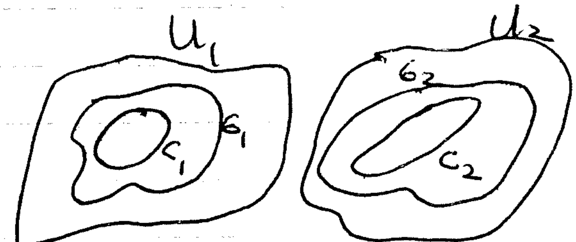
So $g(z)(z-\lambda) = f(z) - \mu$

$$g(A)(A-\lambda) = (A-\lambda)g(A) = f(A) - \mu$$

But $(A-\lambda)$ is not invertible $\Rightarrow$

$f(A)-\mu$ is not invertible or $\mu \in \sigma(f(A))$.

Done with pf.



3. <u>Spectral Projections.</u>

$\not{\mathrm{S}}$ $\sigma = \sigma_1 \cup \sigma_2$ disjoint closed sets,

$C = C_1 \cup C_2$, $U = U_1 \cup U_2$ where U_i disjoint ~~connected~~ open sets containing σ_i, C_i curves in σ_i.

Define $f_i(z) = \begin{cases} 1 & z \in U_i \\ 0 & \text{o/w} \end{cases}$. Then $f_i \in H_A$.

$P_i = \frac{1}{2\pi i} \oint_C f_i(z)(z-A)^{-1}\,dz \in \mathcal{A}$ commutes with A.

$P_i^2 = \frac{1}{2\pi i} \oint_C f_i^2(z)(z-A)^{-1}\,dz = \frac{1}{2\pi i} \oint_C f_i(z)(z-A)^{-1}\,dz = P_i$.

$\therefore$ P_i is a cont. projection.

$$P_1 P_2 = \oint_C f_1(z) f_2(z) (z-A)^{-1} dz = 0$$

$$P_1 - I = P_2 .$$

In this case, $X \xrightarrow{A} X$ in $B(X)$, we have $P_1, P_2 \ni \ker P_1 = \text{range } P_2$
$\ker P_2 = \text{range } P_1$

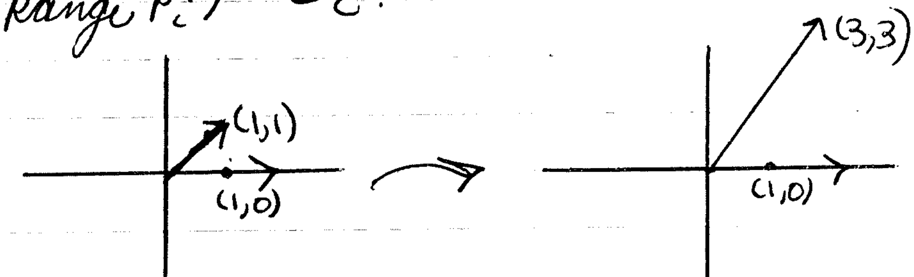
$$P_1 A = A P_1, \quad P_2 A = A P_2, \quad P_1 P_2 = 0.$$

So we can decompose this operator.

If $A : X \to X$ is compact, get whole sequence of projections $\{P_N\}$ that commute w/A & each other $\ni \sigma(A|_{\text{range } P_i}) = \sigma_i$.

Example

$\mathbb{C}^2 \xrightarrow{A} \mathbb{C}^2 \ni$

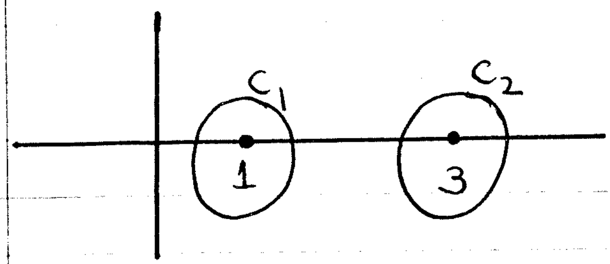


Then operator is $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = A.$

$$\sigma(A) = \{1, 3\}$$

$$\lambda - A = \begin{bmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 3 \end{bmatrix}$$

$$(\lambda - A)^{-1} = \frac{1}{(\lambda-1)(\lambda-3)} \begin{bmatrix} \lambda - 3 & 2 \\ 0 & \lambda - 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda-1} & \frac{-1}{\lambda-1} + \frac{1}{\lambda-3} \\ 0 & \frac{1}{\lambda-3} \end{bmatrix}$$



