MAT. 534. (Notes taken by C. C. Chen)

Instructor: Dr. Bellnot Tel. 644 - 4419

office hours: M W F 12:15 ~ 1.25 & 2.15 - 3.35 (Tues. usually as well)

Thur. 12:15 ~ 3.15 by appointment

Text: Schechter: principle of functional Analysis

Home work: 5 problems every two weeks, Midterm, Final

__Defn.__ A partially ordered set X is a non-empty set with relation $\leq$ such that:

(1) $a \leq a$ (Reflexive)

(2) $a \leq b$ & $b \leq a \Rightarrow a = b$ (Antisymmetric)

(3) $a \leq b$ & $b \leq c \Rightarrow a \leq c$ (transitive)

E.g. Let X be any set, $P(X)$ be the ~~ele~~ collection of subsets of X, "$\subseteq$" be the usual set inclusion, Then $(X, \leq)$ be a partially ordered set.

__Defn.__ If X is partially ordered set. A subset $\mathcal{C} \subset X$ is a chain iff $\mathcal{C}$ is totally ordered i.e. $\forall a, b \in \mathcal{C}$, Then $a \leq b$ or $b \leq a$

__Defn.__: An upper bound of $\mathcal{C} \subseteq X$ is any element $b \in X \ni$:

$c \in \mathcal{C} \Rightarrow c \leq b$.

Defn: A maximal element of X is any element $m \in X \ni$:

$x \in X$, & $m \leq x \Rightarrow m = x$.

Zorn's Lemma: If X is a partially ordered set and every chain contained

X has an upper bound in X, Then X has a maximal element

Note: Zorn's Lemma equivalent to Axiom of choice and Well ordering principle

Before we give application of Zorn's Lemma. Let us review some definition in linear Algebra:

Let V be a vector space over $\mathbb{R}$.

Defn: $A \subset V$ is linearly Independent iff for each $a_1, a_2, \cdots a_n$ finite distinct collection of elements from A. and any scalars $\alpha_1, \alpha_2, \cdots \alpha_n$. Then

$$\sum_{i=1}^{n} \alpha_i a_i = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Defn: Let $A \subset V$, Span A = set of all linear Combination of A.

i.e. $\text{span } A = \{ a \mid a = \sum_{i=1}^{n} \alpha_i a_i, \; a_i \in A, \; \alpha_i \in \mathbb{R} \}$

Defn: $A = \{a_\lambda\}_{\lambda \in \Lambda} \subset V$ is a Hamel basis for V if it is an independent set and span $A = V$.

Defn: Let $A \setminus G = \{ x \in F : x \notin G \}$

Lemma A: If $A \subset V$ is linearly independent and $w \in V \setminus \text{Span } A$. Then

$B = A \cup \{w\}$ is independent.

Pf: Let $a_1, a_2, \cdots a_n$ be distinct element of B & suppose

$$\sum_{i=1}^{n} \alpha_i a_i = 0$$

Case 1. no a_i is w, Then $\sum_{i=1}^{n} \alpha_i a_i = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ (by independent of A)

Case 2. all others. i.e. $\exists i, \ni: a_i = w$ say $w = a_n$

i.e. $\sum_{i=1}^{n-1} \alpha_i a_i + \alpha_n w = 0$ ---- (B).

subcase (i). $\alpha_n = 0$, Then (B) $\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$ (again, follows from Independence of A)

sub case (ii) $\alpha_n \neq 0$. Then (B) $\Rightarrow$ $w = \sum_{i=1}^{n-1} \left(\alpha_i / -\alpha_n\right) a_i \Rightarrow w \in \text{Span } A$ $\times$.

Application of Zorn's Lemma:

<u>Thm</u>. Every vector space has a Hamel basis.

Pf: Let $B \subset V$, B is linearly independent.

Claim: B can be extended to a Hamel Basis.

Let $\mathcal{X} = \{A \subset V \mid A \text{ is linearly independent and } A \supseteq B\}$

$X \neq \phi$, since $B \in X$

partially ordered X by $A_1 \leq A_2 \iff A \subseteq A_2$.

Let $\mathcal{C}$ be a chain of elements of X.

Want to show: $\exists\ V \in X$ with $A \leq V \quad \forall A \in \mathcal{C}$

Let $V = \bigcup_{A \in \mathcal{C}} A = \{a \mid a \in A, \text{ for some } A \in \mathcal{C}\}$

~~Thus,~~ $A \subseteq V \quad \forall A \in \mathcal{C}$, Thus, $V \supseteq B$

In order to show that $V \in X$, it is enough to show that V is independent.

Claim: V is linearly independent.

Let $a_1, \cdots, a_n$ be distinct elements of V.

for each a_i, $\exists\ A_i \in \mathcal{C}$ with $a_i \in A_i$

$A_1, A_2, \cdots, A_n \in \mathcal{C}$ so there is a largest $A_i = A^*$.

We have $a_1, a_2, \cdots a_n \in A^* \in X$.

$\sum_1^n \alpha_i a_i = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Thus, V is independent, $V \supseteq B \Rightarrow V \in X$.

by Zorn's Lemma $\exists$ a maximal element $M \in X$.

Claim: M is Hamel basis for V.

suppose not, $\Rightarrow \text{Span } M \neq V$, Let $w \in V \setminus \text{Span } M$. and Let $M^* = M \cup \{w\}$, Then from Lemma A, it follows M^* is a independent set. and $M^* \supseteq B$. this contradict the maximality of M. Q.E.D.

9-22-76 MAT 534 Notes in S.C. Chen

Vector Space: Axioms (1) – (9) & (15) on pp 8-9 of Text.

Vector addition

Scalar multiplication

<u>Subvector space</u>:

$W \subset V$, V vector space, is a subvector space if it is a vector space in its own right with the same vector addition & scalar multiplication.

<u>Theorem</u>. $W \subset V$ is a subvector space iff

(1) $f, g \in W \Rightarrow f + g \in W$

(2) α scalar, $f \in W \Rightarrow \alpha f \in W$

<u>Examples of vector spaces</u>

1. $\mathbb{R}$, $\mathbb{R}^n$ $n = 1, 2, 3, \cdots$

2. "function spaces"

a. Let X be a set.

$F(X) = \{ f : X \to \mathbb{R}$ all real valued functions on $X \}$

/

$f, g \in F(X) \quad \alpha \in R$

$(f+g)(x) = f(x) + g(x)$

$(\alpha f)(x) = \alpha(f(x))$

We have proved $F(X)$ is a vector space.

3. Axiom (3) $\Rightarrow \{0\}$ is a vector space,

but ϕ is not a vector space.

Remark. Every vector space is a subvector space of some $F(X)$.

Def. A norm is a function $\|\cdot\|$ on vector space V s.t. $\|\cdot\| : V \to R^+ = \{r \geq 0\}$, and

(1) $v \in V, v \neq 0 \Rightarrow \|v\| > 0$

(2) $v \in V, \alpha \in R \Rightarrow \|\alpha v\| = |\alpha| \, \|v\|$

(3) $v, w \in V, \|v + w\| \leq \|v\| + \|w\|$.

Remarks: A. (2) $\Rightarrow \|0\| = 0$

indeed $\|0\| = \|0 \cdot 0\| = |0| \, \|0\| = 0$
scalar ↑ ↑ vector

so that (1) can be restated as

$\|v\| \geq 0$, and $\|v\| = 0$ iff $v = 0$.

B. (2) is called "homogeneality"

(3) is called "triangle inequality".

<u>Def.</u> A <u>normed space</u> is a vector space X with a norm $\|\cdot\|$ and we give X the topology defined by the metric $d(f, g) = \|f - g\|$.

<u>Def.</u> A subspace of a normed space X is a subvector space with the restricted norm. "inherits the topology as well as the vector addition & scalar multiplication."

<u>Def.</u> $d(\cdot, \cdot)$ is a metric if

(1) $d(f, g) \geq 0$, $d(f, g) = 0$ iff $f = g$

$(\|f - g\| \geq 0$, $\|f - g\| = 0$ iff $f - g = 0)$

(2) $d(f, g) = d(g - f)$

$(\|f-g\| = |-1|\,\|g-f\| = \|g-f\|)$

(3) $d(f,g) \le d(f,h) + d(h,g)$

$(\|f-g\| \le \|f-h\| + \|h-g\|).$

<u>Def.</u> A <u>Banach space</u> is a normed space which is complete in the metric $d(\cdot,\cdot)$. (i.e. if $\langle f_n \rangle \subset \mathcal{B}$ and $\lim_{n\to\infty} \|f_n - f_m\| = 0$, then $\exists\, f_\infty \in \mathcal{B}$ s.t. $\lim_{n\to\infty} \|f_n - f_\infty\| = 0$.

<u>Examples of Banach spaces</u>

(1). Let X be a compact topological space and

$C(X) = \{ f \in F(X) : f \text{ is continuous} \}$,

define $\|f\| = \sup_{x\in X} |f(x)|$.

(i) Sum of two continuous functions is continuous. scalar multiple of continuous function is continuous.

$\therefore$ $C(X)$ is a vector space.

(ii) $\|f\| = \sup_{x \in X} |f(x)| \geq 0$

If $f \neq 0$, $\exists\, x \in X$ with $f(x) \neq 0 \Rightarrow \|f\| > 0$

$$\|\alpha f\| = \sup_{x \in X} |\alpha f(x)| = \sup_{x \in X} |\alpha|\,|f(x)| = |\alpha| \sup_{x \in X} |f(x)|$$
$$= |\alpha|\,\|f\|$$

$$\|f+g\| = \sup_{x \in X} |f(x) + g(x)|$$
$$\leq \sup_{x \in X} \left(|f(x)| + |g(x)|\right)$$
$$\leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)| = \|f\| + \|g\|.$$

We have not used compactness as yet nor have we use continuity. We need compactness to give a maximal real value to $|f(x)|$, i.e. so that $\|f\|$ is a real number.

$B(X)$ — the bounded functions on X, i.e. f for which $\sup_{x \in X} |f(x)| < +\infty$.

$B(X)$ also forms a normed space.

$\mathcal{B}(X) \supseteq C(X)$

Sketch to show $C(X)$ & $\mathcal{B}(X)$ are Banach spaces.

(a) start with $\langle f_n \rangle$ (Cauchy sequence).

(b) for each $x \in X$, $\langle f_n(x) \rangle \subseteq \mathbb{R}$ is a Cauchy sequence of real numbers.

note that $0 \le |f_n(x) - f_m(x)| \le \|f_n - f_m\| \longrightarrow 0$

(c) Define the function $f_\infty : X \to \mathbb{R}$ by

$$f_\infty(x) = \lim_{n\to\infty} f_n(x)$$

(d) we want to show $f_\infty \in C(X)$ or $\mathcal{B}(X)$

(e) finally we want to show

$$\|f_n - f_\infty\| \longrightarrow 0$$

In this case we note that (e) $\Rightarrow$ (d)

Uniform limit of continuous functions is continuous

Uniform limit of bounded functions is bounded.

$\langle f_n \rangle$ all bounded, $\|f_n - f_\infty\| \to 0$

want to show f_∞ is bounded

pick f_n s.t. $\|f_n - f_\infty\| < 1$

$$|f_\infty(x)| \le \|f_\infty\| \le \|f_n\| + \|f_\infty - f_n\| < +\infty$$

$C = C([0,1])$

$c = C(X)$ where X is the subset $\{\frac{1}{n} : 1, 2, \cdots\} \overset{\cup\{0\}}{\wedge} \subseteq \mathbb{R}$

$c = \{ \langle x_n \rangle \mid \lim_{n\to\infty} x_n \text{ exists} \}$

$l_\infty = \mathcal{B}(\mathbb{N}) = \{ \langle x_n \rangle \subseteq \mathbb{R} \mid \sup_n |x_n| < +\infty \}$

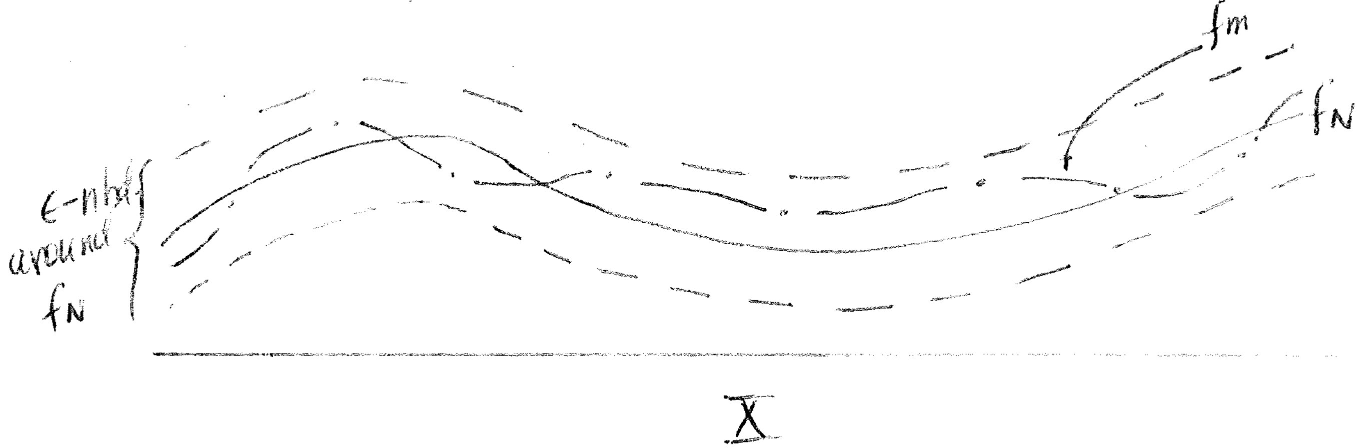
$c_0 = \{ \langle x_n \rangle \in c \mid \lim_{n\to\infty} x_n = 0 \}$

MAT 534
Bellenot

Notes taken by Ed Cooley 9/24/76

We want to show (p.10) that $\|f_n - f_\omega\| \to 0$.

Proof: let $\epsilon > 0$. We will show $\exists\, N \ni m \geq N \Rightarrow$
$|f_m(x) - f_\omega(x)| < \epsilon$ for $x \in X$ (see picture).



Since $\langle f_n \rangle$ is a Cauchy sequence $\exists\, N \ni$
$$\|f_n - f_m\| < \frac{\epsilon}{2} \quad \text{for } m, n \geq N.$$

Thus for $x \in X$, $n \geq N$ we have
$$|f_n(x) - f_N(x)| < \frac{\epsilon}{2}$$
$$|f_\omega(x) - f_N(x)| \leq \frac{\epsilon}{2}, \text{ because of the}$$

Lemma: for real numbers, if $a_n \to a$ and $|a_n - b| < \epsilon$, then $|a-b| \le \epsilon$ for each n.

Proof: $a_n - b \to a - b$. Since $|\cdot|$ is a continuous function we have

$$|a_n - b| \to |a-b|. \qquad \text{QED (lemma).}$$

Now let $m \geq N$; we have

$$|f_\infty(x) - f_m(x)| \leq |f_\infty(x) - f_N(x)| + |f_N(x) - f_m(x)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \text{QED}$$

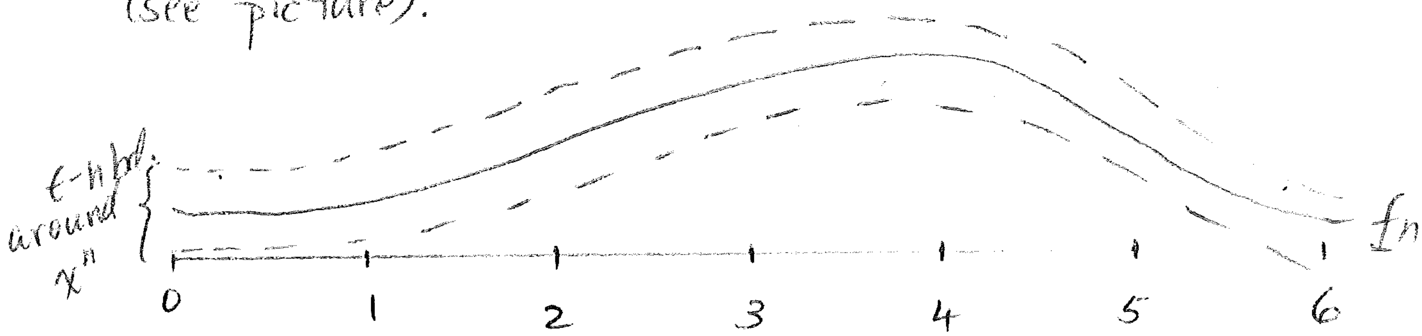
Proposition: a closed subspace of a Banach space is a Banach space.

Proof: let $X \subseteq Y$ as in hypothesis; we need to show X is complete. Let $\langle x_n \rangle$ be a Cauchy sequence in X; i.e., $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. But $\langle x_n \rangle$ is also a Cauchy sequence in the B-space Y, and thus converges to some $y \in Y$; i.e., $\|x_n - y\| \to 0$ as $n \to \infty$. Now $y \in X$ because X is closed (closed sets contain all limit points).

QED

<u>Claim</u>: c_0 is a closed subspace of c, and c is a closed subspace of l_∞.

<u>Proof</u>: suppose $\langle x^n \rangle \subseteq c_0$, $x^n \to y \in c$, $y = \langle y_m \rangle$. It is not hard to show that $y_m \to 0$ as $m \to \infty$ (see picture).



<u>Remark</u>: every B-space is a closed subspace of some $C(X)$, where X is a compact Hausdorff space.

<u>Proof</u>: much later.

Let X be an open subset of $\mathbb{R}$. Define

$$C^k(X) = \{ f \in C(X) \text{ with } k \text{ continuous derivatives} \}$$

and
$$\|f\| = \sup_{x \in X} |f(x)| + \sup_{x \in X} |f'(x)| + \cdots + \sup_{x \in X} |f^{(k)}(x)|.$$

The proof that $C^k(X)$ is a B-space is easy, by repeating much of what we have done for $C(X)$;

We need to recall the following from advanced calculus

<u>THEOREM</u>: if f_n converges uniformly to f and f_n' converges uniformly to g, then $f' = g$.

Consider $\mathbb{R}^n = C(\{1, 2, \dots, n\})$ (discrete topology)

$$\|(x_1, \dots, x_n)\| = \sup_{1 \le i \le n} |x_i|.$$

<u>THEOREM</u>. $(x_1^m, \dots, x_n^m) \to (y_1, \dots, y_n)$ in $\mathbb{R}^n$ with this norm $\iff x_i^m \to y_i$ for $1 \le i \le n$.

In $X \times Y$ (both metric spaces)

$$(x^n, y^n) \to (x_0, y_0) \iff x^n \to x_0,\ y^n \to y_0.$$

<u>THEOREM</u>: if X is a normed space, then

$$+ : X \times X \to X \qquad (x, y) \xrightarrow{+} x + y$$

and

$$\cdot : \mathbb{R} \times X \to X \qquad (\alpha, x) \xrightarrow{\cdot} \alpha x$$

are continuous.

Proof: we need to show that

$$x_n \to x, \quad y_n \to y, \quad \alpha_n \to \alpha$$

$$\Rightarrow x_n + y_n \to x + y, \quad \alpha_n x_n \to \alpha x.$$

But $\|x_n + y_n - (x+y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0.$ ✓

And $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x\|$

$$\leq |\alpha_n - \alpha| \|x_n\| + |\alpha| \|x_n - x\|.$$

Lemma: if $\langle x_n \rangle$ is a convergent sequence in a normed space, then $\exists\, M < \infty \ni \|x_n\| \leq M \quad \forall n.$

Proof: $\langle \|x_n\| \rangle$ converges, and a convergent sequence of real numbers is bounded. QED (lemma)

Hence we have

$$|\alpha_n - \alpha| \|x_n\| + |\alpha| \|x_n - x\| \leq |\alpha_n - \alpha| M + |\alpha| \|x_n - x\| \to 0.$$

Aside: $\big| \|x\| - \|y\| \big| \leq \|x - y\|.$ $(*)$

Proof: $\|x\| \leq \|x-y\| + \|y\|$ by triangle ineq., since $x = (x-y) + y$.

$\therefore \|y\| \leq \|x-y\| + \|x\|$, since $y = -1(x-y) + x$

QED

<u>Note</u>: we use this in proof of lemma above, (*) shows that $\|\cdot\| : \underline{X} \to \mathbb{R}^+$ is continuous.

<u>DEF^n</u>: a <u>topological vector space</u> $(\underline{V}, \xi)$ is a vector space $\underline{V}$ with topology ξ $\ni$ the maps

vector add^n: $\underline{V} \times \underline{V} \to \underline{V}$ $\qquad (v, w) \to v + w$

and scalar mult^n: $\mathbb{R} \times \underline{V} \to \underline{V}$ $\qquad (\alpha, v) \to \alpha v$

are continuous.

<u>Corollary</u>: normed spaces are TVS's.

<u>DEF^n</u>: real inner product is a real-valued function $\langle \cdot, \cdot \rangle$ on $\underline{V} \times \underline{V}$, where $\underline{V}$ is a vector space, $\ni$

(1) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ (linear in its 1st entry)

(2) $\langle f, g \rangle = \langle g, f \rangle$ (symmetry)

(3) $f \neq 0 \Rightarrow \langle f, f \rangle > 0$ (positive definite).

<u>Examples</u> – next time.

PROBLEM SET 1 Due Monday 11 Oct.

1. Let K be a subset of the normed space X
   K is <u>convex</u> if $a, b \in K$ $0 \le t \le 1 \Rightarrow ta + (1-t)b \in K$
   K is <u>balanced</u> if $a \in K$ $|t| \le 1 \Rightarrow ta \in K$
   K is <u>absorbing</u> if for each $x \in X$ there is $\varepsilon > 0$
   s.t. $|t| < \varepsilon \Rightarrow tx \in K$.
   (the set $U = \{x \in X : \|x\| \le 1\}$ has each of these properties

   Show the closure of a convex set is convex, the closure of a balanced set is balanced, the closure of an absorbing set is absorbing, and the closure of a subspace is a subspace.

2. Problem 2 page 26 in Text (HINT to show $\|A\| \ge \|B\|$ write $A = B + C$ and expand $\langle B+C, B+C \rangle$)

3. Problem 3 page 26 in Text.

4. Show c_0 is a closed subspace of c and c is a closed subspace of ℓ_∞.

5. Let K be an absorbing balanced convex set in a vector space V. Let, for $v \in V$.
   $$\|v\| = \inf\{\lambda > 0 : v \in \lambda K\}$$ (where $\lambda K = \{\lambda x : x \in K\}$)
   Show $\|\cdot\|$ has all the properties of a norm except perhaps $\|v\| = 0 \Rightarrow v = 0$.

EX A: [illegible] $\langle (x_1, \dots, x_n), [illegible] \rangle$ [illegible]
THE [illegible] INNER PRODUCT ON [illegible]

EX B: [illegible] $\langle f, g \rangle$ [illegible]

EX C: LET Γ = { $f \in F(\Gamma)$: $f(x) \neq 0$ FOR [illegible]
OF TIMES }

$$\langle f, g \rangle = \sum_{x \in \Gamma} f(x) g(x). \quad \Gamma \text{ MAY BE } \text{[illegible]}$$

DEF: IF $\langle \cdot, \cdot \rangle$ IS AN INNER PRODUCT ON A VECTOR SPACE [illegible]

$\|f\| = \langle f, f \rangle^{1/2}$. THIS IS CALLED THE HILBERT [illegible]

THEOREM: (CAUCHY SCHWARZ) C.B.S. INEQUALITY.
$f, g \in V$, V IS AN INNER PRODUCT SPACE THEN

$|\langle f, g \rangle| \leq \|f\| \|g\|$.

PF: [illegible] $F(t) = \langle f + tg, f + tg \rangle$.
$F : \mathbb{R} \to \mathbb{R}^+$.
$F(t) = \langle f, f \rangle + \langle tg, f \rangle + \langle f, tg \rangle + \langle tg, tg \rangle$
$= \langle f, f \rangle + 2t \langle f, g \rangle + t^2 \langle g, g \rangle$

WE CAN WRITE $F(t) = At^2 + Bt + C$ WHERE A, B, AND C ARE
REAL NOS. (NOTE $A \geq 0$.)

CASE 1: SUPPOSE $A = 0$:
$A = \langle g, g \rangle \Rightarrow g = 0$.

[illegible] THE INEQUALITY BECOMES $0 \leq$ [illegible] WHICH IS TRUE [illegible]

[illegible] $\langle \cdot, \cdot \rangle$ [illegible]

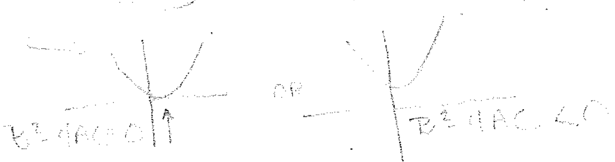
[illegible] $= \langle 0 \cdot 0, f \rangle = 0 \langle 0, f \rangle = 0$

SCALAR

VECTOR

[illegible]

[illegible] QUADRATIC [illegible] HAS [illegible] AT MOST ONE ROOT



IN EITHER CASE [illegible], WE HAVE $B^2-4AC\le 0$

$\Rightarrow B^2 \le 4AC$. IF $B = 2\langle f,g\rangle$,

[illegible] $4\langle f,g\rangle^2 \le 4\|g\|^2\|f\|^2$

[illegible] 4 AND TAKING SQ. ROOT,

$\rightarrow |\langle f,g\rangle| \le \|f\|\,\|g\|$ Q.E.D.

THEOREM: $\|f\| = \langle f,f\rangle^{1/2}$ IS A NORM

PF: ① $\|f\|$ IS REAL SINCE $\langle f,f\rangle\ge 0$. IF $f\ne 0$ THEN [illegible] AND $\|f\|>0$.

② $\|\alpha f\| = \langle \alpha f,\alpha f\rangle^{1/2}$

$= (\alpha^2\langle f,f\rangle)^{1/2}$

$= |\alpha|\,\langle f,f\rangle^{1/2}$

$= |\alpha|\,\|f\|$.

③ $\|f+g\|^2 = \langle f+g, f+g\rangle$

$= \langle f,f\rangle + 2\langle f,g\rangle + \langle g,g\rangle$: BY LINEARITY IN BOTH COORDINATES

$\le \|f\|^2 + 2\|f\|\|g\| + \|g\|^2$: BY [illegible]

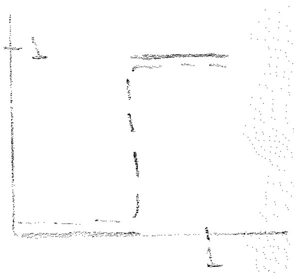
$= (\|f\| + \|g\|)^2$

$\Rightarrow \|f+g\| \le \|f\| + \|g\|$: TAKING SQ. ROOT [illegible] THE NORM $\|f\|$ [illegible]

[illegible] EXAMPLES:

EX A: $\mathbb{R}^n$ [illegible] HILBERT SPACE. IN [illegible] OF [illegible] IN $\mathbb{R}^n$, AND $(x_1^n, \dots$ [illegible] IN THE HILBERT NORM IFF $x_i^n \to$ [illegible]

EX B: THIS IS NOT A HILBERT SPACE.



$$f(x) = \begin{cases} 0 & x \le 1/2 \\ 1 & x > 1/2 \end{cases}$$

$f(x)$ IS THE LIMIT IN THE [illegible] CONTINUOUS FUNCTIONS.

