[illegible] FINAL DO ALL PROBLEMS

1. Let $A : X \to X$ be a bdd lin operator on the B-space X. Show $B = \sum_{n=0}^{\infty} A^n/n!$ is a bdd op $: X \to X$ and that B^{-1} exists as a bdd operator $: X \to X$.

2. Let X & Y be B-spaces & let $B(X, Y)$ be the B-space of all bdd lin maps $X \to Y$. Consider $\varphi : X^* \times Y \to B(X, Y)$ defined by $\varphi(f, y) = T_{f,y}$ where $T_{f,y} : X \to Y$ given by $T_{f,y}(x) = f(x)y$. Show φ is bilinear and $\|T_{f,y}\| = \|f\| \, \|y\|$.

3. Suppose X is reflexive and $\{f_n\} \subseteq X^*$ has the property $\forall x \in X$ if $f_n(x) = 0$ for $n = 1, 2, \ldots,$ then $x = 0$. Show that the linear span of $\{f_n\}$ is norm dense in X^*.

4. Show that \#3 is false for $X = \ell_1$

5. Use Zorn's Lemma to show that for any normed space X and $\delta > 0$ there is a maximal set $\{x_\alpha\}_{\alpha \in \Gamma}$ with the properties
   (1) $\forall \alpha \in \Gamma \quad \|x_\alpha\| \le 1$
   (2) $\forall \alpha, \beta \in \Gamma \quad \|x_\alpha - x_\beta\| \ge \delta$.

6. Use \#5 ~~to~~ in showing that any normed space X with the property that $\forall \delta > 0$ any set $\{x_\alpha\}_{\alpha \in \Gamma}$ satisfying (1) & (2) above is countable then X is separable.

7. State the contrapositive of \#6. In the nonseparable space ℓ_∞ find such a set with $\delta = 2$ (or at least $\delta = 1$)

8. Suppose $\{x^n\}$, x^0 are elements of ℓ_1 and $\|x^n\| = \|x^0\| = 1$. Let $x^j = (x^j_1, x^j_2, \ldots)$.
If $x^n \to x^0$ in the weak-<u>star</u> topology $\sigma(\ell_1, c_0)$ then $\forall \varepsilon$ $\exists M, N$ such that

$$\sum_{j=1}^{N} |x^0_j| > 1-\varepsilon \quad \text{and} \quad \sum_{j=1}^{N} |x^k_j| > 1-\varepsilon \quad \text{for } k \geq M.$$

And thus $x^n \to x^0$ in the ℓ_1 norm.

9. ~~Use~~ Reduce the following thm to the special case in #8
in ℓ_1 if $x^n \to x^0$ in $\sigma(\ell_1, c_0)$ and $\|x^n\| \to \|x^0\|$ then $x^n \to x^0$ in the ℓ_1 norm.

10. Let X be a B-space w/ X^* separable. Let $\{y_n\}$ be a basic sequence in X and let $Y = \text{cl lin span}\{y_n\}$ then Y^* is separable [this is easy] ~~B~~ Let $\{\varphi_n\}$ be a countable set with dense lin span in Y^* with $\|\varphi_n\| = 1$ for all n.
Let $\varepsilon > 0$ then there is $\{b_m\}$ a block basic sequence of $\{y_n\}$ and $\{z_m\}$ in Y with $\|b_m\| = 1$ & $\|z_m\| < \varepsilon/2^m$ such that $\varphi_n(b_m + z_m) = 0$ for $m \geq n$.
Show $\{b_m\}$ is a shrinking basic sequence.