$$P_1 = \frac{1}{2\pi i} \oint_{C_1} \begin{bmatrix} \frac{1}{\lambda-1} & -\frac{1}{\lambda-1}+\frac{1}{\lambda-3} \\ 0 & \frac{1}{\lambda-3} \end{bmatrix} d\lambda = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$P_2 = \frac{1}{2\pi i} \oint_{C_2} \begin{bmatrix} \frac{1}{\lambda-1} & -\frac{1}{\lambda-1}+\frac{1}{\lambda-3} \\ 0 & \frac{1}{\lambda-3} \end{bmatrix} d\lambda = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Verify P_1, P_2 are projections:

$$P_1 P_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = P_1$$

$$P_2 P_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = P_2.$$

And P_i commutes w/ A:

$$P_1 A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = AP_1.$$

Similarly, $P_2 A = A P_2$.

Also $\quad \text{range}\, P_1 = \ker P_2 = \{(x,0) : x \in \mathbb{C}\}$
$\quad\quad\quad \text{range}\, P_2 = \ker P_1 = \{(x,x) : x \in \mathbb{C}\}$

Problem set #4.

#1. 2. Alternate proof

Every B-sp. has a Basic seq.

Let $\varepsilon_i > 0$ s.t. $\prod_{i=1}^{\infty} \frac{1}{1-\varepsilon_i} = 2$

Inductively choose $e_1, e_2, \ldots$ w/ $\|e_n\| = 1$

and e_1 arbitrary.

Supp. we choosen $e_1, \ldots, e_n$.

Let $X_0 = \text{Span}\{e_1, \ldots e_n\}$, $\varepsilon = \varepsilon_i$

Do 1 and 2, choose $e_{n+1} \in \bigcap_{j \le N} \ker f_j$.

And show $\|\sum_1^p \alpha_i e_i\| \le 2 \|\sum_1^{p+q} \alpha_i e_i\|$

by using $\frac{1}{1-\varepsilon_{p+1}} \cdot \frac{1}{1-\varepsilon_{p+2}} \cdot \ldots \cdot \frac{1}{1-\varepsilon_{p+q}} \le 2$.

#6. If $\{x_n\}$ monotone shrinking & bdd. complete, then we know

$$X^{**} = \{(\xi_i) ; \sup_N \|\sum_1^N \xi_i x_i\| < \varepsilon\}$$

By bdd complete, $(\xi_i) \in \mathfrak{X}$.

If $\{x_n\}$ is basis for reflexive sp, and supp. $\{x_n\}$ is not shrinking. Then

$F_1 \in \mathfrak{X}^*$ s.t. $F_1 \notin$ cl. lin. sp $\{f_n\}$

Define $\varphi \in \mathfrak{X}^{**}$, $\varphi(F_1) = 1$, $\varphi(f_n) = 0$.

Extend linear & continuously by H.B.

Clearly nothing in $\mathfrak{X}$ goes to φ.

So $\{x_n\}$ is shrinking.

Again, by same Thm, can show $\{x_n\}$ is bdd complete.

A : commutive Real B-alg. w/1.

$K = [0, 1]$. pt-wise operation.

$C(K)$: conti. fcns. on K w/ $\|f\| = \sup_{t \in K} |f(t)|$

$\|fg\| \le \|f\| \|g\|$

$C^1(K) = f \in C(K)$ w/ $f' \in C(K)$.

pt-wise operation.

$$\|f\| = \sup_{t\in K} \left(|f(t)|, |f'(t)| \right).$$

clearly. $C^1(K) \underset{set}{\subset} C(K)$ and subalgebra.

However, it is not a subspace.

(topology do not agree)

It can be shown that C^1 is not B-alg. w/ $\|\cdot\|$.

$$\|fg\| = \sup_{t\in K} \left(|f(t)\, g(t)|, |f'(t)\, g(t) + f(t)\, g'(t)| \right)$$

$$\leq 2\|f\|\|g\|$$

Introduce equiv. renorm on $C^1(K)$ to make it a B-alg.

$$|||f||| = \sup_{\|g\| \leq 1} \|fg\|$$

let $g \equiv 1$ $\therefore$ $\|f\| \leq |||f|||$.

Since $\|fg\| \le 2\|f\|\|g\|$, $|||f||| \le 2\|f\|$.

So $|||\cdot||| \underset{\text{equiv.}}{\simeq} \|\cdot\|$.