CONTINUOUS FUNCTIONS AREN'T COMPLETE [illegible] THE COMPLETION OF THIS SPACE IS L_2 OR [illegible] (SPACE OF LEBESGUE INTEGRABLE FUNCTIONS)

EX C: DEFINE $\ell_2(\Gamma) = \{ f(\gamma) : \sum_{\gamma \in \Gamma} |f(\gamma)|^2 < +\infty \}$

$$= \sup_{F \in \Gamma} \sum_{\gamma \in F} |f(\gamma)|^2 \text{ WHERE } F \text{ IS FINITE SET.}$$

NOTE: $\ell_2 = \ell_2(\mathbb{N})$ IF Γ IS COUNTABLE.

THIS HAS INNER PRODUCT,

$\langle f, g \rangle = \sum_{\gamma \in \Gamma} f(\gamma) g(\gamma)$. THIS IS REAL BY [illegible] QUALITY

THESE ARE HILBERT SPACES. THE EXAMPLES (EX C) BEFORE [illegible] NOT [illegible] IF Γ IS INFINITE.

EX D: $\ell_2 = \ell_2(\mathbb{N}) = \{ (x_n)$ OF REAL SEQUENCES $: \sum_{n=1}^{\infty} |x_n|^2 < +\infty \}$

$$\langle x_n, y_n \rangle = \sum_{n=1}^{\infty} x_n y_n$$

[illegible] SEQUENCE IN INFINITE [illegible]

[illegible]: $\xi = \{x_n\}$, $x_n = 1/n$. $\xi \in \ell_2(\mathbb{N})$ BUT NOT IN [illegible] 1st

DEF: THE $\measuredangle$ BETWEEN f & g IS ARCCOS $\dfrac{\langle f, g \rangle}{\|f\|\,\|g\|}$

REMARK: THIS YIELDS THE USUAL ANGLES IN $\mathbb{R}^n$.

EXAMPLE: $\mathbb{R}^2$ WITH UNIFORM NORM, $(0,1)$ HAS NORM 1.

$(0,t)$ HAS NORM $|t|$. (SUPREMUM OF ABS. VALUE)

SO IF $|t| \leq 1$, $(1,t)$ HAS NORM 1.

DEF: $f \perp g$ OR "f IS ORTHOGONAL TO g" IF $\langle f, g \rangle = 0$.

DEF: IF N & M ARE SUBSETS OF AN INNER PRODUCT SPACE, WE SAY $N \perp M$ IF FOR EACH $f \in N$ & $g \in M$, $\langle f, g \rangle = 0$.

11 copies

To say Γ is an index set says nothing other than Γ is a set. No order or topology is implied.

$\{X_\gamma\}_{\gamma\in\Gamma}$ – this is just a function "f"
$f(\gamma) = X_\gamma$

e.g. $\{X_n\}_{n\in\mathbb{N}}$ is just a function $n \mapsto X_n$

Sums of scalars –

Suppose Γ is an index set. We want to know: what do we mean by $\sum_{\gamma\in\Gamma} X_\gamma$ where X_γ is a scalar, $\gamma\in\Gamma$

Remarks

① If all but finitely many X_γ are zero, there is exactly one scalar that could be $\sum_{\gamma\in\Gamma} X_\gamma$

② If $\Gamma = \mathbb{N}$ – Now the index set has some order

usually define $\sum_{\gamma\in\Gamma} X_\gamma = \sum_{n=1}^{\infty} X_n$ if it converges

Problems: (a) may not exist (includes $\pm\infty$)

(b) if you re-order the integers, you may get different answers. In fact, $\{(-1)^n/n\} = \{X_n\}$, $\forall r \in \mathbb{R}$ there is a permutation of $\mathbb{N}$ $\ni$ $\sum_{i=1}^{\infty} X_{\pi(i)} = r$.

A series is unconditionally convergent if you get the same limit any way you order the integers.

A series $\{X_n\}$ is absolutely convergent if $\sum_{n=1}^{\infty} |X_n|$ exists - absolute convergence of scalars is equivalent to unconditional convergence.

$\{X_n\}$ is absolutely convergent iff $\sup_{\substack{F \subset N \\ F \text{ finite}}} \sum_{n \in F} |X_n| < +\infty$.

LEMMA: Suppose $X_\gamma \geq 0$, $\sup_{\substack{F \subset \Gamma \\ F \text{ finite}}} X_\gamma < +\infty$. Then for each $\varepsilon > 0$, there exist a finite number of X_γ which satisfy $X_\gamma \geq \varepsilon$

Proof: If not you can add ε's indefinitely, then $\sup_{\substack{F \subseteq \Gamma \\ F \text{ finite}}} X_\gamma = \infty$. $\square$

Cor: Only countably many X_γ's are not equal zero

Proof:

$A_n = \{\gamma : X_\gamma \geq \frac{1}{n}\}$. $\{\gamma : X_\gamma \neq 0\} = \cup A_n$

Hence only countably many X_γ's are $\neq 0$

Thm: If Γ is unordered, $\sum_{\gamma \in \Gamma} X_\gamma$ has meaning exactly when $\sup_{\substack{F \subseteq \Gamma \\ F \text{ finite}}} \sum_{\gamma \in F} |X_\gamma| < +\infty$.

Let X be an inner product space.
$A \subseteq X$ is orthogonal if

(i) $0 \notin A$

(ii) $a, b \in A \underset{a \neq b}{} \Rightarrow \langle a, b \rangle = 0$

$A \subseteq X$ is orthonormal if A is orthogonal and $a \in A \Rightarrow \langle a, a \rangle = 1$.

For example, in $\mathbb{R}^2$, $\{(2,0), (0, \frac{1}{2})\}$ is an orthogonal set but not an orthonormal set.

Remark: we can "normalize" any orthogonal set via A - orthogonal

$B = \{\frac{a}{\|a\|} : a \in A\}$ orthonormal

(1) not dividing by zero. $a \in A \Rightarrow a \neq 0 \Rightarrow \|a\| \neq 0$

(2) $\langle \frac{a}{\|a\|}, \frac{b}{\|b\|} \rangle = \frac{1}{\|a\|\|b\|} \langle a, b \rangle = 0$

(3) $\|\frac{a}{\|a\|}\| = \frac{1}{\|a\|} \|a\| = 1$

THEOREM: If $x \perp y$, then $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

Proof: $\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle =$
$\|x\|^2 + 2\langle x, y \rangle + \|y\|^2$.
Since $x \perp y$, $\langle x, y \rangle = 0$. Hence
$\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Cor: If $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is orthonormal, then for scalars $\alpha_1, \dots, \alpha_n$

$$\|\alpha_1 \varphi_1 + \dots + \alpha_n \varphi_n\|^2 = \alpha_1^2 + \dots + \alpha_n^2$$

Proof: by induction
$\langle \alpha_1\varphi_1, \alpha_2\varphi_2 + \ldots + \alpha_n\varphi_n \rangle =$
$\langle \alpha_1\varphi_1, \alpha_2\varphi_2 \rangle + \ldots + \langle \alpha_1\varphi_1, \alpha_n\varphi_n \rangle =$
$0 + \ldots + 0 = 0.$

By the theorem, $\|\sum \alpha_i\varphi_i\|^2 = \|\alpha_1\varphi_1\|^2 +$
$\|\alpha_2\varphi_2 + \ldots + \alpha_n\varphi_n\|^2.$
$\|\alpha_1\varphi_1\|^2 = \alpha_1^2.$
Result follows by induction.

Let $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal set.
Suppose $f \in X$.
Claim: (1) $\sup_{F \subseteq \Gamma} \sum_{\gamma \in F} \langle f, \varphi_\gamma \rangle^2 < +\infty$
(2) If X is a Hilbert space then
$\sum_{\gamma\in\Gamma} \langle f, \varphi_\gamma \rangle \varphi_\gamma \in X$.
(3) $f - \sum_{\gamma\in\Gamma} \langle f, \varphi_\gamma \rangle \varphi_\gamma$ is orthogonal to each φ_γ, $\gamma \in \Gamma$.

Proof:
(1) Consider scalars $\alpha_1, \ldots, \alpha_n$ and $\varphi_1, \ldots, \varphi_n$
$\in \{\varphi_\gamma\}_{\gamma\in\Gamma}$
$$\left\|\sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i\right\| = \left[\sum_{i=1}^{n} \langle f, \varphi_i \rangle^2\right]^{1/2}$$

By the triangle inequality,
$$\left[\sum_{i=1}^{n} \langle f, \varphi_i \rangle^2\right]^{1/2} = \left\|\sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i\right\| \leq$$
$$\left\|f - \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i\right\| + \|f\| \leq 2\|f\|$$
by problem #2 in HW.
$2\|f\| < +\infty.$

(2) Note that $\langle f, \varphi_\delta \rangle \neq 0$ at most countably many times. Label them $\varphi_1, \varphi_2, \ldots$

$$\sum_{i=1}^{\infty} \langle f, \varphi_i \rangle^2 < +\infty$$

Let $f_N = \sum_{i=1}^{N} \langle f, \varphi_i \rangle \varphi_i$.

If $N < M$, $\| f_N - f_M \| = \| \sum_{i=N+1}^{M} \langle f, \varphi_i \rangle \varphi_i \| =$

$$\left[\sum_{i=N+1}^{M} \langle f, \varphi_i \rangle \right]^{1/2} \longrightarrow 0 \text{ as } M \text{ and } N \to \infty.$$

That is, $\{f_N\}$ is a Cauchy sequence. Since we are in a Hilbert space, they converge to g.

LEMMA: $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is continuous (proof posposed)

$\langle g, \varphi_\delta \rangle = \langle \lim_{N\to\infty} f_N, \varphi_\delta \rangle = \lim_{N\to\infty} \langle f_N, \varphi_\delta \rangle =$

0 if φ_δ is not one of the φ_i, $i = 1, 2, \ldots$

$\langle f, \varphi_j \rangle$ if $\varphi_j = \varphi_\delta$

This implies $g = \sum_{\delta \in \Gamma} \langle f, \varphi_\delta \rangle \varphi_\delta$

[illegible]

[illegible] $y_n \to y$.

[illegible] $|\langle x_n, y_n\rangle - \langle x, y\rangle|$ [illegible]

[illegible] $|\langle x, y\rangle - \langle x_n, y_n\rangle| = |\langle x, y\rangle -$ [illegible]

$$\le |\langle x - x_n, y\rangle| + |\langle x_n, y - y_n\rangle|$$

$$\le \|x - x_n\|\,\|y\| + \|x_n\|\,\|y - y_n\|$$

Since x_n is convergent $\exists M \ni \|x_n\| \le M$ [illegible]

Then $|\langle x, y\rangle - \langle x_n, y_n\rangle| \le \|x - x_n\|\,\|y\| + M\|y - y_n\| \to 0$ as $n \to \infty$.

Thus the lemma is proved.

Let $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ be an orthonormal set, $f \in H$ [illegible]

Let $g = \sum_{\gamma \in \Gamma} \langle f, \varphi_\gamma\rangle \varphi_\gamma = \lim f_N$ where $f_N = \sum_{i=1}^{N}$ [illegible]

with the γ_i's those $\ni \langle f, \varphi_{\gamma_i}\rangle \ne 0$.

Then $f - g$ is orthogonal to $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ since

$$\langle f - g, \varphi_\gamma\rangle = \langle f, \varphi_\gamma\rangle - \langle g, \varphi_\gamma\rangle = 0.$$

Possible answers:

(1) $f - g = 0$

(2) $f - g \neq 0$, in which case let $h = \frac{f-g}{\|f-g\|}$

Then $A = \{h\} \cup \{\varphi_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal set.

Def. $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ is said to be a complete orthonormal set for H (a Hilbert space) or an orthonormal basis if

(1) $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ is orthonormal, and

(2) $\{\sum_{\gamma\in F} a_\gamma \varphi_\gamma : a_\gamma \text{ scalars}, F \text{ finite} \subseteq \Gamma\}$ is dense in H.

By homework problem #3, (2) $\Leftrightarrow$ (2') $\forall f \in H$, if $\forall \gamma\in\Gamma$, $\langle f, \varphi_\gamma\rangle = 0$, then $f \equiv 0$.

Thm. Every Hilbert space has an orthonormal basis.

Pf. Proof is by Zorn's Lemma.

Let $\mathcal{I} = \{A : A \text{ is orthonormal} \subseteq H\}$.

Partially order $\mathcal{I}$ by set inclusion, $\subseteq$.

We must show each chain in $\mathcal{I}$ has an upper bound in $\mathcal{I}$.

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a chain.

Let $A = \bigcup_{\alpha\in\Lambda} A_\alpha$.

Claim: A is orthonormal.

Let g, ... are ...

for H.

Claim: ...

$\langle f, \phi_\alpha \rangle = 0$ if $\phi_\alpha \neq$ the ...

... $\{\phi_i\}$

$\langle g, \phi_\alpha \rangle = 0$ if $\phi_\alpha \neq \phi_i$ for ...

... a sequence $\{\phi_k\}$ so ...

If $\phi_\alpha \neq \phi_k$ for some k, both

$\langle f, \phi_\alpha \rangle = \langle g, \phi_\alpha \rangle = 0.$

Let $f_N = \sum_{k=1}^{N} \langle f, \phi_k \rangle \phi_k,$

$g_N = \sum_{k=1}^{N} \langle g, \phi_k \rangle \phi_k.$

We know $f_N \to f_0$ & $g_N \to g_0$.

<u>Claim</u>: $f_0 - f = 0$.

It suffices to show $\forall h \in H$

(*) $\langle f_0 - f, h \rangle = 0.$

Thus it suffices to show (*) for a dense subset of H.

Since $\{\sum_{\gamma \in F} a_\gamma \phi_\gamma : a_\gamma$ scalars, F finite $\}$ is dense, $\langle f_0 - f, \sum_{\gamma \in F} a_\gamma \phi_\gamma \rangle = \sum_{\gamma \in F} a_\gamma \langle f_0 - f, \phi_\gamma \rangle = 0$

Done with claim.

$\lim_{N\to\infty} \langle S_N, \ldots \rangle$

$\lim_{N\to\infty} \langle \sum_{i=1}^{N} \langle f, \phi_i \rangle \phi_i, \sum \ldots \rangle$

$\lim_{N\to\infty} \sum_{i=1}^{N} \langle f, \phi_i \rangle \langle g, \phi_i \rangle \langle \phi_i, \phi_i \rangle$

$= \lim_{N\to\infty} \sum_{i=1}^{N} \langle f, \phi_i \rangle \langle g, \phi_i \rangle$

$= \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle \langle g, \phi_\gamma \rangle$ since $\sum_{\gamma\in\Gamma} \langle f, \ldots \rangle$

so by Cauchy Schwarz the sum $\sum_{\gamma\in\Gamma} \ldots$
is unconditionally convergent

$L_2[-\pi, \pi]$ completion of $C[-\pi, \pi]$ with

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t)\, dt.$$

Famous orthogonal set
$1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$

Normalize this set
It turns out to be an orthonormal basis
for L_2.

Look at $L_2[0, 1]$, $\gamma : [0, 1] \to L_2[0, 1]$
$\Rightarrow \gamma(t) = \chi_{[0,t)}$.

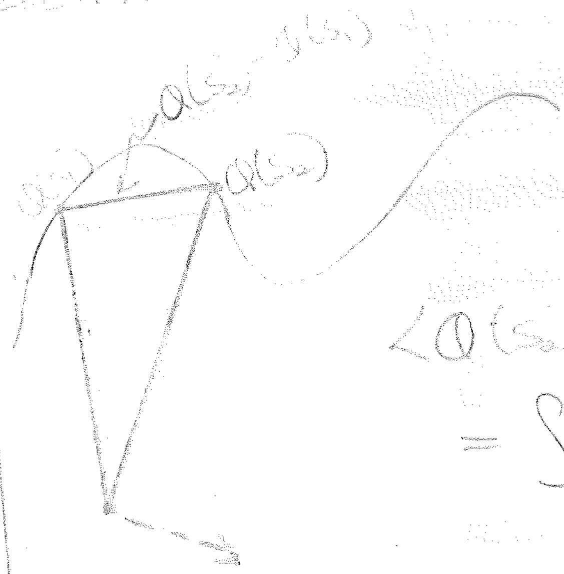
So γ is a nice, continuous curve.

for $s < t$ $\|\gamma(t) - \gamma(s)\| = \|\chi_{[s,t)}\| = \sqrt{t-s}$.

two ...

$s_1 < s_2 < s_3 < s_4$

$\Phi(s_2)-\Phi(s_1)$ + $\Phi(s_4)-\Phi(s_3)$ are ...
... ...



$\langle \Phi(s_2)-\Phi(s_1), \Phi(s_4)-\Phi(s_3) \rangle$

$= \int_0^1 \chi_{(s_1,s_2]} \chi_{(s_3,s_4]}$

$l_2 \ni e_i = (0, \cdots, 0, 1, 0, \cdots) \quad \forall i,$
ith place

Claim: $\{e_i\}$ is an orthonormal basis for l_2 but not a Hamel basis.

Let $f = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$

$f \in l_2, \; \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$

f is not a linear combination of $\{e_i\}$,

V & W ; two vector spaces

Defn a function $T : V \longrightarrow W$ is linear if

(1) $T(x+y) = Tx + Ty$

(2) $T(\alpha x) = \alpha Tx$

Note (1) $T0 = 0$

(2) $\ker T = \{ x \in V \mid Tx = 0 \} \subseteq V$

$\operatorname{Im} T = \{ Tx \mid x \in V \} \subseteq W$

These are both vector spaces.

Examples

(1) $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear iff $A = \begin{pmatrix} m\times n \\ \text{matrix} \end{pmatrix}$

s.t. $Tx = Ax$ where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ column vector

(2) $t \in [0,1]$

$C[0,1] \longrightarrow \mathbb{R}$ is linear

$f(x) \longmapsto f(t)$

↑ evaluation at t

(3) Suppose H is Hilbert sp.

Let $\{\varphi_1, \cdots, \varphi_n\}$ be othonormal set

Then $f \longmapsto \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i$ is linear

(4) $M_\lambda : l_2 \longrightarrow l_2$ where $\lambda = (\lambda_n) \in l_\infty$

(i.e $\sup_n |\lambda_n| < \infty$)

Define M_λ s.t.

$M_\lambda (x_n) = (\lambda_n x_n)$

(5) $T : C^k[0,1] \longrightarrow C[0,1]$

$u \longmapsto u^{(k)} + a_1 u^{(k-1)} + \cdots + a_{k-1} u'$

where $C^k[0,1]$ is space of all functions on $[0,1]$ which have continuous k^{th} derivative.

<u>Thm</u> If X & Y are normed spaces and $T : X \longmapsto Y$ is linear, then following are equivalent.

(1) T is continuous.

(2) T is continuous at 0

(3) T is bounded (maps bounded sets to bdded sets)

(4) $\|T\| < \infty$, where $\|T\| = \sup_{x \in X, \|x\| \leq 1} \|Tx\|$

(5) $\exists M \geq 0$ s.t. $\forall x \in \underline{X}$, $\|Tx\| \leq M \cdot \|x\|$

<u>proof</u>

* (1) $\Longrightarrow$ (2) obvious
* (2) $\Longrightarrow$ (3) First we need a definition

(<u>Defn</u> A set $B \subseteq \underline{X}$ is bounded if $\exists M$ s.t. $b \in B \Rightarrow \|b\| \leq M$)

Suppose not (3), then $\exists B$, bdd s.t.
$T(B) = \{ Tb \mid b \in B \}$ is not bdd.
Thus, $\exists (x_n) \subseteq B$ s.t. $\|Tx_n\| \geq n^2$
also, $\exists M$ s.t. $x \in B \Longrightarrow \|x\| \leq M$
Consider $(\frac{x_n}{n})$. Then

$$\left\| \frac{x_n}{n} \right\| \leq \frac{M}{n} \longrightarrow 0$$

$\therefore$ $\frac{x_n}{n} \longrightarrow 0$ in norm $\therefore$ $\frac{x_n}{n} \longrightarrow 0$

$\therefore$ By Continuity of T at 0, $\| T(\frac{x_n}{n}) \| \longrightarrow 0$

But $\| T(\frac{x_n}{n}) \| = \frac{1}{n} \|Tx_n\| \geq \frac{1}{n} \cdot n^2 = n$

contradiction.

(*) (3) $\Rightarrow$ (4) (4) is the special case of (3)

(*) (4) $\Rightarrow$ (5) Let $M = \|T\|$. Let $x \in \mathbb{X}$.

If $x = 0$, then $\|Tx\| = 0$ o.k.

If $x \neq 0$, let $\varphi = \frac{x}{\|x\|}$ so that $\|\varphi\| = 1$.

Then $$\frac{\|Tx\|}{\|x\|} = \left\| T\left(\frac{x}{\|x\|}\right)\right\| = \|T\varphi\| \leq \|T\| = M$$

Thus $\|Tx\| \leq M \cdot \|x\|$

(Turns out $\|T\| = \inf\left[M \text{ that work in (5)}\right]$)

(*) (5) $\Rightarrow$ (1) Let $x_n \longrightarrow x$. Want to show $Tx_n \to Tx$.

or $\|Tx_n - Tx\| = \|T(x_n - x)\| \longrightarrow 0$

But, by (5), $\|T(x_n - x)\| \leq M \cdot \|x_n - x\| \longrightarrow 0$

Remark All previous examples are conti.

a linear $T : \mathbb{X} \longrightarrow \mathbb{R}$ is sometimes called a linear functional.

Thm For $T : X \to R$ a linear functional, following are equivalent.

(1) T is continuous

(2) $\ker T$ is closed.

$\boxed{pf}$ (1) $\Rightarrow$ (2) From Topology, we know that the inverse image of a closed set by a conti. function is closed.

Thus $T^{-1}\{0\} = \{x \in X \mid Tx \in \{0\}\}$ is closed since $\{0\}$ is closed in R.

(2) $\Rightarrow$ (1) Defn A vector sp. $H \subseteq X$ is a hyperplane or a subsp. of co-dimension one if $\exists\, y \in$ ~~X~~ $X \setminus H = \{x \in X : x \notin H\}$ s.t. each $x \in X$ can be written $x = h + \lambda y$ λ: scalar, $h \in H$.

Fact T is non-zero linear functional, then $H = \ker T$ is a hyperplane.

<proof of Fact> let $y \in X$ s.t. $Ty \neq 0$

let $z = \frac{y}{T(y)}$. Then $T(z) = \frac{T(y)}{T(y)} = 1$

let $x \in X$. Write $x = h + T(x) z$ or

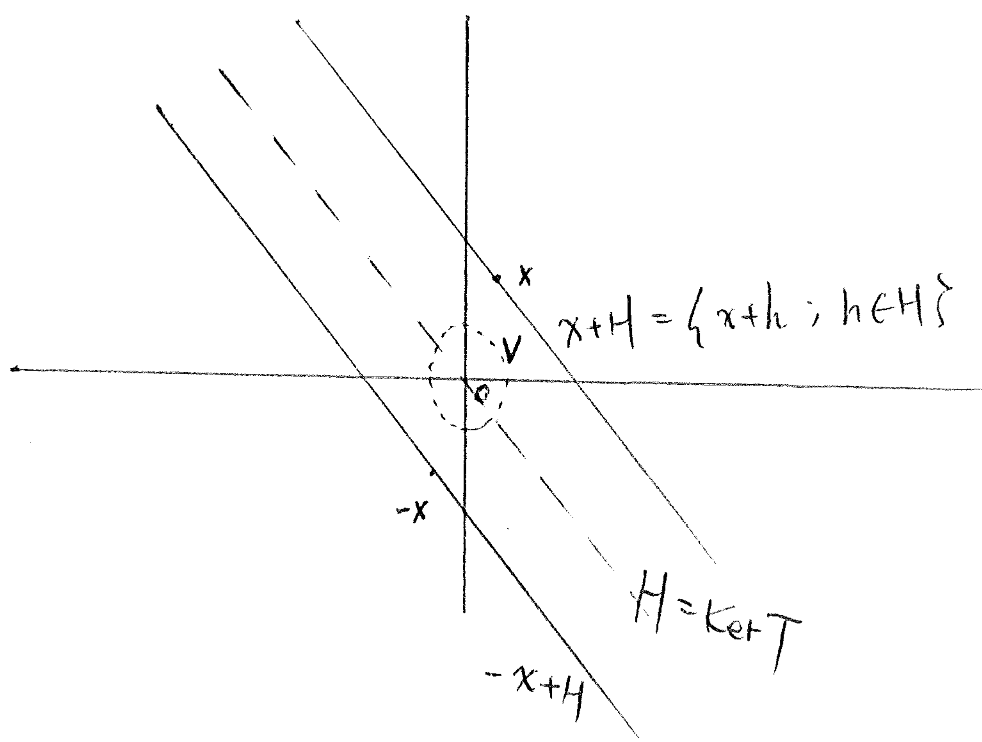
$x - T(x) z = h$

Then $T(h) = T(x - T(x)z) = Tx - Tx\,Tz = 0.$

Thus $h \in H$. and $z \in \cancel{X\backslash H}$ $X \backslash H$ $(\because T(z)=1)$

which finishes the proof of Fact.

Intuitive Picture



To show (1), it is enouff to show $\forall \varepsilon > 0$,

$\exists$ open nhbd (neighberhood) of 0, V s.t

$|T(v)| < \varepsilon$ for $v \in V$.

Fix x s.t. $Tx = \varepsilon$.

We can do this if $T \neq 0$. but

if $T \equiv 0$, then T is continuous.

Since $H = \ker T$ is closed and $x \notin H$,

$\exists\, \delta > 0$ s.t.

$U_{2\delta} = \{ y \in X ; \|x - y\| < 2\delta \}$.

Then $H \cap U_{2\delta} = \phi$ (*)

Let $H_\delta = \{ h + y ; h \in H, \|y\| < \delta \}$

$U_\delta = \{ x + y : \|y\| < \delta \}$

Note that $H\delta$ is open since

$$H_\delta = \bigcup_{h \in H} \{ h + \{ y : \|y\| < \delta \} \}$$

<u>Claim</u> $H_\delta \cap U_\delta = \phi$.

(pf of claim) If not, $\exists\, y_1, y_2$, $\|y_1\| < \delta$, $\|y_2\| < \delta$

and $\exists\, h \in H$ s.t.

$h + y_1 = x + y_2$ or $h = x + (y_2 - y_1)$

But $\|x - h\| = \|y_2 - y_1\| \le \|y_2\| + \|y_1\| < 2\delta$

Thus $h \in U_{2\delta}$ which is contradiction

to $H \cap U_{2\delta} = \phi$ (*). <u>q.e.d of claim</u>

Claim $T(H_\delta) \subseteq D_\varepsilon = \{ r \in \mathbb{R} \mid |r| < \varepsilon \}$

<pf of claim>

If not, $\exists\, w \in H_\delta$ s.t. $|T(w)| \geq \varepsilon$

Let $|\lambda| \leq 1$ s.t. $z = \lambda w$ satisfies $T(z) = \varepsilon$

Subclaim $z \in H_\delta$.

<pf> $w = h + y$, $h \in H$, $\|y\| < \delta$.

Thus $z = \lambda w = \lambda h + \lambda y$

But $\lambda h \in H$ and $\|\lambda y\| \leq \|y\| < \delta$

q.e.d of subclaim.

Now $z - x \in H$, since $Tz = Tx = \varepsilon$

Thus $x = z + h$ and $z = h_1 + y$ ($\because z \in H_\delta$)

$\therefore$ $x = z + h = h_1 + h + y$, $h_1 + h \in H$, $\|y\| < \delta$

Since $x \in U_\delta$, and $z + h \in H_\delta$,

we have $\cancel{H_\delta \cap U_\delta \ni x}$

$$x = z + h \in U_\delta \cap H_\delta$$

Contradiction to $U_\delta \cap H_\delta = \phi$. q.e.d of claim

Therefore $\exists$ open neighborhood $V \subset H_\delta$ of 0

s.t $|T(v)| < \varepsilon$ for $\forall v \in V \subset H_\delta$.

Q.E.D.

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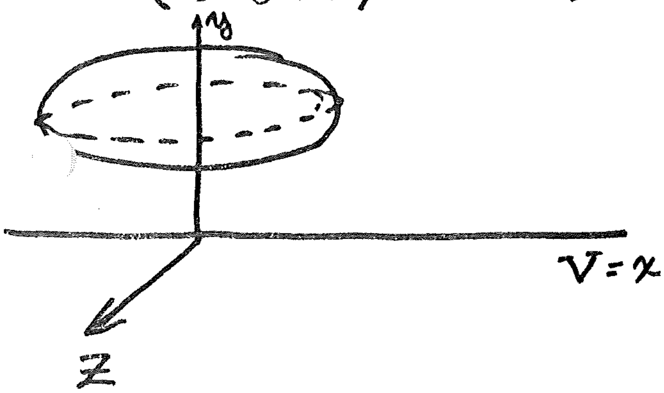
Thm 1 (Geometric Hahn - Banach)

Suppose $\mathfrak{X}$ is a normed space and let A be an open convex set; $A \subseteq \mathfrak{X}$. Let V be a subspace $\subseteq \mathfrak{X}$ with $V \cap A = \phi$.

Then $\exists$ a closed hyperplane $H \subseteq \mathfrak{X} \ni$ $H \supseteq V$ and $H \cap A = \phi$.

i.e. (3 dim.) $\exists$ plane going thru V that misses A.



Thm 2 (Hahn- Banach) (Real case)

Let $\mathfrak{X}$ be a normed space and $M \subseteq \mathfrak{X}$, a subspace. Suppose $f: M \longrightarrow \mathbb{R}$ is a continuous linear functional ($\|f\| = k < +\infty$).

Then $\exists$ $\tilde{f}: \mathfrak{X} \to \mathbb{R} \ni \|\tilde{f}\| = k$ and $\tilde{f}\big|_M = f$, [i.e. $\forall$ $m \in M$; $f(m) = \tilde{f}(m)$].

Proof (1) Suppose $f \equiv 0$, then take $\tilde{f} \equiv 0$. Hence assume $f \not\equiv 0$. Also may assume $\|f\| = 1$, [that is, let $g = \frac{|f|}{\|f\|}$ and we would be able to extend g to $\tilde{g}$ of norm one and then let $f = \|f\| \cdot g$].

There is x_0 ; $\|x_0\| \geq 1$ with $f(x_0) = 1$ since $f \not\equiv 0$.
[i.e. $\exists\, z \ni f(z) \neq 0$. Then take $x_0 = \frac{z}{f(z)}$ & we know $|f(x_0)| \leq \|f\| \cdot \|x_0\|$]
Let $A = \{ x_0 + y \mid \|y\| < 1 \}$; then A is open & convex.
Let $V = \{ x \in M \mid f(x) = 0 \}$; a subspace of X. (note V contains at least 0).

Want to show $A \cap V = \emptyset$. Suppose not, then
$\exists y ; x_0 + y = v$ with $\|y\| < 1$ & $v \ni f(v) = 0$.

But then $1 = f(x_0) = f(v - y) = f(v) - f(y)$

$\Rightarrow \quad 1 = f(y)$

and $\quad 1 = |f(y)| \leq \|f\| \cdot \|y\| < 1 \cdot 1 = 1$ $\times$ contradiction. $(1 < 1)$

Proof (2). By (1) $\exists$ closed hyperplane $H \supseteq V$ & $H \cap A = \emptyset$.

Define $\tilde{f}(x) = \lambda$ where $x = h + \lambda x_0$ is the unique decomposition given by hyperplane H, and where $h \in H$, λ a scalar.

$x_0 \in A \Rightarrow x_0 \notin H$

1) $\tilde{f}$ is linear: for x, y $\quad x = h_1 + \lambda_1 x_0$
$y = h_2 + \lambda_2 x_0$

$\Rightarrow \quad x + y = (h_1 + h_2) + (\lambda_1 + \lambda_2) x_0$

$\Rightarrow \quad \tilde{f}(x+y) = (\lambda_1 + \lambda_2) = \tilde{f}(x) + \tilde{f}(y)$

Similarly for $\tilde{f}(\alpha x) = \alpha \tilde{f}(x)$

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2) $\tilde{f}|_M = f$: Look at M; $V =$ ker f restricted to M, $x_0 \notin V$. V is a hyperplane in M. If $x \in M$, $x = v + \lambda x_0$ and then $f(x) = \lambda$ since $f(x) = f(v + \lambda x_0) = f(v) + \lambda f(x_0) = 0 + \lambda \cdot 1 = \lambda$.

3) $\|\tilde{f}\| = 1$: Suppose not, then $\exists$ z, $\|z\| < 1$ with $|\tilde{f}(z)| \geq 1$. [i.e. know $\|\tilde{f}\| \geq 1$]

Let $w = \dfrac{z}{f(z)}$. We have $f(w) = 1$ and

$$\|w\| = \frac{\|z\|}{|\tilde{f}(z)|} \leq \|z\| < 1.$$

But $x_0 - z \in \ker(\tilde{f}) = H$, and $\|z\| < 1 \Rightarrow x_0 - z \in A$. This contradicts $A \cap H = \phi$.

Applications

Let X be a normed space & define $X^* = \{$cont. linear functionals $X \to \mathbb{R}\}$

1) If $x \in X$, $x \neq 0$, then $\exists$ $f \in X^*$ with $f(x) \neq 0$.

Proof $M = \text{span}\{x\}$ & let $f(x) = 1$. Extend by linearity: $f(\lambda x) = \lambda$; $\|f(\lambda x)\| = \|f(x)\| \cdot |\lambda| = |\lambda|$ and $\|\lambda x\| = |\lambda| \cdot \|x\|$. Thus $\|f(\lambda x)\| \leq \dfrac{\|\lambda x\|}{\|x\|}$ and $\therefore \|f\| = \dfrac{1}{\|x\|}$.

Now extend by H-B to $f \in X^*$.

2) Furthermore $f \in X^*$ can be chosen so that

$$f(x) = 1 \;\&\; \|f\| = \frac{1}{\|x\|} \quad \text{or} \quad \|f\| = 1 \;\&\; f(x) = \|x\|.$$

$$\left[\begin{array}{l} \text{i.e. } f(\lambda x) = \lambda\|x\| \text{ and} \\ |f(\lambda x)| = |\lambda|\cdot\|x\| = \|\lambda\cdot x\| \end{array}\right]$$

3) If V is a closed subspace $\subsetneq X$ and $x \notin V$, then $\exists\, f \in X^*$ with $f(x) = 1$ and $f(v) = 0$, $v \in V$.

Just let $M = \text{span}\{V, x\} = \{v + \lambda x \mid v \in V, \lambda \text{ scalar}\}$ and $f: M \to \mathbb{R}$; $f(v + \lambda x) = \lambda$. Since $f|_M$ has closed kernel it is continuous and thus by H-B, extend to $\tilde{f} \in X^*$. Thus $\tilde{f}|_V = 0$ and $\tilde{f}(x) = f(x) = 1 \neq 0$.

<u>Lemma</u> If A, B convex $\subseteq X$, A open, $A \cap B = \emptyset$, then $\exists\, f \in X^*$ with $f(A) < f(B)$ (i.e. $f(a) < f(b)$; $a \in A, b \in B$)

<u>Corollary</u> If $C \subseteq X$ is a closed convex set and $x \notin C$ then $\exists\, f \in X^*$ with $f(c) < f(x)$ $\forall\, c \in C$.

<u>Proof</u> Let $B = C$ & $\exists A$ (open, convex) with $x \in A$, $A \cap C = \emptyset$. Let f be function given by lemma.

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~~Dr. Bellonet~~

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Notes taken by Atef. M. A-Moneim.

Lemma:

If H is a hyperplane in $\mathfrak{X}$ then H is closed or H is dense.

Pf

If H is not closed then $\overline{H} \supsetneq H$ $\exists\, x \in \overline{H} \setminus H$

$\Rightarrow \forall\, y \in \mathfrak{X}$ $y = h + \lambda x$ $h \in H$ So $y \in \overline{H}$

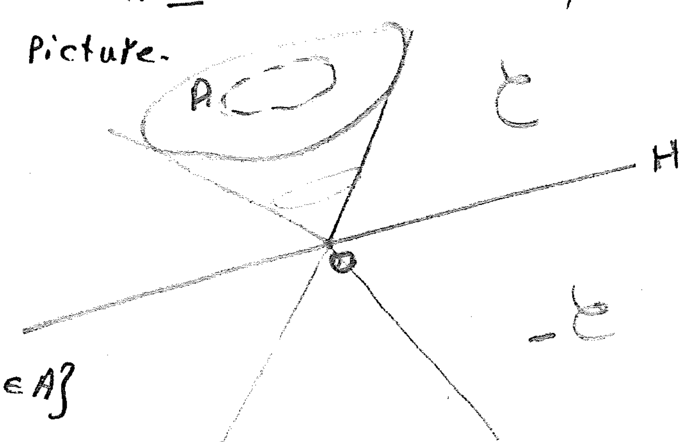
since $\overline{H}$ is a subspace

Now let us prove G.H.B which stated at Page 42 in this Notes.

Pf. of G.H.B

Use Zorn's Lemma to find a subspace H which is Maximal with resp. to. $H \supseteq V$ and $H \cap A = \phi$

let us Now draw a Picture.



let $\mathcal{C} = H + \bigcup_{\lambda > 0} \lambda A$

$= \{h + \lambda a : h \in H, \lambda > 0, a \in A\}$

Also $-\mathcal{C} = \{-c : c \in \mathcal{C}\} = \{h - \lambda a : h \in H, \lambda > 0, a \in A\}$

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Asef. Moneim

$= H - \bigcup_{\lambda>0} \lambda A$

Now claim

$C \cap -C = \phi$

Pf

If not $h_1 + \lambda_1 a_1 = h_2 - \lambda_2 a_2$

then

$(\lambda_1 + \lambda_2)\left[\frac{\lambda_1}{\lambda_1+\lambda_2} a_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} a_2\right] = \lambda_1 a_1 + \lambda_2 a_2 = h_2 - h_1 \in H$

Let $\lambda_1 + \lambda_2 = \mu$, $\left[\frac{\lambda_1}{\lambda_1+\lambda_2} a_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} a_2\right] = a$, $h_2 - h_1 = h$

then $\exists\ \mu a = h$ $\quad \mu > 0$, $h \in H$, $a \in A$

$a = \mu^{-1} h \in H$ $\quad H \cap A \neq \phi$ #.

Claim

$H \cap C = H \cap -C = \phi$

Pf

suppose not then

$h_1 = h_2 + \lambda a$ $\quad \lambda > 0$ $\quad a \in A$, $h_1, h_2 \in H$

$\lambda^{-1}(h_1 - h_2) = a$ $\quad H \cap A \neq \phi$ #.

Now If $\underline{C \cup H \cup -C = X}$ then H is hyperplane

suppose not

$\exists\ a \in C$ with $H_1 = \{h + \lambda a : \lambda \text{ scaler}, h \in H\}$ subspace

but not every thing. $\exists\ b_1 \in X \setminus H_1$ $\quad b_1 \notin H$, $b_1 \in C \Rightarrow -b_1 \in$ $\{-b_1 \notin$

$\exists\ b \in -C$ with $b \notin H$.

hint $C, -C$ are open.

define

$\phi : [0,1] \longrightarrow X$

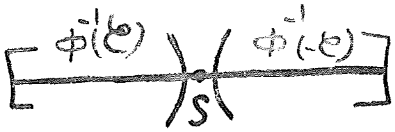
given by $\phi(t) = ta + (1-t)b$, ϕ is continuous.

$\phi^{-1}(\mathcal{C})$, $\phi^{-1}(-\mathcal{C})$ both open & Convex

since $1 \in \phi^{-1}(\mathcal{C})$, $0 \in \phi^{-1}(-\mathcal{C})$

$\phi^{-1}(\mathcal{C}) \cap \phi^{-1}(-\mathcal{C}) = \phi$

$\exists\ S \in [0,1]$ s.t



$\phi(S) \in H$

take

$1 > \lambda > 0$, $\lambda a + (1-\lambda) b = h$

$(1-\lambda) b = h - ha$

$$b = \frac{1}{1-\lambda} h - \frac{\lambda}{1-\lambda} a$$

$b \in H$, #.

Suppose $x_0 \in X \setminus (\mathcal{C} \cup H \cup -\mathcal{C})$

Let $H_1 = \{h + \lambda x_0 : \lambda \text{ scaler}, h \in H\}$

H_1 is subspace $\subseteq X$

and $V \subseteq H_1$

suppose $H_1 \cap A \neq \phi$

$h + \mu x_0 = a$ $\quad h \in H$

μ scaler , $a \in H$. $\mu \neq 0$

$\mu x_0 = -h + a$ $\quad x_0 = -\mu^{-1} h + \mu^{-1} a$

this a contradiction since this element in $\mathcal{C}$ when $\mu > 0$ or in $-\mathcal{C}$ $\mu < 0$.

Thus $H_1 \cap A = \phi$ by this
Contradicts the Maximality of H.
Thus $X = \mathcal{E} \cup H \cup -\mathcal{E}$ & H is a closed
hyperplane.

<u>Lemma</u>

If A open convex & B convex $\subseteq X$ normed space,
$A \cap B = \phi$ then $\exists\, f \in X^*$ with $f(a) > f(b)$
for $a \in A$, $b \in B$.

<u>Pf</u>:- $A - B = \{a - b : a \in A, b \in B\}$
$\bigcup_{b \in B} (A + b)$ open

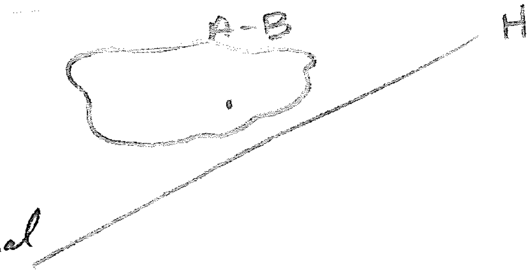
$A-B$ is convex to show that take a convex
combination. let $0 \leq \lambda \leq 1$

$\lambda(a_1 - b_1) + (1-\lambda)(a_2 - b_2)$

$[\underbrace{\lambda a_1 + (1-\lambda)a_2}_{\text{this in } A}] - [\underbrace{\lambda_1 b_1 + (1-\lambda) b_2}_{\text{this in } B}] \in A - B$.

Let $V = \{0\}$ $0 \notin A - B$ since,
if not $a - b = 0 \Rightarrow a = b \Rightarrow A \cap B \neq \phi$.
then by G. H. B
$\exists H$ closed hyperplane $(A-B) \cap H = \phi$.
Let $f \in X^*$ with $\operatorname{Ker} f = H$

Claim $\left.\begin{array}{l} x \in A - B \Rightarrow f(x) > 0 \\ \text{or } \forall x \in A - B \Rightarrow f(x) < 0 \end{array}\right\}$ but not both



Suppose $f > 0$ on $A-B$

$f(a-b) > 0$ or

$f(a) > f(b)$

$\mathbb{X}$ normed, f linear functional

f is non zero.

$A \subseteq \mathbb{X}$ A open then

$f(A) \subseteq \mathbb{R}$ open

let $x \in A$. it is enough to show that

for some $\varepsilon > 0$

$|\lambda - f(x)| < \varepsilon$ then $\lambda \in f(A)$.

Let $y_0 \in \mathbb{X}$ $\ni$ $f(y_0) = 1$

Consider

$U = A - x$ open.

$0 \in U$ $\exists \delta > 0$ $\ni$

$B_\delta(0) = \{z : \|z\| < \delta\} \subseteq U$

for $|\mu| < \dfrac{\delta}{\|y_0\|}$ $\mu y_0 \in U$

this $|\mu| < \dfrac{\delta}{\|y_0\|} = \varepsilon$

$\mu y_0 = a - x$ $a = x + \mu y_0$

$f(a) = f(x) + \mu f(y_0) = f(x) + \mu$.

Corollary:-

If B closed convex $x \notin B$ $\exists f \in \mathbb{X}^*$ and

λ scalar $\Rightarrow$ $\forall\, b \in B$ $\quad f(b) < \lambda < f(x)$

pf

let A open convex $\quad x \in A$

$A \cap B = \phi$

f s.t $\quad f(a) > f(b) \quad a \in A, \; b \in B$

$f(x) > f(a_1) > f(b) \quad b \in B$

and for some $a \in A$

then $\lambda = f(a)$.

Note:-

$X^* = \{ f : X \longrightarrow \mathbb{R},\, f \text{ cont.} \}$ we will show that this X^* is a Banach space under some defined norm.

PROBLEM SET #2 Due 25 Oct

1. Show that there is a sequence of reals $\{a_n\}$ with $a_n \geq 0$ such that $\Sigma_1^\infty a_n^2 < +\infty$ and $\Sigma_1^\infty a_n/\sqrt{n} = +\infty$. HINT $(1/\sqrt{n}) \notin \ell_2$.

2. Do Problem 8 page 54 & use it to show if $1/p + 1/q = 1/s$ then $\{x_n\} \in \ell_p$, $\{y_n\} \in \ell_q \Rightarrow \{x_n y_n\} \in \ell_s$.

3. A. If X is a topological space; f & g continuous functions $X \to X$ s.t. $\forall x \in X$ $f(g(x)) = x = g(f(x))$ then f is a homeomorphism from X onto X.
B. If X is a TVS, $x \in X$, $0 \neq \lambda$ a scalar, then the maps T_x & S_λ are homeomorphisms onto X, where for $y \in X$ $T_x(y) = x + y$ and $S_\lambda(y) = \lambda y$.
C. If X is a normed space show T_x is an isometry of metric which is linear if and only if $x = 0$; and S_λ is an linear isomorphism, which is an isometry if and only if $|\lambda| = 1$.

4. A. Decribe the dual of c.
B. Show that c is isomorphic to c_0
C. Show your isomorphism is not an isometry.

5. Let $\{x_n\} \subseteq X$ & $y \in X$, X a normed space; suppose for all $f \in X^*$ $f(x_n) \to f(y)$. Show that there is an increasing sequence of integers $0 = N_0 < N_1 < N_2 \ldots$ and scalars $t_n \geq 0$ with
$$\sum_{n=N_{i-1}+1}^{N_i} t_n = 1 \quad i = 1, 2, \ldots \quad \text{so that} \quad y_i = \sum_{n=N_{i-1}+1}^{N_i} t_n x_n \quad i = 1, 2, \ldots$$
converges to y in norm. (i.e. $\|y - y_i\| \to 0$) HINT: Show y is in the norm closure of the convex hull of $\{x_n : n = N, N+1, N+2 \ldots\}$.

The Dual of a Normed Space.

$X^* \equiv \{f : X \to R^1\}$, f linear and cont.$\}$ $\|f\| = \sup_{\|x\| \leq 1} |f(x)|$

<u>Theorem</u> X^* is a Banach Space (even if X is not)

Proof: Let $U \equiv \{x \in X : \|x\| \leq 1\}$
$B(U)$, the set of all bounded functions on U, is a Banach Space with the norm $\|f\| = \sup_{u \in U} |f(u)|$. $X^* \subseteq B(U)$ then implies that X^* is a normed space.
To show completeness let $f_n \subset X^*$ and $f_n \to g \in B(U)$. Define $\tilde{g}$ on the whole of X by

$$\tilde{g}(x) = \begin{cases} 0 & x = 0 \\ \|x\| \, g\left(\frac{x}{\|x\|}\right) & x \neq 0 \end{cases}$$

then $\hat{g}(x) = \lim f_n(x) \quad \forall x \in X$

since $\tilde{g}(x) = \|x\| \, g\left(\frac{x}{\|x\|}\right) = \|x\| \lim f_n\left(\frac{x}{\|x\|}\right)$

$= \lim f_n(x)$. So to complete the Proof we only have to show $\tilde{g}$ is linear

$\tilde{g}(x+y) = \lim f_n(x+y) = \lim f_n(x) + \lim f_n(y)$

$= \tilde{g}(x) + \tilde{g}(y)$

$\tilde{g}(\alpha x) = \lim f_n(\alpha x) = \alpha \cdot \lim f_n(x) = \alpha \tilde{g}(x)$

Q.E.D.

$X^{**} \equiv (X^*)^*$ is called the Bidual. This is a Banach space. Moreover there exists a canonical isometry $\varphi : X \to X^{**}$ which allows us to think of X as a subset of X^{**}

For $x \in X$ define $\Phi(x) = F_x$

where $F_x : X^* \to \mathbb{R}'$ is defined by $F_x(f) = f(x)$

(1) Claim F_x is linear

$$F_x(f+g) = (f+g)(x) = f(x) + g(x) = F_x(f) + F_x(g)$$

$$F_x(\alpha f) = (\alpha f)(x) = \alpha(f(x)) = \alpha F_x(f)$$

(2) Claim F is continuous

Let $\|f\| \leq 1$ we need to find $M \leq \infty$ such that $|F_x(f)| \leq M$ for all such f.

$$|F_x(f)| = |f(x)| \leq \|f\| \|x\| \leq \|x\| \equiv M$$

(3) Claim $\|F\| = \|x\|$. of the above shows $\|F\| \leq \|x\|$ we need to show $\|F\| \geq \|x\|$. By the HB theorem $\exists f \in X^*$ such that $\|f\| = 1$ and

$|f(x)| = \|x\|$. then

$|F_x(f)| = |f(x)| = \|x\|$ and $\|F\| \geq \|x\|$

follows.

<u>Definition</u> If X and Y are normed spaces $T: X \to Y$ linear is an isometry if $\|x\| = \|T(x)\|$ $\forall x \in X$

<u>Note</u>: Isometries are homeomorphism

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(4) Claim <u>φ is linear</u>

$[\varphi(x+y)](f) = F_{x+y}(f) = f(x+y) = f(x) + f(y)$

$= F_x(f) + F_y(f) = (F_x + F_y)(f)$

$= [\varphi(x) + \varphi(y)](f)$ This is true $\forall f \in X^*$

so $\varphi(x) + \varphi(y) = \varphi(x+y)$ as desired

Scalar mult. follows similarly

Notation

$$\langle x, f \rangle \equiv f(x) \quad \text{for } x \in \mathfrak{X} \text{ and } f \in \mathfrak{X}^*$$

$$\langle f, F \rangle \equiv F(f) \quad \text{for } f \in X^* \text{ and } F \in \mathfrak{X}^{**}$$

Note that

$$\langle x, f \rangle = \langle f, F_x \rangle \quad \text{in this notation}$$

Theorem <u>Every normed space can be completed</u>

Proof: $\mathfrak{X} \subseteq \mathfrak{X}^{**}$ which is a B-space so the completion of $\mathfrak{X}$ is the closure of $\mathfrak{X}$ in $\mathfrak{X}^{**}$

<u>Definition:</u>

$\ell_1 \equiv \{ (\eta_n)$ scalar sequences such that $\|(\eta_n)\|_1 = \sum_{n=1}^{\infty} |\eta_n| < +\infty \}$

Examples of Dual Spaces

$c_0^* = l_1$, $l_1^* = l_\infty$

i.e., (1) Each $(\eta_n) \in l_1$ $(\in l_\infty)$ defines a bounded linear functional F on c_0 (on l_1) by

$$F((\xi_n)) = \langle (\xi_n)(\eta_n) \rangle \equiv \sum_{n=1}^{\infty} \xi_n \eta_n$$

(2) $\|F\| = \|(\eta_n)\|_1$ $(= \|(\eta_n)\|_\infty)$

(3) Every bounded linear functional on c_0 (on l_1) is obtained as above

Proof: (1) is obvious

Claim
(a) $|\langle (\xi_n)(\eta_n) \rangle| \leq \|(\xi_n)\|_\infty \|(\eta_n)\|_1$
(b) $|\langle (\xi_n)(\eta_n) \rangle| \leq \|(\xi_n)\|_{c_0} \|(\eta_n)\|_1$

Proof of claim a

$$|\langle (\xi_n)(\eta_n)\rangle| \le \sum |\xi_n||\eta_n| \le \sum \|(\xi_n)\|_\infty |\eta_n|$$

$$= \|(\xi_n)\|_\infty \sum |\eta_n| = \|(\xi)\|_\infty \|(\eta_n)\|_1 \quad \blacksquare$$

<u>Proof of claim b</u> Just note that if $(\xi_n) \in c_0$ above then $\|(\xi_n)\|_{c_0} = \|(\xi_n)\|_\infty$ $\blacksquare$

Note that a and b implies that

$\|F\| \le \|(\eta_n)\|_1$ for $F \in c_0^*$

$\|F\| \le \|(\eta_n)\|_\infty$ for $F \in l_1^*$

Now we want to show that equality actually holds.

<u>Claim</u> (c) $\|F\| = \|(\eta_n)\|_1$ $\quad F \in c_0^*$

<u>Proof of claim (c)</u>. It is enough to show that for arbitrary $\epsilon > 0$ $\exists\ (x_n) \in c_0$, $\|(x_n)\|_{c_0} \le 1$ such that $\langle (\eta_n)(x_n)\rangle \geqq \|(\eta_n)\|_1 - \epsilon$.

$\infty \geq \|(\eta_n)\|_1 = \sum |\eta_n| \Rightarrow \exists N$ such that $\sum_{N+1}^{\infty} |\eta_n| < \varepsilon$. choose

$$x_n = \begin{cases} 0 & n \geq N+1 \\ 1 \text{ or } -1 & \text{so that } x_n \eta_n = |\eta_n| \text{ o.w.} \end{cases}$$

Note that $\|(x_n)\|_{c_0} = 1$

Also

$$\langle (x_n), (\eta_n) \rangle = \sum_1^{\infty} x_n \eta_n = \sum_{i=1}^{N} |\eta_n|$$

$$\geq \|(\eta_n)\|_1 - \sum_{N+1}^{\infty} |\eta_n| \geq \|(\eta_n)\|_1 - \varepsilon \quad \blacksquare$$

<u>Claim</u> (d) $\|F\| = \|(\eta_n)\|_\infty \quad F \in l_1^*$

<u>Proof of claim d</u> Let $\varepsilon > 0$ need $(x_n) \in l_1$ $\|(x_n)\|_1 \leq 1$

$\ni \langle (\eta_n)(x_n) \rangle \geq \|(\eta_n)\|_\infty - \varepsilon$

Define $x_n = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

where m is such that

$$|\eta_m| \geq \|(\eta_n)\|_\infty - \varepsilon$$

Note that $\|(x_n)\|_1 = 1$ and that

$$|\langle(\xi_n),(\eta_n)\rangle| = |\xi_m| \geq \|(\eta_n)\|_\infty - \varepsilon$$

as desired ∎

We have shown (1) and (2)

i.e. that

$$c_0^* \supset \ell_1 \qquad \text{and} \qquad \ell_1^* \supset \ell_\infty$$

in the next lecture we will show the other direction.

Mat. 534 (Notes taken by C. C. Chen)

Delayed remark: If X is a Banach space with dual X^*, bidual X^{**}. Let U^* be the closed unit ball of X^*, Then $\exists$ a topology on U^* in which U^* is compact and X considered as subspace of X^{**} are continuous functions.

Def. Let $l_p = \{ (\xi_n) \mid \|\xi_n\|_p = (\sum_{n=1}^{\infty} |\xi_n|^p)^{1/p} < +\infty, \xi_n \in \mathbb{R} \}$.