For each $f \in C'(K)$,

$$C'(K) \xrightarrow{T_f} C'(K) \quad \text{where } T_f(g) = fg$$

$\|T_f\| = |||f|||$

$\|T_f T_g\| = \|T_{fg}\| \le \|T_f\| \|T_g\|$

So $|||fg||| \le |||f||| \cdot |||g|||$

Definition $M \subseteq A$ is an ideal if

(1) $M \neq A$

(2) $m, n \in M \quad \alpha \in \mathbb{C}, f \in A$

$\Rightarrow m+n,\ \alpha m,\ fm \in M$.

a ideal M is maximal if $\forall$ ideals I with $M \subseteq I \subseteq A$, then $M = I$.

If $f \in M$ some ideal, then f^{-1} does not exist in A.

(∵) $\Rightarrow I = f \cdot f^{-1} \in M \Rightarrow \forall a \in A,\ a \cdot 1 = a \in M \Rightarrow M = A$

$f \in C$ or C'

Note that f^{-1} exists iff $f(t) \neq 0$ for any $t \in K$.

If $f(t) \neq 0$, $\forall t$, then $\frac{1}{f(t)} \in C$ or C'.

& $f(t) \cdot \frac{1}{f(t)} = 1$.

Supp. $f(t)\, g(t) = 1$.

then for $\forall t$, $f(t) \neq 0$.

---

for $t \in K$, Consider

$$M_{t_0} = \{ f \in C \text{ or } C' : f(t_0) = 0 \}$$

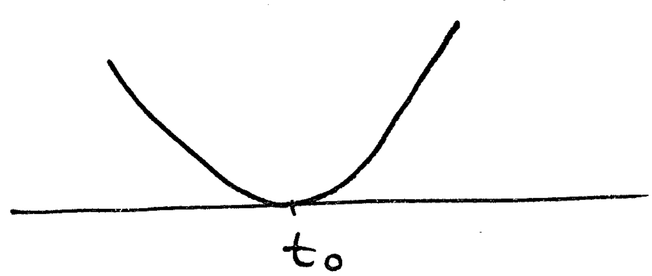
M_{t_0} is an ideal.

also M_{t_0} is a maximal ideal.

Supp. $\exists\, I \supsetneqq M_{t_0}$, then $g \in I$ w/ $g(t_0) \neq 0$.

let $f(x) = (x - t_0)^2 \in M_{t_0}$

$f, g \in I \leadsto f + g^2 \in I$.



$g^2(t) \geq 0$ and $g^2(t_0) > 0$.

So that $f + g^2 > 0$ $\cancel{*}$ $(\Rightarrow I = A)$

<u>Thm</u> Let M be any ideal in C or C'.

Then M is maximal iff $\exists\, t_0 \in K$ w/ $M = M_{t_0}$

Pf ($\Leftarrow$) done

($\Rightarrow$) Supp. M is a maximal ideal

For $f \in M$, define $Z(f) = \{t \in K : f(t) = 0\}$

Then $Z(f)$ is closed and non-empty.

Supp. $f_1, \cdots f_n \in M$.

Then $Z(f_1) \cap \cdots \cap Z(f_n) \neq \phi$

Consider $g(t) = f_1^2(t) + f_2^2(t) + \cdots + f_n^2(t)$

$g(t) = 0$ exactly when all $f_i(t) = 0$.

$Z(g) = \bigcap_{i=1}^{n} Z(f_i)$ and $g \in M$

So g is zero somewhere.

Since K is compact,

$\exists\, t_0 \in \bigcap_{f \in M} Z(f) \neq \phi$ (F.I.P)

So if $f \in M$, $\Rightarrow f(t_0) = 0$.

$\therefore$ $M \subseteq M_{t_0}$ but M is maximal

So $M = M_{t_0}$

A be commutive B-alg. w/1.

Define $\varphi : A \longrightarrow \mathbb{C}$. is a multiplicative linear functional

if (1) φ is linear

(2) $\varphi(ab) = \varphi(a)\,\varphi(b)$

(3) Note φ is not assume to be conti.

Example on $C(K)$ a $C'(K)$.

Let $t_0 \in K$. Define φ s.t $\varphi(f) = f(t_0)$

Lemma Supp. $\varphi \neq 0$ is multiplicative on A.

Then $M = \ker \varphi$ is a maximal ideal of A.

pf Supp. $m, n \in M$, $\alpha \in \mathbb{C}$, $a \in A$.

$\varphi(m+n) = \varphi(m) + \varphi(n) = 0$

$\varphi(\alpha m) = \alpha\,\varphi(m) = 0$

$\varphi(am) = \varphi(a)\,\varphi(m) = 0$

$\therefore$ So M is an ideal since $M \neq A$.

φ is not zero

$\therefore$ ($\varphi(1) = 1$, $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1)\,\varphi(1)$

$\therefore$ $\varphi(1) = 0$ or 1.