Note: l_1 fits, l_∞ fits with sup norm.

Def: Let $l_f = \{ (\xi_n) \mid \xi_n = 0$ except finite number of $n \}$

Note: $l_f \subseteq l_p$ $1 \le p \le \infty$; $l_f \subseteq c_0$

Thm: l_f dense in c_0, l_p. $1 \le p < \infty$, but not l_∞.

Pf: Claim: l_f dense in l_p, $1 \le p < \infty$.

Let $(\xi_n) \in l_p$ Then.

given $\epsilon > 0$, $\exists N \ni: (\sum_{n=N+1}^{\infty} |\xi_n|^p)^{1/p} < \epsilon$.

define $(\eta_n) \in l_f$ as: $\eta_n = \begin{cases} \xi_n & n \le N \\ 0 & \text{o.w.} \end{cases}$

$\|(\xi_n) - (\eta_n)\|_p = (\sum_{n=N+1}^{\infty} |\xi_n|^p)^{1/p} < \epsilon$. Thus l_f dense in l_p.

Claim: l_f dense in c_0.

Let $(\xi_n) \in c_0 \Rightarrow$ given $\epsilon > 0$ $\exists N \ni: |\xi_n| < \epsilon$ $\forall n \ge N$.

define $\eta_n = \begin{cases} \xi_n & n \le N \\ 0 & \text{o.w.} \end{cases}$

Then $(\eta_n) \in l_f$ and $\|(\xi_n) - (\eta_n)\|_\infty = \sup_{n>N} |\xi_n| < \epsilon$ done w/claim.

Claim: l_f is not dense in l_∞.

$l_f \subseteq c_0 \Rightarrow \overline{l_f} \subseteq \overline{c_0} = c_0 \subsetneq l_\infty$ done w/claim.

Def: A Banach space is seperable if it has a countable dense subset.

Remark: l_p & c_0 are seperable where $1 \le p < \infty$

Consider $D = \{(\xi_n) \in l_f \mid \xi_n : \text{rational}\}$.

D is dense in l_p, $1 \le p < \infty$, and c_0.

Last time we have shown that $c_0^* \supseteq l_1$ & $l_1^* \supset l_\infty$. Now, we continue to show the "$\subset$" directions.

Let $f \in c_0^*$, Then $\forall\, x = (\eta_i) \in l_f$. We have $f(x) = \sum_{i=1}^{\infty} \eta_i \xi_i$.

where (ξ_i) be a sequence defined by $\xi_n = f(e_n)$, when $e_n \in c_0$ and $e_n(i) = \begin{cases} 1 & i = n \\ 0 & i \ne n \end{cases}$.

Claim: $(\xi_n) \in l_1$

given N, let $(\eta_n) = \begin{cases} sgn(\xi_n) & n \le N \\ 0 & \text{otherwise} \end{cases}$; $sgn(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$.

Then $\|\eta_n\|_\infty = 1$ and $(\eta_n) \in l_f \subset c_0$.

$\sum_{n=1}^{N} |\xi_n| = |f(\eta_n)| \leq \|f\| \, \|(\eta_n)\|_\infty = \|f\| < \infty \quad \forall N.$

hence $(\xi_n) \in l_1$

Consider $f - \xi_n \in c_0^*$, $f - (\xi_n) = 0$ on l_f. Where l_f dense in c_0 Thus

$f = (\xi_n)$ done w/claim.

Claim: $l_1^* \subset l_\infty$

Let $f \in l_1^*$ Then $\forall\ x = (\eta_i) \in l_f$, we have $f(x) = \sum_1^\infty \eta_i \xi_i$

Where (ξ_i) be a sequence defined by $\xi_n = f(e_n)$ where $e_n \in l_1$ and

$$e_n(i) = \begin{cases} 1 & i = n \\ 0 & i \neq n. \end{cases}$$

now, $|f(e_n)| = |\langle(\xi_n), e_n\rangle| = |\xi_n| \leq \|f\| \, \|e_n\|_1 = \|f\| < \infty \quad \forall n.$

Thus. $(\xi_n) \in l_\infty$ ~~done w/claim.~~ as before
$f - (\xi_n) \equiv 0$ on l_f hence $f = (\xi_n)$ done.

Note: 1). $l_p^* = l_q$ where $\frac{1}{p} + \frac{1}{q} = 1.$ $\forall$ $1 \leq p \leq \infty$, when $p = 1$, $q = \infty$

This follows from the following inequality: $|\sum \xi_n \eta_n| \leq \|(\xi_n)\|_p \cdot \|(\eta_n)\|_q$

2). $L_p^* = L_q$ $p < \infty$ $\frac{1}{p} + \frac{1}{q} = 1$, $q = \infty$ if $p = 1$.

Thm: Let H be a Hilbert space and M closed subspace of H. Then $\exists$ a unique closed subspace N of H $\ni$:

(1) $M \perp N$

(2) $M + N = H$

(3) $\forall\, h \in H$, $\exists$ unique $m \in M$, $n \in N \ni: h = m + n$.

(4). $P: H \longrightarrow H \quad p(h) = m$, then

$\|p\| = 1$, Range $P = M$, kernel $p = N$, $P^2 = P$.

Remarks 1) M & N are called orthogonal Complement.

(2) This Characterizes Hilbert space.

<u>proof of the Thm</u>: M is a Hilbert space $\Rightarrow \exists \{\varphi_\alpha\}_{\alpha \in A}$ an O.N basis for M

and $\exists \{\varphi_\beta\}_{\beta \in B} \ni: \{\varphi_\alpha\}_{\alpha \in A \cup B}$ is an O.N basis for H.

Let $N =$ closed linear span $\{\varphi_\alpha\}_{\alpha \in B}$.

Note: $M \perp N$.

<u>Claim</u>: $N = \bigwedge\limits_{x \in M} \{ y \in H : \langle y, x \rangle = 0 \}$.

"$\subseteq$" since $x \in N \Rightarrow x \perp M$.

"$\supseteq$" let $x \perp M \Rightarrow \langle x, \varphi_\alpha \rangle = 0 \ \forall\, \alpha \in A$.

$\Rightarrow x = \sum\limits_{\alpha \in B} \langle x, \varphi_\alpha \rangle \varphi_\alpha \in N$. done w/claim.

Thus, N is a intersection of closed subspaces hence is a closed subspace.

<u>Claim</u>: $M + N = H$.

If $x \in H$, $h = \sum_{\alpha \in A \cup B} \langle h, \varphi_\alpha \rangle \varphi_\alpha$

$$= \sum_{\alpha \in A} \langle h, \varphi_\alpha \rangle \varphi_\alpha + \sum_{\alpha \in B} \langle h, \varphi_\alpha \rangle \varphi_\alpha$$

Let $m = \sum_{\alpha \in A} \langle h, \varphi_\alpha \rangle \varphi_\alpha \in M$, $n = \sum_{\alpha \in B} \langle h, \varphi_\alpha \rangle \varphi_\alpha \in N$ done w/ claim

Uniqueness: Suppose $m_1 + n_1 = m_2 + n_2 = h$

$\Rightarrow (m_1 - m_2) = (n_2 - n_1) \in M \cap N = \{0\}$. $\Rightarrow m_1 = m_2$, $n_1 = n_2$.

[NOTE $M \perp N \Rightarrow M \cap N = \{0\}$].

Suppose P be another closed subspace of H.

$P \perp M \Rightarrow P \leq \bigcap_{x \in M} \{y : \langle y, x \rangle = 0\} = N$

If $P \subsetneq N$, Then $\exists$ $n \in N \setminus P$ $\ni$: $n \notin M + P$.

thus N is unqiue.

LAST CLASS PERIOD WE PROVED STATEMENTS 1, 2, & 3 OF THE FOLLOWING **THEOREM**: IF M IS A CLOSED SUBSPACE OF H, THEN THERE EXIST A UNIQUE N, CLOSED SUBSPACE OF H SUCH THAT:

① $M \perp N$

② $M + N = H$

③ $\forall h \in H$, $\exists$ UNIQUE $m \in M$, $n \in N$ SUCH THAT $h = m + n$

④ $P: H \to H$ SUCH THAT IF $h = m+n$, $P(h) = m$

THEN $\|P\| = 1$

RANGE $P = M$

KER $P = N$

$P^2 = P$

PROOF OF ④:

P IS LINEAR: $h_1 = m_1 + n_1$ & $h_2 = m_2 + n_2$ THEN

$$h_1 + h_2 = \underbrace{(m_1 + m_2)}_{\in M} + \underbrace{(n_1 + n_2)}_{\in N}$$

$$\lambda h_1 = \underbrace{\lambda m_1}_{\in M} + \underbrace{\lambda n_1}_{\in N}$$

(DECOMPOSITION IS UNIQUE)

$P(h_1 + h_2) = m_1 + m_2 = P(h_1) + P(h_2)$

$P(\lambda h_1) = \lambda m_1 = \lambda P(h_1)$

$P^2 = P$: LET $h = m+n$ THEN

$P^2(h) = P(P(h)) = P(m) = m = P(h)$.

$(P|_M = 1_M$ & $P|_N \equiv 0)$

$\|P\| = 1$: $\|P\| \leq 1$ FOR IF $h \in H$, $\|h\| \leq 1$.

LET $h = m+n$. $m \perp n$ SO THAT $1 = \|h\|^2 = \|m\|^2 + \|n\|^2$.

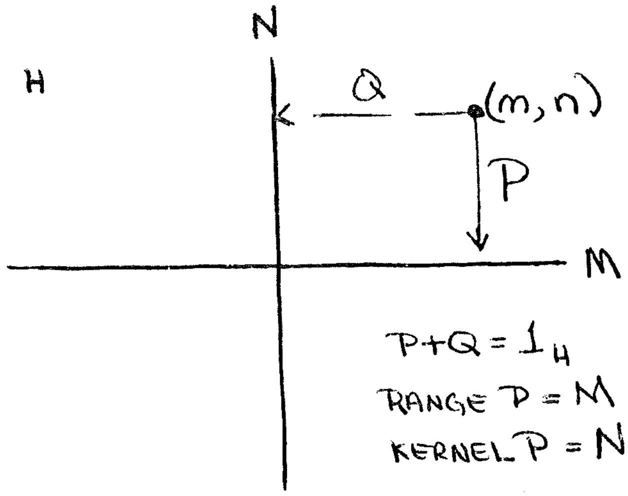
SINCE $\|n\|^2 \geq 0$ $\|m\| \leq 1$.

$\|P(h)\| = \|m\| \leq 1$.

$\|P\| = 1$ FOR IF $m \in M$ & $\|m\| = 1$ THEN $\|Pm\| = \|m\| = 1$

FROM PICTURE WE CAN SEE RANGE OF $P = M$

KER $P = N$.



<u>RIESZ REPRESENTATION THEOREM</u>: LET $f: H \to \mathbb{R}$ BE A CONTINUOUS LINEAR FUNCTIONAL ON A HILBERT SPACE, H, WITH INNER PRODUCT, $\langle \circ \circ \rangle$. THEN THERE EXIST A UNIQUE $y \in H$ SUCH THAT:

① $f(h) = \langle h, y \rangle \quad \forall h \in H$

② $\|f\| = \|y\|$

(CONVERSE IS ALSO TRUE)

PF: LET f BE A CONTINUOUS LINEAR FUNCTIONAL

I. IF $f \equiv 0$ PICK $y = 0$

II. IF $f \not\equiv 0$, LET $M = \ker f$. M IS CLOSED HYPERPLANE. BY PREVIOUS THEOREM, $\exists N$, CLOSED SUBSPACE, SUCH THAT $M \perp N$ AND $M + N = H$. (NOTE THAT N IS ONE-DIM.) LET $w \in N \setminus \{0\}$. $f(w) \neq 0$ BECAUSE $w \notin M = \ker f$. LET $z = \lambda w$ WHERE $\lambda = 1/f(w)$. $f(z) = 1$

$f(m + \lambda z) = f(m) + \lambda f(z) = \lambda$ i.e $f(h) = \lambda$ FOR $h = m + \lambda z$

WANT y SUCH THAT $\|y\| = \|f\|$

LET $y = \dfrac{\|f\|}{\|z\|} \cdot z$

IF $h = m + \lambda z$

$$\langle h, y \rangle = \left\langle m + \lambda z, \frac{\|f\|}{\|z\|} z \right\rangle = \lambda \frac{\|f\|}{\|z\|} \langle z, z \rangle = \lambda \|f\| \|z\|.$$

SO TO COMPLETE THE PROOF IT SUFFICES TO SHOW $\|f\| \|z\| = 1$.

TO SHOW $1 = f(z) = \|f\|\,\|z\|$, IF $\|f\| = 1$ THEN WE WANT TO SHOW $\|z\| = 1$ $\quad 1 = |f(z)| \le \|f\|\,\|z\| = \|z\|$ hence $\|z\| \ge 1$

$\|z\| \geq 1$. IF $\|z\| > 1 \Rightarrow \|f\| < 1$. LET $h = m + \lambda z$, $\|h\| \le 1$

$$\|h\|^2 = \|m\|^2 + \|\lambda z\|^2$$

$$= \|m\|^2 + \lambda^2 \|z\| \le 1. \quad \|\lambda z\|^2 \le 1 \text{ SO THAT } |f(h)| = |f(\lambda z)| = |\lambda|\,|f(z)| = |\lambda|.$$

$$\Rightarrow \lambda^2 \le \frac{1}{\|z\|^2}$$

$$\Rightarrow |\lambda| \le \frac{1}{\|z\|} < 1$$

THEREFORE $\sup\limits_{\substack{h \in H \\ \|h\| \le 1}} |f(h)| \le \frac{1}{\|z\|} < 1 \quad \Rightarrow \|z\| \le 1 \quad \Rightarrow \|z\| = 1$

and $x = \|f\| z$

WE MAY ASSUME $\|f\| = 1$ FROM START SET $g = \frac{f}{\|f\|}$ ~~IF~~

~~$\|f\| = 1$ THEN WE WANT TO SHOW $\|z\| = 1$~~ then $\|g\| = 1$,

~~i.e. NEED $\|z\| \le 1$~~. $g(x) = 1$ thus $\|x\| = 1 = \|f\|\,\|z\|$. done.

LET X & Y BE BANACH SPACES AND T, A CONTINUOUS LINEAR MAP, $T^* : Y^* \to X^*$ IS CALLED THE <u>ADJOINT</u>. IT IS THE UNIQUE MAP SUCH THAT THE FOLLOWING IS TRUE:

$$\langle Tx, y^* \rangle = \langle x, T^* y^* \rangle$$

IF $T: H \to H$ THEN $T^*: H^* \to H^*$

IF P IS THE PROJECTION ONTO M, THE $P^* = P$ (SELF CONJUGATE)

Oct 18, 1976

Trying to show $P^* \equiv P$, where P is the projection $H \longrightarrow H$ onto M.

$$h_1 = m_1 + n_1 \qquad h = m + n$$

$$\langle h_1, P^* h \rangle = \langle P h_1, h \rangle = \langle m_1, h \rangle = \langle m_1, m+n \rangle$$
$$= \langle m_1, m \rangle + \langle m_1, n \rangle$$
$$= \langle m_1, m \rangle = \langle m_1, m \rangle + \langle n_1, m \rangle$$
$$= \langle m_1 + n_1, m \rangle = \langle h_1, m \rangle$$

We have shown $\forall\, h = m+n$, $\forall\, x \in H$, $\langle x, P^* h \rangle = \langle x, m \rangle$. both $P^* h$ & m can be considered linear functions in H^*.

If $P^* h \neq m$, then there would exist $x \in H$ with $(P^* h)(x) \neq m(x)$. Otherwise, $\langle x, P^* h \rangle \neq \langle x, m \rangle$

$\therefore$ $P^* h = m$. Since $Ph = m$, we conclude $P \equiv P^*$

<u>Theorem</u>: If X is normed & Y is a Banach space Then $B(X, Y) = \{A : X \to Y : A \text{ linear bounded}\}$ with norm $\|A\| = \sup\limits_{\substack{\|x\| \leq 1 \\ x \in X}} \|Ax\|$ is a Banach space.

<u>Proof</u> $A+B$, λA defined pointwise

$$\|A+B\| \le \|A\|+\|B\|$$

$$\|\lambda A\| = |\lambda|\cdot\|A\|$$

$$\|A\| \ge 0 \text{ \& } \|A\| = 0 \Leftrightarrow A = 0$$

Finally we show completeness. Choose $\{T_n\} \subseteq B(X,Y)$ where $\{T_n\}$ is a Cauchy sequence s.t.

$$\|T_n - T_m\| \to 0 \quad \text{as } m, n \to \infty$$

Let $x \in X$. $\|T_n x - T_m x\| = \|(T_n - T_m)(x)\|$

$$\le \|T_n - T_m\| \cdot \|x\| \to 0 \text{ as } n, m \to \infty$$

$\therefore \{T_n x\}$ is a Cauchy sequence in Y

$T_n x \to y \in Y$.

Let $Tx = y$. $T: X \to Y$

T is linear since

$$T(x+y) = \lim T_n(x+y)$$
$$= \lim (T_n x + T_n y)$$
$$= \lim T_n x + \lim T_n y$$
$$= Tx + Ty$$

Scalar multiplication is similar.

T is bounded since

$\exists$ M s.t. $\forall n$ $\|T_n\| \leq M < +\infty$ (since $\{T_n\}$ is a Cauchy sequence).

Let $x \in X$, $\|x\| \leq 1$

$$\|T_n x\| \leq M$$

hence $\|Tx\| = \lim \|T_n x\| \leq M$.

This shows $\|T\| \leq M$

Now Only need $\|T_n - T\| \to 0$

Let $x \in X$, $\|x\| \leq 1$.

Let $\varepsilon > 0$, $\exists N$ s.t. $m, n \geq N \Rightarrow \|T_n - T_m\| < \varepsilon$

hence $\|T_n x - T_m x\| < \varepsilon$.

let $m \to +\infty$, $\|T_n x - Tx\| \leq \varepsilon$.

$\|T_n - T\| \leq \varepsilon$ (since x is arbitrary)

The proof is complete.

$$X^* \xleftarrow{T^*} Y^*$$

$$X \xrightarrow{T} Y \qquad T \in B(X, Y)$$

<u>Theorem</u> If $T \in B(X, Y)$, then $T^* \in B(Y^*, X^*)$ and
$\|T\| = \|T^*\|$.

<u>Proof</u> $\underline{\underline{1}}$. $T^*: Y^* \to X^*$

$\forall \varphi \in Y^*$, $T^*\varphi : X \to \mathbb{R}$ is the composite of φ & T^*; hence $T^*\varphi$ is continuous & linear; thus in X^*.

T^* is linear since

$$\begin{aligned} T^*(\varphi + \psi) &= (\psi + \psi)(T) \\ &= \varphi T + \psi T \\ &= T^*\varphi + T^*\psi \end{aligned}$$

The scalar multiplication is similar.

$(\lambda T)^* : Y^* \to X^*$

if $\varphi \in Y^*$, $(\lambda T)^*(\varphi) = \varphi(\lambda T) = \lambda \varphi(T) = \lambda T^*\varphi$

$\therefore (\lambda T)^* = \lambda T^*$

similarly, $(T + S)^* = T^* + S^*$

Estimate $\|T^*\|$:

Let us assume $\|T\| = 1$

Let $y^* \in Y^*$ s.t. $\|y^*\| \le 1$. Want to show

$$\|T^* y^*\| \le 1 \Rightarrow \|T^*\| \le 1$$

$T^* y^* \in X^*$. Let $x \in X$ s.t. $\|x\| \le 1$

$$\|(T^* y^*)(x)\| = \|y^*(Tx)\| \le \|y^*\| \cdot \|Tx\|$$
$$\le \|y^*\| \, \|x\| \cdot \|T\|$$

in this case $\|T\| = 1$

In general $\|T^*\| \le \|T\|$.

Since $\|T\| = 1$, $\forall \varepsilon > 0$ $\exists x \in X$ with $\|x\| \le 1$ s.t.

$$\|Tx\| \ge 1 - \varepsilon$$

$Tx \neq 0$ element of Y

By H-B theorem, $\exists y^* \in Y$ with $\|y^*\| = 1$ and

$$y^*(Tx) = \|Tx\|$$

Evaluate T^* at y^*.

$$\|T^* y^*\| \ge |T^* y^*(x)|$$
$$= |y^*(Tx)|$$
$$= \|Tx\| > 1 - \varepsilon$$

Thus $\|T^*\| > 1 - \varepsilon$; hence $\|T^*\| = 1$.

General case S :-

Case I: $S \equiv 0 \Rightarrow S^* = 0$

$$\|S\| = \|S^*\| = 0$$

Case II: $S \neq 0 \Rightarrow \|S\| \neq 0$

$$T = \frac{1}{\|S\|} S \qquad \|T\| = 1$$

$$T^* = \frac{1}{\|S\|} S^*$$

$$1 = \|T^*\| = \frac{1}{\|S\|}\|S^*\| \Rightarrow \|S\| = \|S^*\|$$

<u>Corollary</u>. $B(X, Y) \to B(Y^*, X^*)$

$$T \longmapsto T^*$$

is a linear isometry into.

$$|\langle Tx, y^*\rangle| = |\langle x, T^* y^*\rangle|$$

$$\le \|T\| \cdot \|x\| \cdot \|y^*\| \qquad X^* \xleftarrow{1_{X^*}} X^*$$

$$|\langle x, y^*\rangle| \le \|x\| \cdot \|y^*\| \qquad X \xrightarrow{1_X} X$$

$$\begin{tikzcd} X^* & Y^* \arrow[l, "T^*"'] \end{tikzcd}$$

$\operatorname{Im} T^*$ everything $\qquad \ker T^*$ $\{0\}$

$$\begin{tikzcd} X \arrow[r, "T"] & Y \end{tikzcd}$$

$\ker T$ $\{0\}$ $\qquad \operatorname{Im} T$ everything

$M \subseteq X$. Let $M^{\perp} \subseteq X^*$ defined by

$$M^{\perp} = \{ f \in X^* \mid f(m) = 0 \quad \forall\, m \in M \}$$

$$M^{\circ} = \{ f \in X^* \mid |f(m)| \le 1 \quad \forall\, m \in M \}$$

Note that if M is a subspace then $M^{\perp} = M^{\circ}$.

$N \subseteq X^*$. Let $N^{\top} \subseteq X$ defined by

$$N^{\top} = \{ x \in X \mid n(x) = 0 \quad \forall\, n \in N \}$$

(book use ${}^{\circ}N$)

If $M \subseteq X$ and $N \subseteq X^*$, then $M^\perp$ is a closed subspace of X^* and $N^\top$ is a closed subspace of X.

proof: Let $f, g \in M^\perp$. Let $m \in M$.

$(f+g)(m) = f(m) + g(m) = 0$

$(\lambda f)(m) = \lambda(f(m)) = 0$

$\therefore M^\perp$ is a subspace of X^*

Let $\{f_n\} \subseteq M^\perp$, let $m \in M$. Suppose $f_n \to f$ in norm.

$\|f_n(m) - f(m)\| \leq \|f_n - f\| \, \|m\| \longrightarrow 0.$

But $f_n(m) = 0$. Hence, $f(m) = 0$.

A similar argument establishes the result for $N^\top$. ∎

If M is a closed subspace of X, then $M = (M^\perp)^\top$.

proof: Let $m \in M$, $f \in M^\perp$. Then $f(m) = 0$. Hence, $m \in M^{\perp\top}$.

Let $m \notin M^\perp$. By the Hahn-Banach Theorem, there is a f in X^* with $f|_M = 0$ and $f(m) \neq 0$.

Note: $f \in M^\perp$

Thus, $m \notin M^{\perp\top}$, since $f(m) \neq 0$. ∎

Remark: This is not true for $M^{\top\perp}$. If $M \subseteq X^*$, with M a closed subspace and $M \subsetneq M^{\top\perp}$

there are examples

Let A be a set. Let $M = \text{span } A$. Then $A^\perp = M^\perp$.

proof: $\left.\begin{array}{l} f \in M^\perp \\ f|_M = 0 \end{array}\right\} \Rightarrow f|_A = 0 \Rightarrow f \in A^\perp$. Thus $M^\perp \subset A^\perp$.

Let $f \in A^\perp$, $m \in M$. Since M = linear span of A, $\exists$ scalars $\alpha_1, \dots, \alpha_n$ and $a_1, \dots, a_n \in A$ such that $m = \sum_{i=1}^{n} \alpha_i a_i$. $\quad f(m) = \sum_{i=1}^{n} \alpha_i f(a_i) = 0$.

$\therefore M^\perp = A^\perp$ ■

$$A \subseteq \mathbb{X} \Rightarrow A^{\perp T} = \cancel{cl(A)} \text{ cl lin span}(A)$$

proof:

(i) Special case – A is a subspace

$A \subseteq \bar{A} \Rightarrow A^\perp \supseteq \bar{A}^\perp \Rightarrow A^{\perp T} \subseteq \bar{A}^{\perp T} = \bar{A}$.

$A \subseteq A^{\perp T} \subseteq \bar{A}$. But $A^{\perp T}$ is a closed subspace.

Hence, $A^{\perp T} = \bar{A}$

(ii) General case: let $M = \text{span } A$

$A^\perp = M^\perp$.

$A^{\perp T} = M^{\perp T}$ = closure of M = closed linear span A ■

Let $\left\{\begin{array}{l} \mathbb{X} \xrightarrow{A} \mathbb{Y} \\ \mathbb{X}^* \xleftarrow{A^*} \mathbb{Y}^* \end{array}\right\}$ where $\mathbb{X}, \mathbb{Y}, \mathbb{X}^*, \mathbb{Y}^*$ are normed spaces and A, A^* are continuous linear maps.

Then (1) $\ker(A) = (\text{range } A^*)^T$
(2) $\ker(A^*) = (\text{range } A)^\perp$
(3) $\text{range } A^* \subseteq (\ker A)^\perp$
(4) $\text{range } A \subseteq (\ker A^*)^T$

Further, if range A is closed, $\text{range } A = (\ker A^*)^T$

proof: (1) $x \in \ker A \Leftrightarrow Ax = 0 \Leftrightarrow \forall y^* \in Y^*, y^*(Ax) = 0$
$\Leftrightarrow (A^*y^*)(x) = 0 \Leftrightarrow x \in (\text{range } A^*)^T.$

(2) $y^* \in \ker A^* \Leftrightarrow A^*y^* = 0 \Leftrightarrow \forall x \in X,$
$(A^*y^*)(x) = 0 \Leftrightarrow \forall x \in X, y^*(Ax) = 0 \Leftrightarrow y^* \in (\text{Range } A)^\perp$

(3) is like (4)

(4) Let $x \in X$, $y^* \in \ker A^*$
$(A^*y^*)(x) = 0 \Leftrightarrow y^*(Ax) = 0 \Longrightarrow Ax \in [\ker A^*]^T$
Thus $(\text{range } A) \subseteq [\ker A^*]^T$

Now suppose range A is closed. We already know range A is a subspace.

$$\text{range } A = \overline{\text{range } A} = [\text{range } A]^{\perp T}. \ \blacksquare$$
$$= [\ker A^*]^T$$

Easy Corollaries

(1) ($\ker A^* = \{0\}$) iff range A is dense in Y

proof: range A is dense in $Y \Rightarrow \text{range}(A) \subseteq \overline{\text{range } A} = (\text{range } A)^{\perp T} = (\ker A^*)^T.$

$\ker A^* = \{0\} \Rightarrow \overline{\text{Range } A}$ is everything $\Rightarrow$ range A is dense in Y

(2) range A is dense iff A^* is 1-1

(3) range A^* is dense $\Rightarrow$ A is 1-1

(3) is <u>not</u> "iff"! Let $\ell_\infty \xleftarrow{A^*} \ell_1$.
$\ell_1 \xrightarrow{A} c_0$

$A((\xi_n)) = (\xi_n)$, $A^*((\xi_n)) = (\xi_n)$

A is continuous, linear, and 1-1 has is A^*
closure Range $A^* \subsetneq \ell_\infty$.
in particular $(1,1,1\ldots) \in \ell_\infty \setminus$ closure Range A^*.