But if $\varphi(1) = 0$, then $\varphi(a.1) = \varphi(a)\,\varphi(1) = 0$)

Supp. $N \supsetneq M$ is an ideal in A.

$\exists\, n \in N$ w/ $\varphi(n) \neq 0$.

We may assume $\varphi(n) = 1$.

Consider $1 - n \in N$ since $\varphi(1-n) = \varphi(1) - \varphi(n)$
$= 0$.

So $1 - n \in M \subset N$.

Then $(1-n) + n \in M$.

$\therefore$ $1 \in N$ ///.

3-10-77

A commutative B-alg w/1.

Lemma.
Suppose $a \in A \ni a^{-1} \notin A$. Then $\exists$ max. ideal containing a.

Pf. $I = \{ax : x \in A\}$.

$I \neq A$ since $1 \notin I$.
$ax + ay = a(x+y)$
$\alpha(ax) = a(\alpha x)$
$(ax)y = a(xy)$.
So I is an ideal.

Claim: Every ideal I is contained in a maximal ideal.

Pf. By Zorn's Lemma: Let $\mathcal{C} = \{J : J \text{ proper ideal} \supset I\}$

Partially order $\mathcal{C}$ by inclusion. Suppose $\{J_\alpha\}$ is a chain of ideals. Then $J = \cup J_\alpha$ is also an ideal.
[$m, n \in J \Rightarrow \exists \alpha, \beta \ni m \in J_\alpha$, $n \in J_\beta$.

with either $J_\alpha \subseteq J_\beta$ or $J_\beta \subseteq J_\alpha$.
S $J_\alpha \subseteq J_\beta$. Then $m, n \in J_\beta \Rightarrow m+n \in J_\beta \subseteq J$, $\alpha m \in J_\beta \subseteq J$, $mf \in J_\beta \subseteq J$ $\forall f \in A$, so J is an ideal.]

Also $1 \notin J$ because $1 \notin J_\beta$. So $J \in \mathcal{C}$.

By Zorn's Lemma, $\exists$ maximal elt. M of $\mathcal{C}$.

Clearly $I \subseteq M$ & M is a maximal ideal.

Lemma. Suppose $M \subseteq A$ is a maximal ideal. Then A/M is a field and hence $\mathbb{C}$.

Pf. Suppose $a \in A \setminus M$. Then $\exists\, b \in A$ $\ni ab - 1 \in M$. For suppose not.

Let $I = \{ax + m \mid x \in A,\ m \in M\}$.
By assumption, $I \neq A$.

So $1 \notin I$ ow $1 = ab + m$
$m = ab - 1$

$M \subseteq I$, $a(0) + m = m$

$$(ax+m) + (ay+n) = a(x+y) + (m+n) \in I$$

$$\alpha(ax+m) = a(\alpha x) + (\alpha m) \in I$$

$$(ax+m)y = a(xy) + (my) \in I$$

$\therefore$ I is an ideal containing M.
By maximality, $I = M \Rightarrow a = a \cdot 1 + 0 \in M$ #

So A/M is a B-alg. with 1.

w/ every non-zero element invertible.
$\therefore$ it is $\mathbb{C}$.

$(a+M)(b+M) = ab + M$
$(1+M)(a+M) = (a+M)$

So $1+M$ is iden. for A/M.

If $a \notin M \Leftrightarrow a + M \neq M$

$\exists\, b$ with $ab - 1 \in M$

$(a+M)(b+M) - (1+M) = M$ or $(ab+M) = 1+M$.

__Lemma__. *Maximal ideals are closed.*

Pf. We have $\overline{M} \supseteq M$. $\overline{M}$ is an ideal:
$x, y \in \overline{M} \Rightarrow \exists\, (x_n), (y_n) \subseteq M$ with $x_n \to x$, $y_n \to y$.

Then $(x_n + y_n) \to x+y \Rightarrow x+y \in \overline{M}$
$\alpha(x_n) \to \alpha x \Rightarrow \alpha x \in \overline{M}$
$f \in A$, $x_n f \to xf \Rightarrow xf \in \overline{M}$.

$1 \notin \overline{M}$ since there is an open nbhd of 1 containing only invertible elements, U and $U \cap M = \emptyset$.