Definition: A set V in a normed space is a barrel if V is closed, absorbing, balanced, and convex.

Theorem: In a Banach space, every barrel contains 0 as an interior point.

proof: Let V be a barrel in X.

claim: $X = \bigcup_{n=1}^{\infty} nV$.

proof: $x \in X$, V absorbing $\Rightarrow \exists\ \varepsilon > 0$ such that if $|\lambda| < \varepsilon$, $\lambda x \in V \Rightarrow \exists\ n \in N$ such that $\frac{1}{n} x \in V \Rightarrow x \in nV$.

The Baire Category Theorem says $\exists$ an interior point of closure nV.
V closed $\Rightarrow nV$ is closed.
$x \in$ interior $(nV) \Rightarrow \frac{x}{n} \in$ interior V

Notes taken by Ed Cooley 10/22/76

We continue with the proof of the theorem (p. 79); we have shown that $\exists\, x \in \operatorname{int}(V)$ (using the fact that V is closed and absorbing). To show $0 \in \operatorname{int}(V)$, we note that

$S_{-1} : \mathbb{X} \to \mathbb{X}$ defined by $S_{-1}(x) = -x$ is a homeomorphism and $S_{-1}(V) = V$ (since V is balanced); thus $-x \in \operatorname{int}(V)$. Hence $\exists\, \epsilon > 0 \ni$ if $y \in \mathbb{X}$ and $\|y\| < \epsilon$, then $-x+y \in V$ & $x+y \in V$.

Claim: $\|z\| < \epsilon \Rightarrow z \in V$.

Proof of claim: $\|z\| < \epsilon \Rightarrow x+z \in V$ & $-x+z \in V$ $\Rightarrow$ (since V is convex) $z = \frac{1}{2}(x+z) + \frac{1}{2}(-x+z) \in V$. QED

BANACH OPEN MAPPING THEOREM: If $\mathbb{X}$ & $\mathbb{Y}$ are B-spaces and $T : \mathbb{X} \to \mathbb{Y}$ is 1-1 onto & continuous, then T^{-1} is continuous (T is open).

Proof: for each $\epsilon > 0$ define $B_\epsilon^{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq \epsilon\}$. Then $\exists\, \delta > 0 \ni$ $cl\left[T(B_\epsilon^{\mathbb{X}})\right] \supset B_\delta^{\mathbb{Y}}$. In particular, $T(B_1^{\mathbb{X}})$ is a convex balanced set.

<u>Claim:</u> $T(B_1^X)$ is also absorbing.

<u>Proof of claim:</u> Let $y \in Y$. Since T is onto, $\exists$ $x \in X$ with $Tx = y$. Letting $\epsilon' = \frac{1}{\|x\|}$ we have $\lambda x \in B_1^X$ if $|\lambda| \le \epsilon'$; hence $T(\lambda x) = \lambda Tx \in T(B_1^X)$ for $|\lambda| \le \epsilon'$, so $T(B_1^X)$ is absorbing. QED.

Thus $K = cl.[T(B_1^X)]$ is a barrel. Hence $\exists\, \delta > 0 \ni B_\delta^Y \subset K$. Now $B_\epsilon^X = \epsilon B_1^X$, so $T(B_\epsilon^X) = \epsilon T(B_1^X)$

$\Rightarrow \overline{T(B_\epsilon^X)} = \epsilon\, \overline{T(B_1^X)}$. Now if δ works for $\varepsilon = 1$ then

$\therefore\ \overline{T(B_\epsilon^X)} \supset \epsilon B_\delta^Y = B_{\delta\epsilon}^Y$.

To finish the proof it suffices to show $\exists\, \delta' > 0 \ni T(B_1^X) \supset B_{\delta'}^Y$. Note that, this would imply, for $y \in Y$, $\|y\| \le \delta' \Rightarrow \|T^{-1}(y)\| \le 1$, and hence T^{-1} is cont. Let $y \in B_{\delta'}^Y$. Pick a sequence $\{x_n\} \subset X$ and a sequence $\{y_n\} \subset Y$. We want $y_n \to y$, and we want $\{x_n\}$ to be a Cauchy sequence in $X \ni Tx_n = y_n$. Hence $\exists\ x = \lim_{n\to\infty} x_n$, and $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y$.

IDEA OF PROOF DETAILS DIFFER A LITTLE

Note $\|x_n\| \le 1 \Rightarrow \|x\| \le 1$. Let $\delta' = \frac{1}{2}\delta$. Since $T(B_{\frac{1}{2}}^X)$ is dense in $B_{\delta'}^Y = B_{\frac{\delta}{2}}^Y$ we can pick $x_1 \in B_{\frac{1}{2}}^X$ (i.e.,

$\|x_1\| \le \frac{1}{2}$) $\ni$ $y_1 = Tx_1$ satisfies $\|y_1 - y\| < \delta/4$. Pick $x_2 \in B^{X}_{1/4}$ $\ni$ $y_2 = Tx_2$ and $\|y - y_1 - y_2\| < \delta/8$. In general $x_n \in B^{X}_{\frac{1}{2^n}}$ $\ni$ $y_n = Tx_n$ satisfies

$$\|y - y_1 - y_2 - \cdots - y_n\| < \frac{\delta}{2^{n+1}} \quad \text{and} \quad \|x_n\| \le \frac{1}{2^n}.$$

Let $x = \sum_{n=1}^{\infty} x_n$. This exists by the Cauchy criterion, since $\sum_{n=1}^{\infty} \|x_n\| \le \sum_{1}^{\infty} \frac{1}{2^n} = 1$. $\therefore \|x\| \le 1$. Thus

$$\left\| y - T\left(\sum_{i=1}^{N} x_i\right) \right\| = \|y - y_1 - y_2 - \cdots - y_N\| < \frac{\delta}{2^{N+1}} \to 0 \text{ as } N \to \infty.$$

Hence $Tx = T\left(\lim_{N\to\infty} \sum_{1}^{N} x_n\right) = \lim_{N\to\infty} T\left(\sum_{1}^{N} x_n\right) = y$.

QED.

<u>Corollary</u>: if $A: X \to Y$ is a continuous linear 1-1 map between B-spaces, then A has closed range $\Leftrightarrow$ A is an isomorphism into.

Consider $M_\lambda : l_2 \to l_2$, where $\lambda = (\lambda_n)_n$ with each $\lambda_n \neq 0$ defined by $M_\lambda(\xi_n) = (\lambda_n \xi_n)$. We have noted that M_λ is continuous if $\sup_n |\lambda_n| < \infty$.

NOTE: if $(\lambda_n) \to 0$ then M_λ is not onto, because if it were then M_λ^{-1} would be continuous; i.e., there would exist $K \ni \|M_\lambda^{-1} x\| \le K\|x\|$. Equivalently,

$$\|y\| \le K\|M_\lambda(y)\| \qquad (*).$$

But if we consider $e_n = (0, \dots, 0, \underset{n^{\text{th}}\text{ spot}}{1}, 0, \dots, 0)$ then $\|e_n\| = 1$, $M_\lambda(e_n) = \lambda_n e_n$ and $\|\lambda_n e_n\| = |\lambda_n| \to 0$; this contradicts $(*)$.

Consider the following ordinary differential equation with constant coefficients:

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' = f.$$

Suppose we know that for continuous f and initial values $y(0), y'(0), \dots, y^{(n-1)}(0)$ $\exists$ a unique solution.

Consider $L : C^n[0,1] \to C[0,1] \times \mathbb{R}^n$ defined by

$$L(y) = \left(y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y', \; y(0), y'(0), \dots, y^{(n-1)}(0)\right),$$

where the norm on $C[0,1] \times \mathbb{R}^n$ is defined by

$$\|f, \vec{a}\|_\infty = \max\left(\|f\|_\infty, \|\vec{a}\|_\infty\right).$$

<u>Claim</u>: L is continuous, linear, 1-1 and onto.

Proof of claim: continuous because we can rescale the a's to have norm ≤ 1; linear since our differential equation is linear; onto because of the existence of a solution; 1-1 since this solution is unique. "QED"

We can then conclude L^{-1} is continuous. If

~~$f_1, y_1(0), y_2(0$~~

$$\| (f_1, y_1(0), \ldots, y_1^{(n-1)}(0)) - f_2, y_2(0), \ldots, y_2^{(n-1)}(0) \| < \epsilon$$

then the solutions y_1 & y_2 satisfy $\|y_1 - y_2\| \leq \|L^{-1}\| \epsilon$.

This is sometimes called continuity with respect to data.

QUOTIENT SPACES. Let X be a normed space and let $M \subset X$ be a closed subspace. The quotient normed space X/M is the set of equivalence classes

$$\{ M + x : x \in X \}.$$

This is a vector space (we will show this later) with norm $\| M + x \| = \inf_{m \in M} \| m + x \|$.

Th. If M is a closed subspace of a normed space X, then X/M with $\|M+x\| = \inf\{\|m+x\| : m \in M\}$ is a normed space.

Pf. There are lots of details to check. We will only verify that $(M+x)+(M+y)$ is well-defined and that $\|\cdot\|$ is a norm.

note: $(M+x)+(M+y) = M+x+y$.

For if $m_1+x \in M+x$, $m_2+y \in M+y$, then

$$(m_1+x)+(m_2+y) = (m_1+m_2)+x+y \in M+x+y.$$

So $(M+x)+(M+y) \subseteq M+x+y$.

Also $M+x+y \subseteq (M+x)+(M+y)$.

We must show that if $M+x_1 = M+x$ and $M+y_1 = M+y$, then $M+x+y = M+x_1+y_1$.

We have $x = m+x_1$, $y = m'+y_1$, $m, m' \in M$.

Then
$$\begin{aligned} M+x+y &= M+(m+x_1)+(m'+y_1) \\ &= (M+m+m')+x_1+y_1 = M+x_1+y_1 \end{aligned}$$

since for $m \in M$, $m+M = M$.

We now show $\|\cdot\|$ is a norm.

$\|M+x\| \geq 0$ since it is the inf. of non-negative numbers.

$$\|M+\lambda x\| = \inf\{\|m+\lambda x\| = |\lambda|\cdot\|\tfrac{m}{\lambda}+x\| : m\in M \Leftrightarrow \tfrac{m}{\lambda}\in M\}$$

$$= |\lambda|\cdot\inf\{\|n+x\| : n\in M\}$$

$$= |\lambda|\cdot\|M+x\|.$$

$$\|M+x+y\| = \inf\{\|m+x+y\| : m\in M\}$$

$$\le \inf\{\|m+x\|+\|n+y\| : n,m\in M\}$$

$$\le \inf\{\|m+x\| : m\in M\} + \inf\{\|n+y\| : n\in M\}$$

$$= \|M+x\| + \|M+y\|.$$

$\ni$ $\|M+x\| = 0$.

Then $\operatorname{dist}(M,x) = \inf\{\|x-m\| : m\in M\} = 0$
$\Rightarrow x\in\overline{M} = M$.

Th. The quotient map $Q : X \to X/M \ni$
$Q(x) = M+x$ is continuous, linear, onto, and open with $\|Q\| = 1$ (if $M\ne X$).

Pf. Q is linear since
$Q(x+y) = M+x+y = (M+x)+(M+y) = Q(x)+Q(y)$.
Scalar multiplication is similar.

It is obvious that Q is onto.

If $\|x\| \le 1$, $\|M+x\| \le \|0+x\| = \|x\| \le 1$, since $0 \in M$.
Hence $\|Q\| \le 1$. Thus Q is continuous.

To show Q is open, it is sufficient to show $Q\{x \in \mathfrak{X} : \|x\| < 1\} = \{M+x : \|M+x\| < 1\}$.

Let $\|M+x\| < 1$.
Since $\|M+x\| = \inf\{\|m+x\| : m \in M\}$,
$\exists\, m \in M$ with $\|m+x\| < 1$.

$Q(m+x) = M+m+x = M+x$.
So Q is open.

We now show that if $M \subsetneqq \mathfrak{X}$, then $\|Q\| = 1$.
[Note: If $M = \mathfrak{X}$, then $\mathfrak{X}/M = \{0\}$ and $\|Q\| = 0$.]

Let $M+x \ne 0$, and $\|M+x\| = 1$.
$\forall \epsilon > 0\ \exists\, m \in M$ with $\|m+x\| < 1+\epsilon$.
Since Q is cont., $\|Q(m+x)\| \le \|Q\| \cdot \|m+x\|$.
Hence $1 \le \|Q\| \cdot \|m+x\| \le \|Q\| \cdot (1+\epsilon)$,
and $\dfrac{1}{1+\epsilon} \le \|Q\|$.

Since this is true $\forall \epsilon > 0$, we have $1 \le \|Q\|$
$\Rightarrow \|Q\| = 1$.

<u>Th</u>. If $\mathfrak{X}$ is a B-space, so is $\mathfrak{X}/M$.

<u>Pf</u>. It suffices to show that if $\{M+x_n\}$ is a sequence in $\mathfrak{X}/M \ni \sum_{n=1}^{\infty}\|M+x_n\| < +\infty$, then $\exists\, y \in \mathfrak{X}$ with $M+y = \lim_{N\to+\infty} \sum_{1}^{N}(M+x_n)$.

Let $\{M+x_n\}$ be such a sequence.
Choose $y_n \in \mathfrak{X} \ni y_n = m_n + x_n$ and $\|y_n\| \le 2\cdot\|M+x_n\|$.

Since $\mathfrak{X}$ is complete and
$$\sum_{n=1}^{\infty}\|y_n\| \le 2\sum_{n=1}^{\infty}\|M+x_n\| < +\infty,\ \exists\, y = \lim_{N\to\infty}\sum_{1}^{N} y_n.$$

Then
$$Q(y) = Q\Big(\lim_{N\to\infty}\sum_{1}^{N} y_n\Big)$$
$$= \lim_{N\to\infty}\Big(\sum_{1}^{N} Q(y_n)\Big) \qquad \text{(since } Q \text{ is cont.)}$$
$$= \lim_{N\to\infty}\Big(\sum_{1}^{N}(M+x_n)\Big).$$

<u>Th</u>. Suppose $\mathfrak{X}, Y$ are B-spaces, $T: \mathfrak{X} \to Y$ is a continuous linear map which has a closed range. Then Range T is isomorphic to a quotient of $\mathfrak{X}$.

<u>Pf</u>.
$$\begin{tikzcd} \mathfrak{X} \arrow[r, "T"] \arrow[d, "Q"'] & \text{Range } T \arrow[r, "\subset"] & Y \\ \mathfrak{X}/\ker T \arrow[ur, "\tilde{T}"'] & & \end{tikzcd}$$

$T = \tilde{T}\circ\Phi$

Range T is a B-space since it is closed.

$\tilde{T}$ exists by homework problem.

$\tilde{T}$ is continuous, onto, and one-one.
So by the open mapping theorem,

$$\mathfrak{X}/_{\ker T} \approx \text{Range } T.$$

Th. Suppose $\mathfrak{X}$ with $\|\cdot\|_1$ is a B-space and $\mathfrak{X}$ with $\|\cdot\|_2$ is a B-space, and suppose $\exists M \ni \forall x \in \mathfrak{X}$, $\|x\|_1 \leq M \cdot \|x\|_2$.
Then $(\mathfrak{X}, \|\cdot\|_1) \approx (\mathfrak{X}, \|\cdot\|_2)$, or
$\exists k$ with $x \in \mathfrak{X} \Rightarrow \|x\|_2 = k \cdot \|x\|_1$.

Pf. If $1_{\mathfrak{X}}$ is the identity map on $\mathfrak{X}$, then we have

$$(\mathfrak{X}, \|\cdot\|_1) \xrightarrow{1_{\mathfrak{X}}} (\mathfrak{X}, \|\cdot\|_2),$$

where $1_{\mathfrak{X}}$ is one-one, onto, and continuous (since $\|x\|_1 \leq M \cdot \|x\|_2$).
So by the open mapping theorem, its inverse is continuous.

Th. (Closed Graph Theorem).
Let $\mathfrak{X}$ & $\mathcal{Y}$ be B-spaces, $T: \mathfrak{X} \to \mathcal{Y}$ linear, and T has closed graph. Then T is continuous.

Def. The graph of T is the set $\{(x, Tx) \in \mathfrak{X} \times \mathcal{Y}\}$.
Give $\mathfrak{X} \times \mathcal{Y}$ the norm $\|(x,y)\| = \sup\{\|x\|, \|y\|\}$.

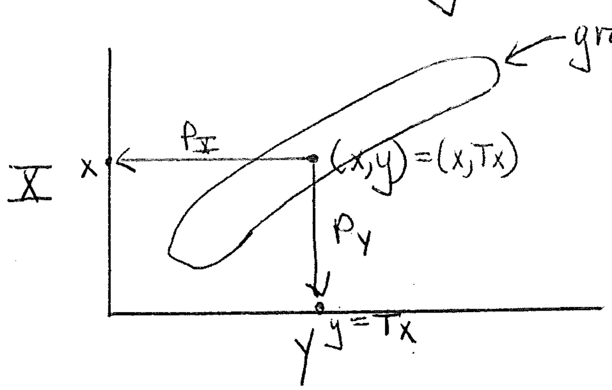
This makes $\mathfrak{X} \times Y$ a B-space.

Pf. of Th.

T has a closed graph $\Leftrightarrow \{(x, Tx)\}$ is closed in $\mathfrak{X} \times Y$

$$\Leftrightarrow \begin{cases} (x_n, Tx_n) \to (x, y) \\ \Rightarrow Tx = y \end{cases}$$

$$\Leftrightarrow \begin{cases} x_n \to x \text{ and } Tx_n \to y \\ \Rightarrow Tx = y. \end{cases}$$



Note: $P_{\mathfrak{X}} : \mathfrak{X} \times Y \to \mathfrak{X}$
$\ni (x, y) \longmapsto x$
& $P_Y : \mathfrak{X} \times Y \to Y$
$\ni (x, y) \longmapsto y$
are cont. linear fcns.

Further, graph of T is a closed subspace of $\mathfrak{X} \times Y$, hence a B-space.

Restrict $P_{\mathfrak{X}}$ and P_Y to the graph of T. Then you obtain cont. linear fcns.

$P_{\mathfrak{X}}|$graph of T is 1-1, onto, cont. $\Rightarrow$ by the open mapping th.,

$S : X \to$ graph $T \ni x \mapsto (x, Tx)$ is cont.

$P_y \circ S(x) = P_y(x, Tx) = Tx$. So,

$T = P_y \circ S$ - thus T is continuous.

<u>Principle of Uniform Boundedness</u> (Banach-Steinhaus Th.):

If X is a B-space,

$W = \{T : X \to Y$ cont., linear maps$\}$

$\ni \forall x \in X$, $\sup_{T \in W} \|Tx\| < +\infty$,

then $\exists$ k with $\|T\| < K$ for all $T \in W \overset{df}{\Leftrightarrow}$ W is uniformly bounded.

Pf. Let U = unit ball in Y,

$V = \bigcap_{T \in W} T^{-1}(U)$.

U is closed, balanced, and convex $\Rightarrow$ $T^{-1}(U)$ is closed, balanced, and convex $\Rightarrow$ V is closed, balanced, and convex.

If V is absorbing, then V is a barrel.

So $\exists \epsilon > 0 \ni \|x\| < \epsilon \Rightarrow x \in V$, $\|Tx\| \leq 1 \; \forall T \in W$.

So $\|T\| \leq \frac{1}{\epsilon}$.

It remains to show T is absorbing

<u>proof that V is aborbing.</u>

let $x \in X$. Want $\exists\, \varepsilon > 0$. s.t. $|\lambda| < \varepsilon \Rightarrow \lambda x \in V$.

let $\sup_{T \in W} \|Tx\| = \frac{1}{\varepsilon}$. Then

$$|\lambda| < \varepsilon \Rightarrow \sup_{T \in W} \|T\lambda x\| = |\lambda| \sup_{T \in W} \|Tx\| < 1$$

$\Rightarrow$ $T(\lambda x) \in U$ for each $T \in W$.

$\Rightarrow$ $\lambda x \in V$

$\Rightarrow$ V is absorbing. q.e.d

Thus V is a barrel.

$\Longrightarrow$ $\exists\, \varepsilon > 0$, s.t $\|x\| < \varepsilon$ implies $x \in V$

$\Longrightarrow$ $\|Tx\| \leq 1$ $\forall\, T \in W$.

$\Longrightarrow$ $\|T\| \leq \frac{1}{\varepsilon}$ Q.E.D of Banach, Steinhaus

\#

<u>Thm</u> If $\mathbb{R}^n$ $n = 0, 1, 2, \cdots$ is given any T_2 (Hausdorff) T.V.S topology & Y is any T.V.S. and $T : \mathbb{R}^n \longrightarrow Y$ is linear. Then T is continuous.

<u>Cor</u> If <u>Thm</u> is true for n, then there is only one TVS T_2 topology on $\mathbb{R}^n$.

i.e $(\mathbb{R}^n, \mathcal{J}_1) \xrightarrow{1_{\mathbb{R}^n}} (\mathbb{R}^n, \mathcal{J}_2)$; homeomorphism.

<u>pf of Thm</u>

By induction.

$n=0$, $\mathbb{R}^0 = \{0\}$ So <u>Thm</u> is obvious true.

<u>Lemma</u> If <u>Thm</u> is true for $\mathbb{R}^n$, then every functional f on $\mathbb{R}^{n+1}$ is continuous.

<u>pf of Lemma</u>

If $f \equiv 0$, f is continuous.

So assume $f \not\equiv 0$.

Let $K = \ker f$.

Note that it is a n-dimensional subspace

So, by the induction hypothesis, the restriction of the topology on $\mathbb{R}^{n+1}$ to K is homeomorphic to $\mathbb{R}^n$ with usual topology.

Since $\mathbb{R}^n$ with usual Top. is complete. and complete sets are closed, $K = \ker f$ is closed. (proof delayed)

[~~This doesn't really prove~~ ~~In fact,~~ ~~no~~ statement above ~~has~~ a ~~halt~~ ~~since completeness is more than Topological Property~~]

Thus f is continuous. done with lemma #

Let $y \in Y$. consider $f \in (\mathbb{R}^{n+1})^*$

<u>claim</u> map $x \in \mathbb{R}^{n+1} \longmapsto f(x)\, y \in Y$ is continuous

<u>pf of claim</u>

$$\mathbb{R}^{n+1} \xrightarrow{f} \mathbb{R} \xrightarrow{\text{injection into } 1^{st} \text{ factor } \mathbb{R}\times\{y\}} \mathbb{R}\times Y \xrightarrow{\text{scalar multi}} Y$$

$$x \longmapsto f(x),\quad r \longmapsto (r, y),\quad (\lambda, y) \longmapsto \lambda y$$

f is continuous by lemma.

$\therefore$ $x \longmapsto f(x) \longrightarrow (f(x), y) \longrightarrow f(x)\, y$

is continuous since Y is TVS.

On $\mathbb{R}^{n+1}$, define functional f_i, $i = 1, 2, \cdots n+1$.

s.t $f_i(x_1, \cdots x_{n+1}) = x_i$: linear. Conti.

Let $T : \mathbb{R}^{n+1} \longrightarrow Y$.

Let $y_i = T((0, 0 \cdots \underset{i\text{th}}{1}, 0, \cdots 0))$.

Then $T(X) = \sum_{i=1}^{n+1} x_i y_i = \sum_{i=1}^{n+1} f_i(X)\, y_i \qquad X = (x_i)$

Thus T is sum of conti. ftion.

$\therefore$ T is continuous. Q.E.D of Thm

Note ; That if X is anormed space of infinite dimension, there are lots of linear functionals $X \longrightarrow \mathbb{R}$ that are not continuous

Hence there are a lot of dense Hyperplane.

<u>Thm</u> If $f: X \longrightarrow \mathbb{R}^n$, X: normed sp. is continuous iff $\ker T$ is closed

pf Next time.

Thm. Let X be a normed space and $T: X \to \mathbb{R}^n$ be linear. Then T is continuous iff $\ker T$ is closed.

Proof Suppose T is continuous and consider the inverse image of a closed set.

Conversely we can assume T is onto [i.e. the range of T is finite dimensional].

Then consider $X \xrightarrow{T} \mathbb{R}^n$ and note:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\varphi"'] & \mathbb{R}^n \\ X/\ker T \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

1) $\tilde{T}: X/\ker T \longrightarrow \mathbb{R}^n$ is onto since T is.

2) $\tilde{T}$ is 1–1 since $\ker T$ is divided out.

Thus $X/\ker T$ is finite dimensional $(\dim = n)$ since $\tilde{T}$ is 1–1 and onto.

And hence $\tilde{T}$ is an isomorphism which implies $T = \tilde{T} \circ \varphi$ is continuous.

OPERATORS (continuous linear functions ≡ 'maps').

Let X and Y be normed spaces and then the maps $T: X \to Y$ with $T(X)$ finite dimensional are called <u>operators of finite rank.</u>

<u>Finite rank operators</u>

<u>example:</u> 1) $f \in X^*$ and $y \in Y$, then $T: X \to Y \ni$
$T(x) = f(x) \cdot y$ is a finite rank operator [has 1-dim range iff $f \neq 0$].

2) $\{f_i\}_1^n$ with $f_i \in X^*$ $i = 1, 2, \dots n$
and $\{y_i\} \in Y$. Then $T(x) = \sum_1^n f_i(x) y_i$ $(*)$
has range dimension $\leq n$.

<u>Proposition</u> EVERY finite rank operator can be written in the form $(*)$.

<u>Proof</u> Consider $T: X \to Y$ a finite rank operator.
Let $Z = T(X)$ where Z has dimension n.
Let $\{y_1, y_2, \dots y_n\}$ be Hamel basis for Z.

Let $g_1, g_2, \ldots g_n$ be elements of Z^* where $g_i\left(\sum_{j=1}^{n} \alpha_j y_j\right) = \alpha_i$, $i = 1, 2, \ldots n$. Then extend g_i to $\tilde{g}_i \in Y^*$ by Hahn-Banach and let $f_i = T^* \tilde{g}_i$ for $i = 1, 2, \ldots n$, $[f_i \in X^*]$.

Now show $T(x) = \sum_{i=1}^{n} f_i(x) y_i$ for $x \in X$. Notice that $\tilde{g}_i(T(x)) = T^* \tilde{g}_i(x) = f_i(x)$. But $T(x) = \sum_1^n g_i(T(x)) y_i$, and that g_i and $\tilde{g}_i$ are the same on Z.

Hence $T(x) = \sum_{i=1}^{n} f_i(x) y_i$.

<u>Notation</u>. $T(x) = f(x) y$ is sometimes written $T = f \otimes y$.

(i.e. above becomes $T(x) \equiv T = \sum_{i=1}^{n} f_i \otimes y_i$

Also note the abundance of finite rank operators (consider the collection of linear functionals).

<u>Thm</u>
Let $T : X \to Y$ have finite rank. Then $T^* : Y^* \to X^*$ has finite rank.

<u>Proof</u> Let $T(x) = \sum_{i=1}^{n} f_i(x) y_i$ and $y^* \in Y^*$.

Then $\langle T(x), y^* \rangle = \langle x, T^*(y^*) \rangle$ which implies

$$\langle \sum_1^n f_i(x)\, y_i, y^* \rangle = \sum_{i=1}^n f_i(x) \langle y_i, y^* \rangle$$
$$= \sum_{i=1}^n \langle f_i, x \rangle \langle y_i, y^* \rangle \text{ where}$$

$f_i \in \mathbb{X}^*$ and $y_i \in \mathbb{Y}^{**}$ [i.e. $= \sum_1^n y_i(y^*) f_i(x) = \langle x, \sum_1^n y_i(y^*) f_i \rangle$]

Thus $\forall x \in \mathbb{X}$; $T^*(y^*) = \langle x, \sum_1^n y_i(y^*) f_i \rangle$ and

hence $T^*(y^*) = \sum_1^n y_i(y^*) f_i$.

$\therefore$ T^* has finite rank.

<u>Lemma</u> Let $T: \mathbb{X} \to \mathbb{Y}$ be an operator, and consider

$$T^*: \mathbb{Y}^* \longrightarrow \mathbb{X}^*$$
$$T^{**}: \mathbb{X}^{**} \longrightarrow \mathbb{Y}^{**}.$$

Note: can think of $\mathbb{X}$ as a subspace of $\mathbb{X}^{**}$ and then taking T^{**} restricted to $\mathbb{X}$ gives T. (i.e. $T^{**}\big|_{\mathbb{X}} = T$)

<u>Proof</u> Let $x \in \mathbb{X}$, then $x \in \mathbb{X}^{**}$ and thus $\underset{\substack{\uparrow \\ \in \mathbb{X}^{**}}}{x}(x^*) = x^*(x)$.

Now $\underset{\substack{\uparrow \\ \in \mathbb{Y}^{**}}}{\langle T^{**}(x)}, y^* \rangle = \langle \underset{\substack{\uparrow \\ \in \mathbb{X}^{**}}}{x}, T^* y^* \rangle = \langle \underset{\substack{\uparrow \\ \in \mathbb{X}}}{x}, T^* y^* \rangle = \langle \underset{\substack{\uparrow \\ \in \mathbb{Y}}}{T(x)}, y^* \rangle.$

And $\langle T(x), y^* \rangle = \langle T(x)\ y^* \rangle$ ∎.

$\uparrow \in Y$ $\qquad$ $\uparrow \in Y^{**}$

$\left[\text{i.e } T(x^{**}) = T(x) = T(x)\right].$

$\uparrow \in Y^{**}$ $\quad$ $\uparrow \in Y$

<u>Corollary</u>

The operator T has finite rank iff T^* has finite rank.

<u>Proof</u> $\Longrightarrow$ Use <u>Thm</u>.

$\Longleftarrow$ Consider T^{**} with finite rank (by Thm).

Then $T = T^{**}\big|_X$ and hence has finite dimensional range.

<u>Example</u> Consider c_0 and it's dual l_1.
Let T be an operator $T: l_1 \to l_1 \ni$ for all $S: c_0 \to c_0$; $S^* \neq T$.

Let $\left.\begin{array}{l} y = (1, 0, 0, \dots) \in l_1 \\ f = (1, 1, 1, \dots) \in l_\infty \end{array}\right\}$ and consider $T: l_1 \to l_1$
such that $T(x) = f(x)\, y$. $\left[\text{i.e. } T \text{ takes } \{\xi_n\} \to \left(\sum_1^\infty \xi_n, 0 \dots\right)\right]$

Thm. Finite rank operators have closed range.

(i.e. Range has finite dimension and complete set in finite topology is closed).

Thus $\text{Range}(T') = (\ker T'^{*})^{\top}$.

Thm. Let X be a normed space. Then for every finite ~~dimensional~~ dimensional supspace; or closed subspace of finite co-dimension, say M, there is a continuous linear operator $P : X \longrightarrow X \ni P^2 = P$ and the $\text{Range}(P) = M$; while $P|_M = 1_M$. Also $\ker(P) + M = X$, with $\ker P \cap M = \{0\}$.

Proof M finite dim. and $\{y_i\}$ a Hamel Basis implies $g_j(\sum \alpha_i y_i) = \alpha_j \quad g_j \in M^*$; and by the Hahn-Banach thm g_i can be extended to $\tilde{g}_i \in X^*$.

Let $P(x) = \sum_1^n \tilde{g}_i(x) y_i$. Then $\text{Range}(P) = M$ and hence $P|_M = 1$. It should be clear that P is continuous and one can check that $P^2 = P$.

"Back to page 100"

If one consider the map $T: \ell_1 \longrightarrow \ell_1$ s.t.

$T: (\xi_n) \longrightarrow \left(\sum_{n=1}^{\infty} \xi_n, 0, 0, \ldots \right)$. That is to say,

$T(x) = f(x)y$, $f = (1,1,1, \ldots)$, $y = (1,0,0,\ldots)$.

Claim: If $S: c_0 \longrightarrow c_0$, then $S^* \neq T$

Proof: suppose $S: c_0 \longrightarrow c_0$ and $S^* = T$, then $S^{**} = T^*$, and

$$T^*(x) = y(x) f. \quad : \ell_\infty \longrightarrow \ell_\infty$$

though of ℓ_∞

But since $S^{**} = T^*$ then $T^*|_{c_0} = S$

i.e $T^*(c_0) \subseteq c_0$ since $S(c_0) \subseteq c_0$

Now look at
$x = (1,0,0,0, \ldots) \in c_0$

$T^*(x) = y(x) f = f = (1,1,1,1,\ldots) \notin c_0$
which contradict $T^*(c_0) \subseteq c_0$ then, $S^* \neq T$

Let Us Now Return To What We Just Start in P101.

$M \subseteq \underset{\text{normed}}{X}$ finite dim. and consider

$P : X \longrightarrow X$ s.t

Rang $P = M$, $P/M = 1$, $P^2 = P$ (idempotent)

and P continuas.

We had shown $\ker P \cap M = \phi$.

Our aim now is to show $\ker P + M = X$

Proof :-

Let $x \in X$ consider $x - P_x \in X$,

$x - P_x \in \ker P$ since $P(x - P_x) = P_x - P_x^2 = P_x - P_x = 0$.

But

$x = x - P_x + P_x \implies x \in \ker P + M$ since "Rang $P = M$".

<u>Lemma</u> : If $P : X \to X$ is continuos Proj. , $P^2 = P$. Then X is isomorphic to $\ker P \times \text{rang}\, P$

(i.e we can divide the space X into two subspaces such that X is isomorphic to the cross product of them.)

Proof

Define

$S : \ker P \times \operatorname{rang} P \longrightarrow X$ s.t

$S(x,y) \longrightarrow x+y$.

It is Clear that S is linear & Continuous.

Consider $Q = I - P$, and define

$T : X \longrightarrow \ker P \times \operatorname{rang} P$ s.t

$T(x) \longrightarrow (Qx , Px)$.

We need to Prove 1) T is the inverse of S
2) T is Cont, T is linear.

The linearity of T is Clear. To show T is cont., let us Look to Q^2,

$$Q^2 = (I-P)^2 = I^2 - PI - IP + P^2 = I - P - P + P = I - P = Q,$$

then Q is Continuous.

T is Cont. since if $x \in X$ with $\|x\| \leq 1$ then

$$\|(Qx, Px)\| = \max(\|Qx\|, \|Px\|) \leq \max(\|Q\|, \|P\|) = M < +\infty.$$

Now let us look at TS, ST

$$TS(x,y) = T(x+y) = \big(Q(x+y), P(x+y)\big) = (x,y)$$

since $\left.\begin{array}{l} x \in \ker P \\ y \in \text{Rang}\, P \end{array}\right\} \Rightarrow \begin{array}{l} Q(x+y) = (x+y) - P(x+y) = (x+y) - y = x, \\ P(x+y) = y. \end{array}$

Then
$$TS = 1_{\ker P \times \text{Rang}\, P}.$$

Also
$$ST(x) = S(Qx, Px) = Qx + Px$$
$$= (1-P)x + Px = x.$$

Then
$$ST = 1_X.$$

Thus T is isomorphism and we done.

Now; the case of <u>finite Co-dimension</u>:-

Suppose M has finite Co-dim. & is closed.

Inductively define, $x_1, x_2, \ldots\ldots x_{\text{Co-dim}\,M}$, and
$f_1, f_2, \ldots\ldots f_{\text{Co-dim}\,M}$.

Such that
$$f_i(M) = 0, \quad f_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

by the following:-

Let $x_1 \notin M$, let g_1 be defined as a function on $M + \lambda x_1$

$g_1(m + \lambda x_1) = \lambda$

g is cont. on span $[M, x_1]$, extend to f_1 on $\mathfrak{X}$.

Now span $[M, x_1]$ is closed in $\mathfrak{X}$

Pf: of theorem by induction on co-dim of M.

Start out of co-dim. of $M = 1$

$$P : \mathfrak{X} \longrightarrow \mathfrak{X}$$
$$m + \lambda x \longrightarrow \lambda x$$

$\ker P = M$ so P is cont. (finite-dim range).

$$P^2(m + \lambda x_1) = P(\lambda x_1) = \lambda x_1 = P(m + \lambda x_1).$$

Let us consider

$Q = I - P$ $\qquad Q^2 = Q$

$Q(m + \lambda x) = (m + \lambda x) - \lambda x = m.$ So

Range $Q = M$ & $\ker Q = \operatorname{Span} x_1$

*** Back to the ranch :-

Suppose $m_n + \lambda_n x_1 \xrightarrow{\text{converges}} y$.

$m_n = Q(m_n + \lambda_n x_1) \longrightarrow Qy = m_0$ (say),

on the other hand

$\lambda_n x_1 = P(m_n + \lambda_n x_1) \longrightarrow Py = \lambda_0 x_1$.

Thus $y = m_0 + \lambda_0 x_1 \in \text{span}[M, x_1]$

So that $\text{span}[M, x_1]$ is closed.

LEMMA

If $[M, x_1] \neq X$ & f_1 s.t $f_1|M = 0$,

$f_1(x_1) = 1$. Then

$\exists\, x_2 \in \ker f_1$ with $x_2 \notin \text{span}[M, x_1]$.

Pf:

Let $y_2 \notin \text{span}[M, x_1]$

Let $x_2 = y_2 - f_1(y_2)x_1$.

$f_1(x_2) = f_1(y_2) - f_1(y_2) f_1(x_1) = 0$

i.e $x_2 \in \ker f_1$. We need to show $x_2 \notin \text{span}[M, x_1]$

Now if $x_2 = m + \lambda x_1$ then

$$y_2 = x_2 + f_1(y_2) x_1$$
$$= m + (\lambda + f_1(y_2)) x_1 \in \text{span}[M, x_1]$$

which contradict $y_2 \notin \text{span}[M, x_1]$ then $x_2 \notin \text{span}[M, x_1]$.

Define:-

$g_2 : \text{span}[M, x_1, x_2]$ such that

$g_2(m + \lambda x_1 + \mu x_2) = \mu$.

g_2 is cont. since $\text{span}[M, x_1] = \ker g$ is closed.

<u>extend to F_2 on $\underline{X}$</u>

Recall the definition of $x_1, \ldots, x_{\text{co-dim}\, M}$, $f_1, f_2, \ldots, f_{\text{co-dim}\, M}$.

Let

$$P(y) = \sum_{i=1}^{\text{co-dim}\, M} f_i(y) x_i$$

$$P^2(y) = P\left(\sum_{i=1}^{\text{co-dim}\, M} f_i(y)\, x_i\right)$$

$$= \sum_{i=1}^{\text{co-dim}\, M} f_i(y)\, x_i = P(y)$$

P is Cont. Projection.

$\operatorname{Ker} P \supseteq M$ since $\operatorname{Ker} f_i \supseteq M$

suppose $x \notin M$ then

$x = m_0 + f_1(x) x_1 + \cdots + f_n(x) x_n$ then we have two cases

1) If $f_i(x) = 0 \quad \forall i$
$\Longrightarrow x = m_0$ i.e $x \in M$ (contradiction)

2) If $f_i(x) \neq 0$ for some i, then

$$P(x) = \sum f_i(x) Y_i \neq 0$$
$\Longrightarrow x \notin \operatorname{Ker} P$ since x_i are lin. indep.

$\sum c_i x_i = 0$, $c_j = f_j\left(\sum c_i x_i\right) = 0$

$\operatorname{Ker} P = M$ So $Q = I - P$ is the Projection Required.

Note that if $T : X \to Y$ is a finite rank operator and U = unit ball in X then;

$T(x)$ is compact in $\underline{Y}$.

The following are equiv. For

$T : X \to Y$. (normed spac) , and

U the unit ball of X.

1) $T(U)$ is relatively compact (closure is compact)

2) $cl(T(U))$ is compact

3) $T(U)$ is relatively sequnilly compact

4) For all $\{x_i\} \subseteq X$ with $\|x_i\| \leq M < +\infty$
$\{Tx_i\}$ has convergent subsequ.

5) $T(U)$ is totally bounded {If Y is complete}

Such mapping is called compact oparator.
operator satisfying (5) (Y complete or not) is called precompact
we will show next time
precompact opar. $\Rightarrow$ cont.

FACTS FROM TOPOLOGY

In a metric space X the following are equivalent for a set $A \subseteq X$

(1) A is rel compact (rel ≡ relatively)

(2) $\mathrm{cl}\,A$ is compact

(3) $\forall \{U_\alpha\}$ open cover of $\mathrm{cl}\,A$ $\exists\, U_1, \dots, U_n$ finite subset of $\{U_\alpha\}$ which covers $\mathrm{cl}\,A$.

(4) A is rel sequencely compact

(5) $\mathrm{cl}\,A$ is seq compact

(6) $\{x_n\} \subset A$ has a convergent subsequence

(7) $\{x_n\} \subset \mathrm{cl}\,A$ has a subsequence which convergences to element of $\mathrm{cl}\,A$

(8) Infinite subsets of A have limit points

Furthermore each of the above imply the following equivalent statements. The converse is also true if $\mathrm{cl}\,A$ is complete (in particular, if X is complete.)

(9) A is precompact (totally bounded)

(10) $\forall \varepsilon > 0\ \exists\, x_1, \dots, x_n \in A$ s.t. $\forall y \in A$ $\exists\, x_i$ s.t. $d(y, x_i) < \varepsilon$

(11) (10) with A replaced by $\mathrm{cl}\,A$

(12) The closure of A in the completion of X is compact.

The linear map $T: X \to Y$, X, Y normed spaces. $U = \{x : \|x\| \leq 1\}$, is called <u>precompact</u> if $T(U)$ satisfies any of (9) to (12). It is called <u>compact</u> if $T(U)$ satisfies any of (1) to (5).

Note that the two conditions are equivalent if Y is a B-space.

Example: Operator of finite rank are compact, because the usual topology on $\mathbb{R}^n$ is locally compact. In a norm space this is equivalent to saying that the closed unit ball is compact.

$T: l_f \longrightarrow l_f$ (l_f both with norm $\|\cdot\|_\infty$)

$(\xi_n) \xrightarrow{T} (\xi_n/n)$. T is precompact but not compact.

$T(U) = \{(\eta_n) : \eta_n = 0$ except finitely often & $|\eta_n| \le 1/n\}$. The sequence $x^n = (1, 1/2, \ldots, 1/n, 0, 0, \ldots)$ belongs to $T(U)$ but does not have a convergent subsequence in l_f because $x^n \to (1, 1/2, \ldots, 1/n, 1/n+1, \ldots) \in c_0$. $\overline{T(U)}$ in $c_0 = \{\xi_n \in c_0$ with $|\xi_n| \le 1/n\} \equiv B$.

Pick $\varepsilon > 0$ and N such that $\frac{1}{N} < \varepsilon$.

Consider $B_N = \{\xi_n \in B : \xi_n \equiv 0 \quad n \ge N\}$.

$B_N \subseteq \mathbb{R}^{N-1}$ & bounded there. Thus compact. So there exists $x_1, \ldots, x_m \in B_N$ such that for $\forall x \in B_N$ $\exists\, i$ for which $\|x_i - x\| < \varepsilon$. Pick $y \in B$ let $y^N = \begin{cases} y_i & i < N \\ 0 & i \ge N \end{cases}$ then

$\|y - y^N\| < \frac{2}{N} < 2\varepsilon$. $\exists\, x_i$ with $\|x_i - y^N\| < \varepsilon$ so $\|y - x_i\| < 3\varepsilon$. ∎

<u>Theorem</u> If $T: \mathfrak{X} \to \mathfrak{Y}$ is precompact then T is continuous.

Proof: $T(U)$ is precompact. Let $\varepsilon > 0$ and $x_1, \dots, x_m \in T(U) \ni y \in T(U) \Rightarrow \exists x_i$ with $\|x_i - y\| < \varepsilon$. Let $M = \max(\|x_i\| \; i=1,\dots,m) + \varepsilon$. Claim $\|T\| \leq M < \infty$. If $y \in T(U)$ then $\exists x_i$ with $\|x_i - y\| < \varepsilon$ and $\|y\| \leq \|x_i\| + \|y - x_i\| \leq M$. Thus T is cont. ∎

Let $K(X,Y) = \{T \in B(X,Y) : T \text{ is compact}\}$ then $K(X,Y)$ is a closed subspace of $B(X,Y)$. Thus if Y is a B-space then $K(\mathfrak{X}, Y)$ is a B-space.

Proof: Let $K: \mathfrak{X} \to \mathfrak{Y}$ precompact. means $K(U)$ is precompact. We want to show $\lambda K(U)$ is precompact.

If $\lambda = 0$ $\lambda K(U) = \{0\}$ and there is not problem. If $\lambda \neq 0$. Let $\varepsilon > 0$. Since $\varepsilon/|\lambda|$ $\exists x_1, \ldots, x_m$ such that $y \in K(U) \Rightarrow$ $\exists x_i$ with $\|x_i - y\| < \varepsilon/|\lambda|$. Let $w \in \lambda K(U)$ $w = \lambda y$ some $y \in K(U)$. $\exists x_i$ with $\|x_i - y\| < \varepsilon/|\lambda|$. So $\|w - \lambda x_i\| = \|\lambda y - \lambda x_i\| = |\lambda| \|x_i - y\|$ $< \varepsilon$. So $\lambda x_1, \ldots, \lambda x_n$ works for $\lambda K(U)$. —

Suppose K_1 & K_2 are precompact. Let $\varepsilon > 0$ Show $(K_1 + K_2)(U) = K_1(U) + K(U)$ is precompact. Let $x_1, \ldots, x_m$ work for K_1 and $y_1, \ldots, y_m$ work for K_2. Both with $\varepsilon/2$. Claim $x_i + y_j$ $i = 1, \ldots, m$, $j = 1, \ldots, n$ work for $K_1 U + K_2 U$ with $\varepsilon > 0$. Let $y = z_1 + z_2$ $z_i \in K(U)$ Let x_i be such that $\|x_i - z_i\| < \varepsilon/2$

and y_j be such that $\|y_j - z_i\| < \epsilon/2$

Then $\|y - (x_i + y_j)\| \leq \|x_i - z_1\| + \|y_j - z_2\| < \epsilon$ ∎

<u>Corollary</u>: The sum of two precompact sets is precompact. Also the scalar multiplication of precompact set is precompact.

Suppose T_n are precompact $: \mathfrak{X} \to \mathfrak{Y}$ & $\|T_n - T\| \to 0$ as $n \to \infty$. Let $\epsilon > 0$. and N be such that $\|T - T_N\| < \epsilon/3$

Let $x_1, \ldots, x_m \in U$ = unit ball of $\mathfrak{X}$ such that $\forall y \in U$ $\exists x_i$ $\|T_N(y) - T(x_i)\| < \epsilon/3$. Consider $y \in U$ pick x_i as above. then $\|Ty - Tx_i\| \leq \|Ty - T_N y\| + \|T_N y - T_N x_i\| + \|T_N x_i - Tx_i\| < \|T - T_N\| \|y\| + \epsilon/3 + \|T_N - T\| \|x_i\| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$

<u>Lemma:</u> If X is a normed space where U, the unit ball, is precompact. Then $\dim X < +\infty$.

Proof: Let $\epsilon = 1/2$. Pick $x_1, \ldots, x_m \in U$ such that $\forall y \in U$ there exists x_i such that $\|x_i - y\| < 1/2$. Let $Y = \text{span}\{x_1, \ldots, x_m\}$. Note that $\dim Y < +\infty$. So Y is a closed subspace of X. If $Y = X$ we have finish. Otherwise consider X/Y

$\varphi : X \longrightarrow X/Y$. We know $\|\varphi\| = 1$ so $\exists\ u \in U \ni \|\varphi(u)\| > 3/4$ but $\exists\, x_i$ with $\|u - x_i\| < 1/2$. This implies

$3/4 < \|\varphi(u)\| = \varphi(u - x_i)\| < 1/2$ $\times$

Suppose $T : X \to Y$ is precompact and has closed range. Then if X and Y are complete, T is of finite rank.

Proof: We may suppose T is onto

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "q"'] & Y \\ X/\mathrm{Ker}\,T \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

$\tilde{T}$ is cont, 1-1, and onto. Since $\tilde{T}(q(U)) = T(U)$ it is also precompact. By the Open mapping Theorem $\tilde{T}$ is an isomorphism. Thus Y has a precompact unit ball ($\cancel{\Phi(U)}$). Thus Y is finite dimensional i.e T is of finite rank.

Problem Set 4 Due Monday 22 Nov

1. Let $\lambda = (\lambda_n)$ & Let $M_\lambda : l_2 \to l_2$ be the linear map $M_\lambda((\xi_n)) = (\lambda_n \xi_n)$. Show M_λ is compact $\Longleftrightarrow \lim_{n\to\infty} \lambda_n = 0$.

2. Let X & Y be B-spaces with X a subspace of Y. Show X^* is isometric to $Y^*/X^\perp$ and $(Y/X)^*$ is isometric to $X^\perp$ (a subspace of Y^*).

3. Let J be the set of sequences $(\xi_n) \in c_0$ such $\|(\xi_n)\| < +\infty$ where $\|(\xi_n)\|$ is defined to be the sup of $\left[\sum_1^{n-1} (\xi_{p_{i+1}} - \xi_{p_i})^2\right]^{1/2}$ where $p_1 < p_2 < \cdots < p_n$ are an increasing sequence of integers and $n = 2, 3, 4, \ldots$.
Show $\|\cdot\|$ is a norm, and J is a B-space with this norm.

4. Let $x_n = (\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}, \underbrace{0, 0, \ldots}_{\text{zero thereafter}})$ $n = 1, 2, \ldots$
Show $\{x_n\}$ is a basis for c_0.

5. In J show that $P_N((\xi_n)) = \eta_n = \begin{cases} \xi_n & n \le N \\ 0 & n > N \end{cases}$ are norm one projections.
Suppose we define $(\xi_n) \le (\eta_n)$ iff $\xi_n \le \eta_n$ $n = 1, 2, \ldots$
Show that in J, for each M there are sequences $(\xi_n), (\eta_n) \in J$ with $(0) \le (\xi_n) \le (\eta_n)$ $[0 \le \xi_n \le \eta_n \text{ all } n]$ $\|(\eta_n)\| \le 1$ but $\|(\xi_n)\| \ge M$.
Finally show for $(\xi_n) \in J$ $(\xi_n) = \lim_{N\to+\infty} P_N((\xi_n))$.

Thm: If $T: X \to Y$ is ctn linear map Then

(1) T is precompact $\Leftrightarrow$

(2) T^* is precompact $\Leftrightarrow$

(3) T^* is compact

Pf: (2) $\Leftrightarrow$ (3)

Since X^* is complete, to finish the prove it suffices to show

(1) $\Rightarrow$ (2) for then (2) $\Rightarrow$ (3) via

given T^* precompact $\Rightarrow$ $T^{**}: X^{**} \to Y^{**}$ precompact

$T: X \to Y$

Since $T^{**}|_X = T$

$T(U_X) \subset T^{**}(U_{X^{**}})$ in Y^{**}.

and since subsets of pre-compact set are precompact. so T is precompact.

(1) $\Rightarrow$ (2), Let T be precompact $\varepsilon > 0$, $\exists$ $x_1, x_2, \ldots, x_n$ $\|x_i\| < 1$ $\ni$

$\forall x \in X$, $\|x\| \le 1$, $\exists$ i $\ni$:

$\| Tx - Tx_i \| < \epsilon$ $(*)$

Let $W = \{ y^* \in Y^*, \|y^*\| \le 1 \}$

$S: Y^* \to (\mathbb{R}^n, \sup\|\cdot\|)$

$y^* \to (y^*(Tx_1), y^*(Tx_2), \ldots, y^*(Tx_n))$

S is cts and have finite rank so it is pre-compact.

$\exists\, y_1^*, \cdots y_m^* \in W,\ y^* \in \overline{W} \Rightarrow \exists\, j \ni: \| S y^* - S y_j^* \| < \epsilon$ (i.e. $\forall x,\ | y^*(Tx) - y_j^*(Tx) | < \epsilon$)

Consider $\| T^* y^* - T^* y_j^* \|$ where j as above

$$= \sup_{\substack{x \in X \\ \|x\| \le 1}} | T^*_{y^*}(x) - T^*_{y_j^*}(x) |$$

Note that: $| T^* y^*(x) - T^*_{y_j^*}(x) | = | y^*(Tx) - y_j^*(Tx) |$

$$\le | y^*(Tx) - y^*(Tx_i) | + | y^*(Tx_i) - y_j^*(Tx_i) | + | y_j^* T(x_i) - y_j^*(Tx) |$$

$$\le \|y^*\| \, \|Tx - Tx_i\| + \epsilon + \|y^*\| \, \|Tx - Tx_i\|$$

$$\le 3\epsilon.$$

Thus, T^* is pre-compact.

Open questions:

(1) X & Y be B-spaces, $B(X, Y)$ has compact [nuclear] operators, does it have any more? If $\dim X = \dim Y = \infty$)

Defn: T: nuclear operator $T: X \to Y$, B-spaces if $\exists \{f_n\} \subset X^*, \{y_n\} \subset Y \ni:$

$$\sum_{n=1}^{\infty} \|f_n\| \, \|y_n\| < \infty \ni: T(x) = \sum_{n=1}^{\infty} f_n(x)\, y_n.$$

$T = \lim T^N$ where $T^N(x) = \sum_1^N f_n(x)\, y_n$.

(2) $B(X)$ has operator of the form $\lambda I + K$. λ: scalar, K compact

are any more if $\dim X = \infty$.

(3) Is there a cts projection P on $\mathfrak{X}$ $\ni$:

$\dim(\text{Range } P) = \dim \ker P = +\infty$.

Special type of B-space:

Suppose Banach space $\mathfrak{X}$ has a sequence of cts projections $\{P_n\}_{n=1}^{+\infty}$ $\ni$:

(1) $P_n P_m = P_m \cdot P_n = P_{\min(m,n)}$.

(2) $x \in \mathfrak{X}$ then $x = \lim P_n x$

~~if dim(Range P_n is finite for $n=1,2,$~~

Then $\exists M \ni: \|P_n\| \leq M$. $n = 1, 2, \cdots$

E.g. (1) $\mathfrak{X} = l_p$ or c_0

$$P_N(\xi_n) = (\eta_n) = \begin{cases} \xi_n & n \leq N \\ 0 & \text{otherwise} \end{cases}$$

E.g. (2) l_∞ has property (1) but not (2).

If $\nexists M$ $\|P_n\| \leq M$ $n = 1, 2, \ldots$ then
By the principle of uniform boundedness, $\exists x \in \mathfrak{X}$ with

$\sup_n \|P_n x\| = +\infty$. But $\|x\| = \lim_{n \to \infty} \|P_n x\|$ where $\|P_n x\|$ are bounded since P_n's are continuous, This is impossible done with the proof.