$\therefore \overline{M} = M$ by maximality.
Thus A/M is a B-space.

$$\|ab + M\| = \inf_{m \in M} \|ab + m\|$$

$$\|a + M\| \leq \|a + m\| < \|a + M\| + \epsilon$$
$$\|b + M\| \leq \|b + n\| < \|b + M\| + \epsilon$$

$$\|ab + M\| \leq \|ab + mb + na + mn\|$$
$$\leq \|a + m\| \cdot \|b + n\| \leq \|a + M\| \cdot \|b + M\|.$$

Consider $\phi : A \to \mathbb{C} \ni a \mapsto a + M, \simeq \mathbb{C}$.

α is linear.
$\alpha(ab) = ab + M = (a+M)(b+M) = \alpha(a)\alpha(b)$.

M max. ideal $\Leftrightarrow$ $\alpha: A \to \mathbb{C}$ is multiplicative $\neq 0$ linear functional.

Since M is always closed, α is always cont.

Prop.
Suppose $a \in A$. $\sigma(a)$ spectrum of a, then $\lambda \in \sigma(a) \Leftrightarrow \exists\, \alpha$ multiplicative linear functional on A with $\alpha(a) = \lambda$, $\alpha \neq 0$.

Pf. $\lambda \in \sigma(a) \Rightarrow (\lambda - a)^{-1} \notin A$. Hence $\exists$ M max ideal with $\lambda - a \in M$.
Look at $\alpha: A \to A/M = \mathbb{C}$.

$\alpha(1) = 1$, $0 = \alpha(\lambda - a) = \alpha(\lambda) - \alpha(a)$.

So $\alpha(\lambda) = \alpha(a) = \lambda(\alpha(1)) = \lambda$.

Suppose α is non-zero linear functional on A and $\alpha(a) = \lambda$.
$\alpha(\lambda - a) = 0 \Rightarrow \lambda - a \in \ker \alpha \leftarrow$ ideal

So $(\lambda - a)^{-1}$ does not exist.

Let $X = \{M : \text{max ideals of } A\} = \{\alpha \neq 0 \text{ mult lin. fcn}\}$,

$a \longmapsto F(X) =$ fcn on X where

$a \longmapsto f_a \ni f_a(M) = f(\alpha_M) = \alpha(a)$.
$a \longleftrightarrow f_a$ + spectrum of a is exactly the values of

the fcn f_a.

$C(K)$ $\quad C'(K)$ $\quad$ map ideals M_{t_0}

$f \in \Psi$

$f \longmapsto F_f \qquad F_f(M_{t_0}) = f(t_0)$

Prop.
If $\Phi = \{\varphi \neq 0$ mult lin on $A\}$, then
(1) $\|\varphi\| = 1$,
(2) Φ is closed in $\sigma(A^*, A)$, hence compact.
(3) For $a \in A$ the function $f_a : \Phi \ni$
$f_a(\varphi) = \varphi(a)$ is cont. in this topology
(4) $A \to C(\Phi)$ is cont. when $A, \|\cdot\|$
$a \to f_a$ & $C(\Phi)$ has sup norm.

Pf of (1). $\varphi(1) = 1$ so $\|\varphi\| \geq 1$.
S $\exists$ a with $\|a\| = 1$, $|\varphi(a)| > 1$.

We know φ is cont. so $\exists M \ni \|\varphi\| \leq M$.
Consider $\{a^n\}$. $\quad \|a^n\| \leq \|a\|^n = 1^n$,

$$|\varphi(a^n)| = |\varphi(a)|^n \to +\infty \text{ as } n \to +\infty \quad \#$$

hence bigger than M for some n.

So $\|\varphi\| = 1$.
Pf of (2).
S $\psi \in$ cl $\sigma(A^*, A)$ top Φ.
We want to show ψ is multiplicative.
[We know ψ is linear.]

$1 \in A$ so there is $\varphi \in \psi$ such that φ & ψ disagree at 1 by at most $\frac{1}{2}$.

$|\psi(1) - \varphi(1)| < \frac{1}{2}$

$|\varphi(1) - 1| < \frac{1}{2}$ so $\psi \neq 0$.

Let $a, b, ab \in A$. Then $\forall \epsilon > 0\ \exists \varphi \in \Phi \ni$ φ & ψ miss on a, b, ab by less than ϵ.

$$\sup_{\substack{x=a\\ b\\ ab}} |\varphi(x) - \psi(x)| < \epsilon.$$

$$|\psi(ab) - \psi(a)\psi(b)| \leq |\psi(ab) - \varphi(ab)| + |\overset{0}{\varphi(ab) - \varphi(a)\varphi(b)}|$$

$$+ |\varphi(a)\varphi(b) - \psi(a)\psi(b)|$$ which can be as smal as you like.

So $|\psi(ab) - \psi(a)\psi(b)| = 0$ and $\psi \in \Phi$.