Let $x \in X$, $x_{n+1} = (P_{n+1} - P_n)(x)$

$$P_n(x) = P_n(x) - P_{n-1}(x) + P_{n-1}(x)$$

$$= x_n + P_{n-1}x, \text{ so by induction } P_N x = \sum_1^N x_n, \; \text{or } X = \sum_1^\infty X_n$$

Math 534 Nov 8, 1976

Suppose X, Y, Z are normed spaces, T, S are continuous linear

$$X \xrightarrow{T} Y \xrightarrow{S} Z$$

$$\|ST\| \le \|S\| \cdot \|T\|$$

for if $x \in X$, $\|x\| \le 1$

$$\|S(Tx)\| \le \|S\| \cdot \|Tx\| \le \|S\| \cdot \|T\| \cdot \|x\|$$

If S is precompact [compact] or T is precompact [compact], then ST is precompact [compact].

<u>proof</u>. If T is precompact [compact], then

$$ST(U) = S(T(U)) \text{ and}$$

$$\overline{S(T(U))} \subset S(\overline{TU})$$

so suffices to show the continuous linear image of precompact [compact] is precompact [compact].

Let A be precompact set and S be continuous linear. Let $\varepsilon > 0$, pick $x_1, x_2, \dots, x_n$ in A s.t. $\forall x \in A \; \exists i$ s.t.

$$\|x - x_i\| < \varepsilon / \|S\| .$$

then $\|Sx - Sx_i\| < \varepsilon$, so that $Sx_1, \dots, Sx_n$ works for $S(A)$

If S is precompact, then S maps balls about origin into precompact sets.

In particular, $T(U) \subseteq B_{\|T\|}$

$ST(U)$ is contained in the precompact set $S(B_{\|T\|})$

<u>Lemma</u> A precompact & $B \subseteq A \Rightarrow B$ precompact.

<u>Proof.</u> Let $\varepsilon > 0$. Find $x_1, \cdots, x_n$ s.t.

$$x \in A \Rightarrow \exists\, x_i \text{ with } \|x - x_i\| < \varepsilon/2$$

for each x_i let $y_i \in B$ (if one exists) s.t. $\|x_i - y_i\| < \varepsilon/2$. Now $y_1, \cdots, y_n$ (maybe few missing) work for B, since $y \in B$ $\exists\, x_i$ s.t. $\|y - x_i\| < \varepsilon/2$ and hence

$$\|y - y_i\| \le \|y - x_i\| + \|x_i - y_i\| < \varepsilon.$$

$B(X)$ is a ring (a algebra) ~~if (Banach space)~~

$T, S \in B(X) \Rightarrow ST \in B(X)$

In this ring, $K(X)$ = precompact maps is a ideal.

Continuous projections $\{P_n\}$ on Banach space X

$P_n P_m = P_m P_n = P_{\min(m,n)}$

$P_n x \to x \quad \forall x \in X$

We have shown that $\exists\, M$, $\|P_n\| \le M$

Let $Q_1 = P_1$

$Q_n = P_n - P_{n-1} \qquad n = 2, 3, \cdots\cdots$

$\|Q_n\| \le 2M$

Are continuous projections,

$$x = \sum_{1}^{\infty} Q_n x$$

Let $X_n = Q(X)$ ~~be~~ which is a closed subspace of X.

closure of linear space $\bigcup_{n=1}^{\infty} X_n = X$

If $x_n \in X_n$ and $\sum_{n=1}^{\infty} x_n = x$, then $x_n = Q_n(x)$

$$Q_n Q_m = (P_n - P_{n-1})(P_m - P_{m-1})$$

$$= P_n P_m - P_n P_{m-1} - P_{n-1} P_m + P_{n-1} P_{m-1}$$

$$= P_n - P_n - P_{n-1} + P_{n-1} = 0 \quad \text{if } n < m$$

$$Q_n(X_n) = 0$$

Since Q_N is continuous,

$$Q_N(x) = Q_N\left(\sum_{1}^{\infty} x_n\right) = \lim_{M\to\infty} Q_N\left(\sum_{1}^{N} x_n\right) = x_N$$

$$\|$$

$$Q_N\left(\sum_{1}^{\infty} Q_n x\right) = \lim_{M\to\infty} Q_N\left(\sum_{1}^{M} Q_n x\right) = Q_N x$$

Define a sequence of closed subspaces $\{X_n\} \subseteq X$ (Banach space) as decomposition if

$\forall x \in X$, $\exists$ unique sequence $x_n \in X_n$ with $x = \sum_{1}^{\infty} x_n$

<u>Example</u>

Let $A_1, A_2, \ldots$ be infinite subsets of $\mathbb{N}$ with

$$A_i \cap A_j = \phi \text{ for } i \neq j \quad \& \quad \bigcup_{i=1}^{\infty} A_i = \mathbb{N}$$

For $X = l_p$, $1 \le p < \infty$ or $X = c_0$, define

$$P_m(\{\xi_n\}) = \{\eta_m\} = \begin{cases} \xi_n & n \in \bigcup_{i=1}^{m} A_i \\ 0 & \text{o.w} \end{cases}$$

for each of these X's, $\|P_m\| = 1$

$$P_m P_n = P_n P_m = P_{\min(m,n)}$$

If $X \neq l_\infty$, then $P_m x \to x$ for $x \in X$ (ie $X = l_p$ $1 \le p < \infty$ or c_0)

X_n is isometric to X.

$$X_n = \{(\xi_i) \in X : \xi_i \neq 0 \Rightarrow i \in A_n\}$$

$$(\xi_1, \xi_2, \xi_3, \cdots\cdots)$$

$$\downarrow$$

$$(\xi_1, 0, \xi_2, 0, \xi_3, 0, \cdots\cdots)$$

$$l_2 \longrightarrow X = l_2 = \{(\xi_n) : \xi_n = 0 \ n \text{ even}\}$$

This shows (almost)

$$(l_2 \times l_2 \times \cdots\cdots)_{l_2} = l_2$$

<u>Definition</u>

Decomposition is finite dimensional decomposition (f.d.d)
if $\dim X_n < +\infty$, $n = 1, 2, \cdots$

If $\dim X_n = 1$, $n = 1, 2, \cdots$.
Let $e_n \in X_n \setminus \{0\}$, then $x \in X$, $x_n \in X_n \Rightarrow \exists$ scalars α_n
with $x_n = \alpha_n e_n$.

$\forall x \in X$ there is a unique sequence of scalars $\{\alpha_n\}$ with

$$x = \sum \alpha_n e_n$$

Examples: Let

$$e_n = (0, 0, \cdots, 0, 1, 0, \cdots\cdots),$$

$\uparrow$ nth slot

then $\{e_n\}$ is a basis for l_p, $1 \le p < \infty$ or c_0.

Theorem. If $\{e_n\}$ is a sequence of indep vectors with dense linear span in X (a Banach space); $\{e_n\}$ is a basis iff $\exists M$ s.t.

$$(*) \qquad \left\| \sum_1^p \alpha_i e_i \right\| \le M \left\| \sum_1^{p+q} \alpha_i e_i \right\|$$

for each sequence $\{\alpha_i\}$ & integers p, q.

Proof. $(\Leftarrow)$. Let $Y =$ linear span of $\{e_i\}_{i=1}^{\infty}$ on Y

Define P_N by

$$P_N(\sum \alpha_i e_i) = \sum_{i=1}^{N} \alpha_i e_i$$

Clearly, $P_N^2 = P_N$, P_N is linear map

The inequality $(*)$ says exactly that

$$\|P_N\| \le M \quad \text{for } N = 1, 2, 3, \cdots$$

Also it is seen that

$$P_N P_m = P_m P_N = P_{\min(m,N)}$$

Extend P_N to $\tilde{P}_N : X \to X$ by

$$x \in X \Rightarrow \exists (x_n) \subseteq Y \text{ with } x_n \to x \quad \text{and let}$$

$$\tilde{P}_N(x) = \lim_{x_n \to x} P_N(x_n)$$

LAST PROBLEM SET DUE 1 DEC

1. Let $\lambda = (\lambda_n) \in l_\infty$ and let $M_\lambda : l_2 \to l_2$ be the map $M_\lambda(\{\xi_n\}) = \{\lambda_n \xi_n\}$.
Find the set $\sigma(M_\lambda) = \{\mu \in \mathbb{R} : \mu I - M_\lambda$ does not have an cont inverse. $\}$

2. Suppose H is a Hilbert space and $\{x_n\}$ is a sequence of norm one vectors in H with dense linear span. Suppose $T : H \to H$ is a cont linear map, and $\{\lambda_n\}$ is a sequence of reals with $\lambda_n \neq \lambda_m$ if $m \neq n$ s.t. $Tx_n = \lambda_n x_n$. Show
(1) $\{x_n\}$ is an orthonormal basis for H
(2) $T = M_\lambda$ $\lambda = \{\lambda_n\}$ with resp to this basis.

3. Let $\mathfrak{X}_n$ $\|\cdot\|_n$ be a sequence of B-spaces show Y is a B-space where $Y = (\mathfrak{X}_1 \times \mathfrak{X}_2 \times \dots)_{l_2}$
$\stackrel{df}{=} \{(x_n) : x_n \in \mathfrak{X}_n$ and $\|(x_n)\| = \left(\sum_1^\infty [\|x_n\|_n]^2\right)^{\frac{1}{2}} < +\infty\}$

4. Let $\mathfrak{X} = l_f, \|\cdot\|_\infty$
(1) Find $f_n \in \mathfrak{X}^*$ with the property $x \in \mathfrak{X} \Rightarrow \sup_n |f_n(x)| < \infty$ but $\{\|f_n\|\}$ is not bounded
(2) Find $T : \mathfrak{X} \to \mathfrak{X}$ linear 1-1 onto ~~conts~~ ~~but~~ with closed graph but T is not cont.

5. Let $T : \mathfrak{X} \xrightarrow[onto]{} Y$ satisfy $\|x\| = \|Tx\|$ & T maps lines to lines (ie. $T\{tx + sy \mid t + s = 1\} \subset \{\lambda\xi + \mu\eta ; \lambda + \mu = 1\}$ $t\,s\,\lambda\,\mu \in \mathbb{R}$, $x, y \in \mathfrak{X}$ $\xi, \eta \in Y$ that is $\forall x, y\ \exists \xi, \eta$.)
Show T is linear. HINTS (1) Show T maps lines throu 0 into lines throu 0, (2) Show $T(\alpha x) = \alpha T(x)$, (careful $|\alpha| = \pm\alpha$)
(3) Show $T(\frac{1}{2}(x+y)) = \frac{1}{2}(Tx + Ty)$.

MAT 534 Notes taken by Ed Cooley 11/10 & 11/15/76

Note that $\tilde{P}_N(x) \in \bar{X}$ since $x_n \to x$ so that $\{P_N(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence and $\bar{X}$ is complete.

$\tilde{P}_N$ is well-defined: Suppose (x_n) & (y_n) are sequences in $\bar{Y}$ $\ni$ $x_n \to x$ & $y_n \to x$. Then $x_n - y_n \to 0$, hence $P_N(x_n - y_n) \to 0$. $\therefore \lim_{n\to\infty} P_N(x_n) = \lim_{n\to\infty} P_N(y_n)$.

$\|\tilde{P}_N\| = \|P_N\|$: we see easily that $\|\tilde{P}_N\| \geq \|P_N\|$; indeed, if $x \in \bar{Y}$ $\ni$ $\|x\| \leq 1$ then $x \in \bar{X}$, so $\sup \|\tilde{P}_N(x)\|$ is over a bigger set. To show $\|\tilde{P}_N\| \leq \|P_N\|$, let $x \in \bar{X}$ $\ni$ $\|x\| \leq 1$. Then $\exists$ a sequence (x_n) in $\bar{Y}$ $\ni$ $x_n \to x$; hence $\|x_n\| \to \|x\|$ and $\|P_N(x_n)\| \to \|\tilde{P}_N(x)\|$. But $\|P_N(x_n)\| \leq \|P_N\| \|x_n\| \Rightarrow$ for any $\epsilon > 0$ $\exists$ M_0 $\ni$ $\|x_n\| < 1+\epsilon$ for $n \geq M_0$. Hence for $n \geq M_0$ we have $\|P_N(x_n)\| < \|P_N\|(1+\epsilon)$ and thus $\|\tilde{P}_N(x)\| < \|P_N\|(1+\epsilon)$.

$\tilde{P}_m \tilde{P}_N = \tilde{P}_N \tilde{P}_m = \tilde{P}_{\min(m,N)}$: note that $P_N(\bar{X}) = \operatorname{span}\{e_1, \ldots, e_N\}$, which is closed in $\bar{X}$, hence $\tilde{P}_N(\bar{X}) = \operatorname{span}\{e_1, \ldots, e_N\}$.

The result follows since on $\operatorname{span}\{e_1, \ldots, e_m\}$ we have $\tilde{P}_N = P_N$.

As a special case we have $\tilde{P}_N^{\,2} = \tilde{P}_N$.

Now if we can show $x \in X \Rightarrow \tilde{P}_N(x) \to x$ then we're done. So let $\epsilon > 0$ be given. Since the $\{e_n\}$ have a dense linear span $\exists$ a sequence $\{\beta_n\}$ and $N \ni \left\|\sum_{i=1}^{N} \beta_i e_i - x\right\| < \epsilon$. For $M_0 \geq N$ we have

$$\left\|\tilde{P}_{M_0}\left(\sum_1^N \beta_i e_i - x\right)\right\| \leq \|\tilde{P}_{M_0}\| \left\|\sum_1^N \beta_i e_i - x\right\| \leq M\epsilon\ ;\ \text{i.e.,}$$

$$\left\|\sum_1^N \beta_i e_i - \tilde{P}_{M_0}(x)\right\| \leq M\epsilon. \text{ Then}$$

$$\|x - \tilde{P}_{M_0}(x)\| \leq \left\|x - \sum_1^N \beta_i e_i\right\| + \left\|\sum_1^N \beta_i e_i - \tilde{P}_{M_0}(x)\right\|$$
$$< \epsilon + M\epsilon = (1+M)\epsilon.$$

$\therefore\ \tilde{P}_N(x) \to x$, and by previous work (pp. 121-122 & 124-127) we conclude that $\{e_n\}$ is a basis. This finishes the proof of the $(\Leftarrow)$ direction.

To prove $(\Rightarrow)$, suppose $\{e_n\}$ is a basis for X; i.e., suppose $\forall x \in X$ $\exists$ a unique sequence $\{\alpha_n\}$ of scalars $\ni \sum_{i=1}^{\infty} \alpha_i e_i = x$ (equivalently, $\lim_{N\to\infty} \sum_1^N \alpha_i e_i = x$). Define $f_n : X \to \mathbb{R}$ by $f_n(x) = \alpha_n$ (f_n is the n^{th} coefficient functional).

If the (f_n) $n=1,2,\ldots$ are continuous, then the proof is simple by defining $P_N(x) = \sum_1^N f_i(x) e_i$; we would know the P_N are cts. projections, and previous work (same as above) would tell us that the P_N are uniformly bounded.

But it's hard to show that the (f_n) are cts, so we define a new space where they are cts.

Let $\hat{X} = \left\{ (\alpha_n) : \sum_{i=1}^{\infty} \alpha_i e_i \in X \right\}$ with $|||(\alpha_n)||| = \sup_N \left\| \sum_1^N \alpha_i e_i \right\|$

<u>Outline:</u> show (1) $\hat{X}$ is a normed space with the (f_n) cts.

(2) $\hat{X}$ is complete

(3) $T : X \to \hat{X}$ which maps $\sum \alpha_i e_i \longmapsto (\alpha_i)$ is a homeomorphism onto.

We then conclude the (f_n) are cts. on X.

First <u>note</u>: (1) $|||(\alpha_n)||| < \infty$ since $\left\| \sum_1^N \alpha_i e_i \right\| \to \left\| \sum_1^{\infty} \alpha_i e_i \right\|$.

(2) This implies that T^{-1} is cts., since

$$\left\| \sum_1^{\infty} \alpha_i e_i \right\| \leq \sup_N \left\| \sum_1^N \alpha_i e_i \right\| = |||(\alpha_n)|||.$$

(3) T is 1-1 & onto, since for each x there is a unique sequence (α_n) of scalars $\ni$ $\sum_1^{\infty} \alpha_i e_i = x$. It is also easy to show that T is linear.

<u>$|||\cdot|||$ is a norm</u>: (a) $|||(\alpha_n)||| \geq 0$, $= 0 \iff (\alpha_n) = 0$. Easy.

(b) Homogeneity is easy.

(c) Triangle inequality:

$|||(\alpha_n) + (\beta_n)||| = \sup_N \left\| \sum_1^N (\alpha_i + \beta_i) e_i \right\|$. But $\forall N$ we have

$$\left\| \sum_1^N (\alpha_i + \beta_i) e_i \right\| \leq \left\| \sum_1^N \alpha_i e_i \right\| + \left\| \sum_1^N \beta_i e_i \right\|$$

$$\Rightarrow \sup_N \left\| \sum_1^N (\alpha_i + \beta_i) e_i \right\| \leq \sup_N \left\| \sum_1^N \alpha_i e_i \right\| + \sup_N \left\| \sum_1^N \beta_i e_i \right\|.$$

(f_n) are cts. on $\hat{X}$: suppose $|||(\alpha_n)||| \le 1$; then

$$|f_n((\alpha_i))|\,|||(e_n)||| = |\alpha_n|\,|||e_n||| = |||\alpha_n e_n||| = |||\sum_1^n \alpha_i e_i - \sum_1^{n-1} \alpha_i e_i|||$$

$$\le |||\sum_1^n \alpha_i e_i||| + |||\sum_1^{n-1} \alpha_i e_i|||$$

Think of these as sequences in $\hat{X}$

But $\forall m$ we have $|||\sum_1^m \alpha_i e_i||| \overset{\text{def}^n}{=} \sup_{N \le m} \|\sum_1^N \alpha_i e_i\|$ ← in X

$$\le \sup_N \|\sum_1^N \alpha_i e_i\| = |||(\alpha_n)|||.$$

$\therefore |f_n((\alpha_i))|\,|||(e_n)||| \le 2$ and hence

$|||f_n||| \le \dfrac{2}{|||(e_n)|||}$. $\therefore$ the (f_n) are cts. on $\hat{X}$.

Assuming for now that we've shown all of (1), (2) & (3) as outlined above, we complete the proof as follows:

Since T is cts., $\exists\ M\ (= \|T\|) \ni$

$|||(\alpha_n)||| \le M\|\sum_1^\infty \alpha_i e_i\|$, hence

$$\|\sum_1^p \alpha_i e_i\| \le |||(\tilde{\alpha}_n)||| \qquad \text{where } \tilde{\alpha}_n = \begin{cases} \alpha_n & n \le p+1 \\ 0 & \text{o.w} \end{cases}$$

$\le M\|\sum_1^{p+q} \alpha_i e_i\|$ for all integers p, q.

We will show next time that $\hat{X}$ is complete.

SHOW $\hat{X}$ IS COMPLETE

$\hat{X} = \{ (\alpha_n) \mid \sum_{n=1}^{\infty} \alpha_n e_n \in X \}$

$|||(\alpha_n)||| = \sup_n \| \sum_{i=1}^{n} \alpha_i e_i \|$

LET $\alpha^m = \{\alpha_n^m\}$ BE A CAUCHY SEQUENCE IN X.

SINCE b_j IS CONTINUOUS ON X, (j TH COEFFICIENT FUNCTIONAL) $b_j(\alpha^m) = \alpha_j^m$ IS A CAUCHY SEQUENCE IN $\mathbb{R}$.

HENCE CONVERGES TO α_j^0.

CLAIM $\alpha^0 = \{\alpha_n^0\}$ IS THE LIMIT OF $\{\alpha_n^m\}$

LET $\epsilon > 0$. SINCE α^m IS A CAUCHY SEQUENCE, $\exists M \ni$ $m_1, m_2 \geq M \Rightarrow |||(\alpha^{m_1}) - (\alpha^{m_2})||| < \epsilon$. LET N BE ANY INTEGER, THEN $\| \sum_{j=1}^{N} (\underline{\alpha_j^{m_1} - \alpha_j^{m_2}}) e_j \| < \epsilon$ & $m_1, m_2 \geq M$ (ON SPAN $\{e_1, \cdots, e_n\}$ THIS CAN BE ~~THOUGHT OF~~ CONSIDERED AS A CONTINUOUS FUNCTION IN UNDERLINED VARIABLES). THEREFORE, WE HAVE $\| \sum_{j=1}^{N} (\alpha_j^{m_1} - \alpha_j^0) e_j \| \leq \epsilon$. (*)

IF $m_1 \geq M$, $\| \sum_{j=1}^{N} \alpha_j^0 e_j \| \leq \| \sum_{1}^{N} (\alpha_j^{m_1} - \alpha_j^0) e_j \| + \| \sum_{j=1}^{N} \alpha_j^{m_1} e_j \|$

$\leq \epsilon + |||\alpha_j^m|||$ (NOTE. INDEP. OF N)

THEREFORE $\sup_N \| \sum_{j=1}^{N} \alpha_j^0 e_j \| \leq \epsilon + |||\alpha_j^m||| \Rightarrow |||(\alpha_j^0)||| < \infty$

(*) also implies $|||\alpha^m - \alpha^0||| \to 0$

$\Rightarrow (\alpha_j^0) \in \hat{X}$ AND $\hat{X}$ IS COMPLETE.

$\boxed{\text{COROLLARY}}$ IF $\{e_j\}$ IS A BASIS FOR B-SPACE X THEN $\{b_n\}$ (THE COEFFICIENT FUNCTIONS) ARE CONTINUOUS.

<u>DEF:</u> NORMALIZED BASIS IS BASIS WITH $\|e_n\| = 1$ $n = 1, 2, 3, \cdots$

SUPPOSE $T: X \to Y$ IS LINEAR, X, Y ARE B-SPACES WITH NORMALIZED BASIS $\{x_n\}$ & $\{y_n\}$ RESPECTIVELY.
SUPPOSE $T(\sum_{i=1}^{\infty} \alpha_i x_i) = \sum_{i=1}^{\infty} \alpha_i \lambda_i y_{\pi(i)}$ FOR (α_i) FINITELY NON ZERO. IF $\sum \alpha_i x_i \in X$ THEN $\sum \alpha_i \lambda_i y_{\pi(i)} \in Y$ WHERE $\lambda = (\lambda_i) \in l_\infty$, $\pi: N \to N$ IS A PERMUTION, THEN T IS CONTINUOUS.

<u>THEOREM</u>: T IS BOUNDED IF $x \in X \Rightarrow \|T(x)\| < +\infty$

PROOF:
BY CLOSED GRAPH, LET $\xi^n \to \xi^0$ IN X AND $T\xi^n \to \eta^0$ IN Y. NEED ONLY TO SHOW:
IF $y_n^* =$ <u>NTH</u> COEFFICIENT FUNCTION $\{y_i\}$ IN Y
THEN $y_n^*(T\xi^0) = y_n^*(\eta^0)$

$$y_n^*(T\xi^0) = y_n^*(T(\sum \xi_j^0 x_j))$$
$$= y_n^*(\sum \xi_j^0 \lambda_j x_j) = \star$$

IF PICK $m \ni \pi(m) = m$, WE HAVE $\star = \xi_m^0 \lambda_m$

$\Rightarrow y_n^*(T\xi^k) = \xi_m^k \lambda_m$. SINCE $T\xi^k \to \eta^0$ THIS IMPLIES
$y_n^*(T\xi^k) \to y_n^*(\eta^0) = \eta_n^{n}$

$\xi_m^k \lambda_n \to \lambda_n$ BUT $\xi_m^k \to \xi_m^0$ BECAUSE X_n^* IS CONTINUOUS

$$\Rightarrow \xi_m^k \lambda_m \to \xi_m^0 \lambda_m$$
$$\Rightarrow \xi_m^0 \lambda_m = \eta_n$$

NOTE (λ_i) MUST BE IN l_∞.

<u>EXAMPLE</u>: $T: c_0 \to l_1$

$(\xi_n) \to (\gamma_n \xi_n)$ IF $T(c_0) \subseteq l_1$

THEN T WOULD BE CONTINUOUS HENCE BOUNDED.

BUT $T(\underbrace{1,1,\cdots,1}_{n \text{ TIMES}},0,0,\cdots) = (1, 1/2, \cdots, 1/n, 0, \cdots)$

$\|(\underbrace{1,1,\cdots,1}_{n \text{ TIMES}},0,\cdots)\|_\infty = 1$

$\|(1, 1/2, \cdots 1/n, 0, \cdots)\|_1 = \sum_{j=1}^{n} 1/j \to \infty$

T IS NOT CONTINUOUS

$(1/\ln j) \in c_0$ BUT $\left(\frac{1}{j \ln j}\right) \notin l_1$

$(\lambda_j) \in l_\infty \quad \lambda \in l_\infty \quad \|\lambda\|_\infty \le K$

$M_\lambda : l_p \to l_p$ IS CONTINUOUS

$\{x_n\} \in l_p \;, \; \{\lambda_n x_n\} \in l_p$

$\sum |\lambda_n x_n|^p = \sum |\lambda_n|^p |x_n|^p \le \sum K^p |x_n|^p \le K^p (\sum |x_n|^p) < +\infty$

IF X HAS NORMALIZED BASIS $\{x_n\}$ THEN $\hat{X}$ SATISFIES $l_1 \subseteq \hat{X} \subseteq c_0$ AS SETS, HENCE $l_1 \to X \to c_0$ WHERE $(\xi_n) \to \sum \xi_n x_n$ & $\sum \alpha_n x_n \to \alpha_n$ ARE CONTINUOUS MAPS.

IF $(\alpha_n) \in l_1$ THEN $\sum_1^\infty \alpha_n x_n \in X \iff \|\sum_{j=n+1}^{\infty} \alpha_j x_j\| \le \sum_{n+1}^{\infty} |\alpha_j| \to 0$

IF $(\alpha_n) \in \hat{X}$ WE NEED TO SHOW $\alpha_n \to 0$.

SUPPOSE NOT. $\exists\, \epsilon > 0 \ni \forall M \; \exists\, N \ge M$ WITH $|\alpha_N| \ge \epsilon$

BUT $\|\alpha_N x_N\| \le M \|\sum_N^\infty \alpha_n x_n\|$

BASIS $\{x_n\}$ WE HAVE $M \ni \|\sum_{i=1}^{P} \alpha_i x_i\| \le M \|\sum_1^{P+p} \alpha_i x_i\|$

$$\beta_n = \begin{cases} 0 & n < N \\ \alpha_n & n \ge N \end{cases}$$

LET $P = N$

$\|\alpha_n x_n\| \le M \|\sum_N^\infty \alpha_n x_n\| \Rightarrow \|\sum_N^\infty \alpha_n x_n\| \ge \epsilon/M > 0$

SINCE $\sum_1^\infty \alpha_n x_n \in X$.

<u>THEOREM</u>: IF X, Y ARE B SPACES AND Y or X^* ~~ONE OR THE OTHER~~ HAS A BASIS. THEN FOR EVERY COMPACT OPERATOR $K: X \to Y$ $\exists\, T_n : X \to Y$ FINITE RANK OPERATOR WITH $\|T_n - K\| \to 0$ AS $n \to \infty$.

PROOF (NEXTIME)

$$\text{CLOSURE } \underbrace{F(X,Y)}_{\text{FINITE RANK}} \subseteq \underbrace{K(X,Y)}_{\text{COMPACT}} \subseteq \underbrace{B(X,Y)}_{\text{BDD}}$$

IN OTHER WORDS, CLOSURE OF $F(X,Y) = K(X,Y)$

<u>REMARK</u>: $\exists\, X \,\&\, Y$ WITH $\mathrm{cl}\, F(X,Y) \neq K(X,Y)$

<u>COROLARRY</u>: $\exists$ B-SPACES WITHOUT BASIS

Nov. 18.

Let $T : X \to X$ be a conti. linear operator.

Define "<u>Spectrum of T</u>" $= \sigma(T)$ s.t

$$\sigma(T) = \{ \lambda \text{ scalars} \; ; \; \lambda I - T \text{ or } T - \lambda I \text{ does not have a bounded inverse} \}$$

How can an continuous operator fail to have an inverse.

Case I) T not 1-1 i.e $\operatorname{Ker} T \neq \{0\}$

II) T not onto

T may have an inverse but not a bounded one.
(This can't happen if X is complete)

<u>Example</u>

$M_\lambda : l_f \longrightarrow l_f$ with $\lambda_n = \frac{1}{n}$

inverse is not bounded, but M_λ is 1-1 & onto.

<u>Remark</u>

In finite dimension I $\Longleftrightarrow$ II

Since $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\dim \operatorname{Ker} T + \dim \operatorname{Rang} T = n$.

This is not true in infinite dimension.

eg. $M_\lambda : c_0 \longrightarrow c_0$ $\lambda_n = \frac{1}{n}$.

M_λ : continuone

<u>claim</u> M_λ is 1-1

(pf) Suppose $(\xi_n) \in c_0$ and $M_\lambda(\xi_n) = 0$

$\Rightarrow \lambda_n \xi_n = \frac{\xi_n}{n} = 0 \quad \forall n = 1, 2, \ldots$

$\Rightarrow \xi_n = 0 \quad n = 1, 2, \ldots$

$\Rightarrow \ker M_\lambda = \{0\}$. #

Thus M_λ is not onto. If it is onto, then by open mapping Th, M_λ is an isomorphism which is not.

e.g.

$S : l_2 \longrightarrow l_2$ Shift operator on l_2.

$$S(\xi_n) = (\eta_n) = \begin{cases} 0 & n = 1 \\ \xi_{n-1} & n \geq 2 \end{cases}.$$

S is 1-1 but not onto.

eg $T : l_2 \longrightarrow l_2$ Backwards shift

$T(\xi_n) = (\eta_n)$ where $\eta_n = \xi_{n+1} \quad n = 1, 2, \ldots$

$\ker T = \{ (\xi_n) ; \xi_n = 0 \quad n \geq 2 \} \neq \{0\}$

$\therefore T$ is not 1-1

But clearly onto.

Point spectrum of T are the set of $\lambda \in \sigma(T)$

where $\lambda I - T$ is not 1-1 (in case ...)

i.e. $\ker(\lambda I - T) \neq \{0\}$

i.e. $\exists\, x \neq 0$, $x \in X$ with $(\lambda I - T)(x) = 0$

$$\lambda x - Tx = 0 \quad \text{or} \quad Tx = \lambda x$$

In such cases, λ is called eigenvalue and

x is called eigen $\begin{cases} \text{vector} \\ \text{element} \\ \text{function (if } X \text{ is a function sp)} \end{cases}$

Note that $x \neq 0$ but λ may be zero

<u>Remark</u> If X is finite dimensional, then point spectrum = spectrum.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ has $\sigma(T) = \{\lambda_1 \cdots \lambda_n\}$ distinct scalars with eigen vectors $x_1, \cdots x_n$

Then $\{x_i\}$ are linearly independent*. (proof next time)

<u>Pf</u> Write $\widetilde{\mathbb{R}_n}$ as $(\alpha_1, \cdots \alpha_n)$ corresponding $\Sigma \alpha_i x_i$

Then

$$\begin{tikzcd} T : \mathbb{R}^n \arrow[r] \arrow[d, leftrightarrow, "(\sum \alpha_i x_i)"'] & \mathbb{R}^n \arrow[d, leftrightarrow] \\ \tilde{\mathbb{R}}^n \arrow[r, "\tilde{T}"] & \tilde{\mathbb{R}}^n \end{tikzcd}$$

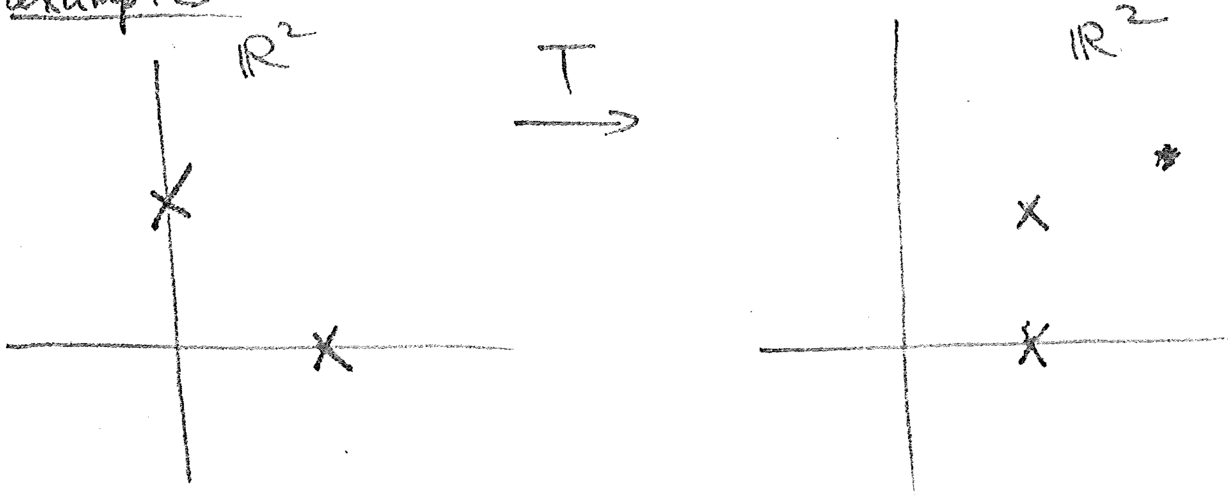
$(\alpha_1, \cdots \alpha_n)$

$$, \quad \tilde{T} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= M_\lambda$$

where $\lambda = (\lambda_1, \cdots \lambda_n)$ #

<u>*examples*</u>



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x+y \\ y \end{bmatrix}$$

$$(I - T)\binom{x}{y} = \binom{y}{0}$$

$$\ker (I - T) = \{ (x, 0) : x \in \mathbb{R} \}$$

$$\ker (I - T)^2 = \mathbb{R}^2$$

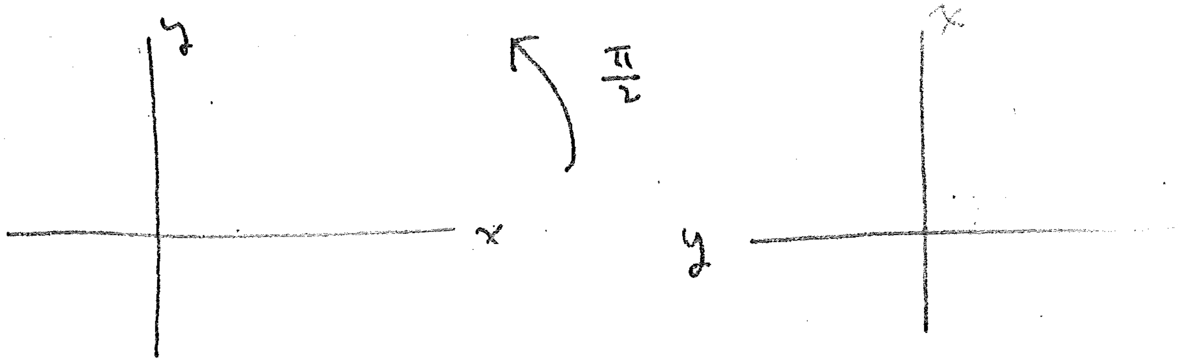
$$\sigma(T) = \{1\}$$

$$\begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ 0 & & \lambda \end{bmatrix}$$

~~Com~~

Consider

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Clearly T has empty spectrum.

$T - \lambda I$ singular

$\Longleftrightarrow \det(T - \lambda I) = 0$

In this case

$\Longleftrightarrow \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \qquad \Longleftrightarrow \quad \lambda = \pm i$

<u>Qu</u> How big spectrum.

finite case $\exists$ max #.

But how about infinite dim?

[<u>Nonsense</u>

$$(I-T)^{-1} = \frac{1}{I-T} = I + \sum_{n=1}^{\infty} T^n \Big]$$

Consider seq. $S_n = I + \sum_{i=1}^{n} T^i$

This is seq. in $B(X)$.

Let X be a B-space.

$$(I-T)S_n = I + T + T^2 + \cdots + T^n$$
$$\qquad - T - T^2 - \cdots - T^n - T^{n+1}$$
$$= I - T^{n+1}$$

$$\therefore \quad (I-T)S_n = I - T^{n+1}$$

Now what is $(I-T)^{-1}$?

$\|T^{n+1}\| = \|T\,T^n\| \le \|T\|\,\|T^n\|$ So by induction

$$\|T^n\| \le \|T\|^n$$

If $\|T\| < 1$, then $\sum_{n=1}^{\infty} \|T^n\| \le \sum \|T\|^n = \frac{1}{1-\|T\|} < \infty$

Hence S_n is a Cauchy seq. in $B(X)$ since

$$\|S_n - S_m\| = \Big\|\sum_{m+1}^{n} T^n\Big\| \le \sum_{m+1}^{n} \|T^n\| \le \sum_{m+1}^{n} \|T\|^n \longrightarrow 0$$

So, $S_n \longrightarrow \sum_{n=0}^{\infty} T^n$

$$\underbrace{\|(I-T)S_n - I\|}_{\uparrow \text{ continuous function of } S_n \text{ on } B(X).} = \|T^{n+1}\| \longrightarrow 0$$

Hence

$$\| (I-T)\sum_{n=0}^{\infty} T^n - I \| = 0$$

or $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$ with $T^0 = I$

(Nonesense makes sense)

If $\overline{\lim_n} \|T^n\|^{\frac{1}{n}} < 1$, then $\sum \|T^n\| < \infty$

So results works as well.

If λ is large enough ($|\lambda| > \|T\|$), then

$\lambda I - T$ is invertible.

$\lambda I - T = \lambda(I - T/\lambda)$.

This will have an inverse if $\|T/\lambda\| < 1$ i.e. $\|T\| < |\lambda|$

Proposition

The set of invertible (conti. inverse) operators in $B(X)$, X : B-sp. is open.

Cor

$\sigma(X)$ is closed

Next time.

Nov 26, 1976
L. Barker

Proposition: Let X be a Banach space. Then the set of all invertible operators in $B(X)$ is open.

Proof: Let T be an invertible operator in $B(X)$. (i.e., $T^{-1} \in B(X)$)

To establish the proposition, we must exhibit an $\varepsilon > 0$ such that for any $S \in B(X)$, where $\|S\| < \varepsilon$, the operator $(T+S)$ is invertible.

We know that if $\|S\| < 1$, $I + S$ is invertible. Rewriting I as $T^{-1}T$, we have $T^{-1}T + S$ invertible.

Hence, if $\|S\| < 1$, $T(T^{-1}T + S)$ is invertible. Then $T + TS$ is invertible.

Let V be an operator such that $\|V\| < 1/\|T^{-1}\|$.

Then $\|T^{-1}V\| < \|T^{-1}\| \, \|V\| =$

$\|T^{-1}\| \, 1/\|T^{-1}\| = 1$.

Hence, if $\|V\| < 1/\|T^{-1}\|$,

$T + T(T^{-1}V) = T + V$ is invertible.

Then $\varepsilon = 1/\|T^{-1}\|$ ∎

Cor: $\sigma(T)$ is compact

Proof: We know $\sigma(T)$ is bounded, so it is sufficient to show $\sigma(T)$ is bounded.

($\sigma(T)$ is bounded since

$|\lambda| > \|T\| \Rightarrow \lambda I - T$ is invertible

so $-\|T\| < \lambda < \|T\|$)

Suppose $\lambda I - T \notin \sigma(\ \ )$

Then there is a $\delta > 0$ such that

$|\lambda| < \delta \Rightarrow \lambda I - T$ is invertible.

If $|\mu| < \delta$, then $\|\mu I\| < \delta$, so $(\mu + \lambda) I - T$ is invertible. Then $(\lambda + \mu) \notin \sigma(T)$. Hence, $\sigma(T)$ is closed. ∎

Remark: If K is a compact subset of scalars, there exists $T: l_2 \to l_2$ such that $\sigma(T) = K$.

From Topology, we know there is a sequence (λ_n) contained in K such that (λ_n) is dense in K - i.e. $cl(\lambda_n) = K$.

By Problem 1, $M_\lambda: l_2 \to l_2$ has spectrum K.

Let $\underline{X}$ be a Banach space. Let $K: \underline{X} \to \underline{X}$ be a compact operator. If $\dim \underline{X} = +\infty$, K is not onto, and hence $0 \in \sigma(K)$.

Proof:
$$\begin{tikzcd} X \arrow[rr, "K"] \arrow[dr, "\varphi"'] & & X \\ & X/\ker K \arrow[ur, "\tilde{K}"'] & \end{tikzcd}$$

Suppose K is onto. $\tilde{K}$ is 1-1 & onto.

Claim: $\tilde{K}$ is compact.

$\tilde{K}(\ker K + x : \|\ker K + x\| < 1) =$

$\tilde{K}(\varphi(x : \|x\| < 1)) \subseteq K(V)$, where V is the unit ball of X.

Now $K(V)$ is a precompact set, so $\tilde{K}$ is precompact. Since X is complete, $\tilde{K}$ is compact.

By the Open Mapping Theorem, $\tilde{K}$ is an isomorphism.
Then $\tilde{K}(\ker K + x : \|\ker K + x\| < 1)$ is an open set in X.
Hence, it is an open precompact set in X.
So X is locally compact, implying $\dim X < +\infty$. ∎

To study the non-zero spectrum of compact operators, it suffices to study operators of the form $I - K$, where K is compact.

Let $A : X \to X$, where X is a B space and $A = I - K$. Then if K is compact, A has closed range.

Proof:
$$\begin{tikzcd} X \arrow[rr, "A"] \arrow[dr, "\varphi"'] & & X \\ & X/\ker A \arrow[ur, "\tilde{A}"'] & \end{tikzcd}$$

$\tilde{A}$ is 1-1

It suffices to show $\tilde{A}^{-1}$, restricted to range $\tilde{A}$ is bounded.
If this holds, $\tilde{A}$ is an isomorphism between $X/\ker A$ and range $\tilde{A}$ = range A. Then since $X/\ker A$ is complete, range A is complete.

Suppose not. Then there is a sequence (x_n) contained in $X/\ker A$ such that $\|x_n\|=1$ but $\|\tilde{A}x_n\| \xrightarrow[n\to\infty]{} 0$

There is a sequence (z_n) in X such that $1 \le \|z_n\| \le 2$, $\varphi(z_n)=x_n$, and $\mathrm{dist}(z_n, \ker A) \ge 1$.

Now $1 \le \|z_n\| \le 2$, but $\|Az_n\| \to 0$.

Since K is compact, there exists a subsequence of (z_n), say (u_n), such that $Kx_n \to w$ for some $w \in X$.

Claim: $Aw=0$
Justification: $u_n = (I-K)u_n + Ku_n$, so

$u_n = Au_n + Ku_n$.

Since (u_n) is a subsequence of (z_n), $\|Au_n\| \to 0$. Then

$$\underset{\substack{\downarrow \\ u_n}}{u_n} = \underset{\substack{\downarrow \\ 0}}{Au_n} + \underset{\substack{\downarrow \\ w}}{Ku_n}$$

Hence, $\lim_n u_n = w$.

Now $Aw = A(\lim_n u_n) = \lim_n A(u_n) = 0$. This is a contradiction, since $w \in \ker A$ and $\operatorname{dist}(u_n, \ker A) \geq 1$. ■

Lemma: Let X be a vector space, and let $T: X \to X$. If $\lambda_1, \ldots, \lambda_n$ are distinct scalars with eigenvectors $X_1, \ldots, X_n$, then $X_1, \ldots, X_n$ is a linearly independent set.

Proof: The proof will be by induction.

Now $\{X_1\}$ is a linearly independent set, since no eigenvector can equal 0.

Suppose $\{X_1, \ldots, X_n\}$ is a linearly independent set. We will show $\{X_1, \ldots, X_{n+1}\}$ is a linearly independent set.

Suppose not. Then there exist scalars $(\alpha_1, \ldots, \alpha_{n+1})$, not all zero, such that

$$\sum_{i=1}^{n+1} \alpha_i x_i = 0.$$

Claim: $\alpha_{n+1} \neq 0$
Justification: If it were, there would be some set of scalars $(\alpha_1, \dots, \alpha_n)$ such that $\sum_{i=1}^{n} \alpha_i x_i = 0$. This would contradict the linear independence of $\{x_1, \dots, x_n\}$.

Hence, there is a $\{\beta_i\}_{i=1}^{n}$ such that
$x_{n+1} = \sum_{i=1}^{n} \beta_i x_i$.

Thus $\lambda_{n+1} X_{n+1} = T(X_{n+1}) = T(\sum_{i=1}^{n} \beta_i x_i) =$

$\sum_{i=1}^{n} \beta_i \lambda_i x_i$.

If $\lambda_{n+1} = 0$, then $\lambda_i \neq 0$ for all $i \leq n$

and $\sum_{i=1}^{n} \lambda_i \beta_i x_i = 0$, where not all

$\lambda_i \beta_i = 0$. This contradicts the independence of $\{X_1, \dots, X_n\}$.

If $\lambda_{n+1} \neq 0$, then $X_{n+1} = \sum_{i=1}^{n} \beta_i \frac{\lambda_i}{\lambda_{i+1}} x_i$

and $X_{n+1} = \sum_{i=1}^{n} \beta_i X_i$. Now $\frac{\lambda_i}{\lambda_{i+1}} \neq 1$, so

we have two representations of X_{n+1}.

This contradicts the independence of $\{X_1, \dots, X_n\}$. ∎

Let X be a Banach space having a basis. Let P_n be partial sum projections - i.e. $P_n(\sum_{i=1}^{N} \alpha_i e_i) = \sum_{i=1}^{n} \alpha_i e_i$, N an extended integer.
Let K be a precompact set contained in X.
Then $\lim_{n\to\infty} \sup_{x\in K} \|(I-P_n)x\| = 0$

Proof: Since K is precompact, given any $\varepsilon>0$, there is a $\{x_1,\dots,x_m\}$, $x_i \in K$ such that for any $y\in K$, there is a x_i such that $\|x_i - y\| < \varepsilon$.

For each i, since $P_n(x_i) \to x_i$, there is a N_i with $n > N_i$ such that $\|P_n(x_i) - x_i\| < \varepsilon$.

Let $N_0 = \max N_1,\dots,N_m$
Let $x\in K$, and let $n \geq N_0$.

Now $\|(I-P_n)x\| = \|x - P_n x\|$.
There is a x_i such that $\|x - x_i\| < \varepsilon$.
Then
$\|x - P_n x\| \leq \|x - x_i\| + \|x_i - P_n x_i\| + \|P_n x_i - P_n x\|$.

Now $\|x - x_i\| < \varepsilon$ and $\|x_i - P_n x_i\| < \varepsilon$.
Since we have a basis, there is a $M>0$ such that $\|P_n\| \leq M$ $\forall n$. Then
$\|P_n x_i - P_n x\| \leq M x$.
Hence,
$\|x - P_n x\| < (2+M)\varepsilon$.

Theorem: If $\mathbb{X}$ is a Banach space and $\{e_n\}$ is a normalized basis for $\mathbb{X}$, then there is an equivalent norm $|||\cdot|||$ in which $|||e_n|||=1$ and $|||P_n|||=1$

Proof: Let $S: \mathbb{X} \longrightarrow \hat{\mathbb{X}}$ where $\hat{\mathbb{X}} = \mathbb{X}$ with $|||\cdot|||$ norm satisfying the above.

S is a homeomorphism, so we can take norms from $\mathbb{X}$ to $\hat{\mathbb{X}}$ and back.

Claim: If T is a finite rank operator,

$$\mathbb{X} \xrightarrow{T} \mathbb{Y} \xrightarrow{P_n} \mathbb{Y}.$$

Now $P_n : T \longrightarrow T$ iff $P_n T - T = (P_n - I)T \longrightarrow 0$,

where "$\longrightarrow$" means converges in norm.

Then $\|(P_n - I)T\| =$

$\sup_{x \in V} \|(P_n - I)\, T(x)\| =$

$\sup_{x \in T(U)} \|(P_n - I)x\| \to 0$, by the Lemma.

$\mathbb{X}$ is a B-space, $K: \mathbb{X} \to \mathbb{X}$ is compact,
$A = I - K$.
We have shown A has closed range.

<u>Proposition</u>: $\ker A$ is finite dimensional.

<u>Pf.</u> $K|_{\ker A} = I_{\ker A}$ since $0 = Ax = x - Kx$ for $x \in \ker A$.
The identity on $\ker A$ is compact hence $\dim A < +\infty$.

$$A^* = (I-K)^* = I^* - K^* = I - K^*$$

$$\mathbb{X}^* \xleftarrow{A^*} \mathbb{X}^*$$

$$\mathbb{X} \xrightarrow{A} \mathbb{X}.$$

We already know K^* is compact.
We now know that A^* has closed range and $\ker A^*$ is finite dimensional.

Now from Duality results,
range $A = (\ker A^*)^{\top}$
$(\text{range } A)^{\perp} = \ker A^*$
$(\text{range } A^*)^{\top} = \ker A$

$$(\mathbb{X}/_{\text{range } A})^* = (\text{range } A)^{\perp} = \underset{\substack{\uparrow \\ \text{finite dim}}}{\ker A^*}$$

So $\mathfrak{X}/\text{range}\,A$ is finite dimensional.

Let $R_1 = A(\mathfrak{X})$, $N_1 = \ker A$
$R_{n+1} = A(R_n) = A^{n+1}(\mathfrak{X})$, $N_n = \ker(A^n)$.

Claim: $R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots$
$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ and $\exists M$ such that
$R_M = R_{M+1} = R_{M+2} = \cdots$
$N_M = N_{M+1} = N_{M+2} = \cdots$
Furthermore $N_M \cap R_M = \{0\}$ and $R_M + N_M = \mathfrak{X}$.

Pf of Claim:
Let $x \in R_{N+1}$. We want to show $x \in R_N$.
$\exists y \in \mathfrak{X} \ni A^{N+1} y = x$.
So $A^N(Ay) = x$, $Ay \in \mathfrak{X}$ & we have $x \in R_N$.

Let $x \in N_n$. Then $A^n x = 0$.
$A^{n+1} x = A(A^n x) = A(0) = 0$.

$$A^n = (I-K)^n = I \boxed{- nK + \frac{n(n-1)}{2} K^2 + \cdots + K^n}$$

compact operator

dim $N_n < +\infty$ for all n, codim $R_n < +\infty$ for all n, R_n are closed subspaces,

$\not\exists$ $R_1 \supsetneqq R_2 \supsetneqq R_3 \supsetneqq \cdots$.

In general then, $R_n \supsetneqq R_{n+1}$.

Let $\epsilon > 0$. Choose $x_i \in R_i \setminus R_{i+1}$ $\ni$

$$(1+\epsilon) > \|x_i\| \geq \operatorname{dist}(x_i, R_{i+1}) = 1.$$

Since R_i/R_{i+1} is nonzero, pick $\|z + R_{i+1}\| = 1$ in R_i/R_{i+1}. Pick $x_i \in z_i + R_{i+1}$ to approx. norm.

Consider $\{x_i\}$ in $\mathfrak{X}$ bounded sequence. K is compact so $\exists$ a subsequence $\{x_{n_i}\}$ such that Kx_{n_i} converges.

Let $i<j$.

Since $K = I - A$, we have

$$\|x_{n_i} - Ax_{n_i} - x_{n_j} + Ax_{n_j}\| = \|Kx_{n_i} - Kx_{n_j}\| \to 0.$$

But $x_{n_i} \in R_{n_i}$ so that $Ax_{n_i} \in R_{n_i+1}$.

$x_{n_j} \in R_{n_j} \subseteq R_{n_i+1}$, $\quad Ax_{n_j} \in R_{n_j+1} \subseteq R_{n_i+1}$.

So $(-Ax_{n_i} - x_{n_j} + Ax_{n_j}) \in R_{n_i+1}$ and

$\|x_{n_i} - Ax_{n_i} - x_{n_j} + Ax_{n_j}\| \geq \operatorname{dist}(x_{n_i}, R_{n_i+1}) = 1$ #

contradiction. So we have an $M \ni$

$$R_M = R_{M+1} = R_{M+2} = \cdots.$$

Assume for the minute that we have proved the following:

<u>Thm</u>: for such A, $\dim \ker A = \dim \ker A^*$ $=$ codim range A $=$ codim range A^*.

Then $N_M = N_{M+1} = N_{M+2} = \cdots$.

<u>Pf of Th.</u>

<u>Case I</u>: $\ker A = \{0\}$ iff $\ker A^* = \{0\}$.

($\Rightarrow$) For suppose $\ker A = \{0\}$ but $\ker A^* \neq \{0\}$.

Then $\ker A = \{0\} \Leftrightarrow A$ is 1-1,

$\ker A^* \neq \{0\} \Leftrightarrow$ range $A \subsetneq \mathfrak{X}$.

So A is 1-1 but not onto.

Let $x_1 \in \mathfrak{X} \setminus R_1$. $\quad Ax_1 \in R_1 \setminus R_2$.

In general, $A^n x_1 \in R_n \setminus R_{n+1}$.

By induction, $A^n x \in R_n \setminus R_{n+1}$.

Clearly $A^{n+1} x_1 \in R_{n+1}$.

S $A^{n+1} x_1 \in R_{n+2}$. Then $\exists\, y \ni A^{n+1} x_1 = A^{n+2} y$

so $A(A^n x_1) = A(A^{n+1} y)$.

A is 1-1 $\Rightarrow A^n x_1 = A^{n+1} y \Rightarrow x \in R_{n+1}$ #

contradiction.

So we have $A^n x_1 \in R_n \setminus R_{n+1}$.

But this is a contradiction since it implies $R_1 \supsetneq R_2 \supsetneq R_3 \supsetneq \cdots$ which is not so.

<u>Cor.</u> If A is not onto, then A is not 1-1.

If K is compact, $0 \neq \lambda \in \sigma(K)$,

{bdd $(\lambda I - K)^{-1}$ does not exist}. Then λ is in the point spectrum.

<u>Pf of Th (cont).</u>

($\Leftarrow$) Assume $\ker A^* = \{0\} \Rightarrow \ker A^{**} = \{0\}$.

If $Ax = 0$, $x \in \mathfrak{X}$, then $A^{**} x = 0$.

So $x=0$. Thus $\ker A=\{0\}$.

This implies the following:

<u>Thm</u>: (Fredholm Alternative)

If $\mathfrak{X}$ is a B-space, $K:\mathfrak{X}\to\mathfrak{X}$ is compact, then either

(1) $x-Kx=y$ has a unique solution $\forall y\in\mathfrak{X}$

or (2) $x-Kx=0$ has a non-trivial solution.

<u>Pf. (cont.)</u>

<u>Case II</u>. S $\dim \ker A=n$ and $\dim \ker A^*=m$.

Let $x_1\cdots x_n$ be independent spanning for $\ker A$. Let $x_1^*\cdots x_m^*$ be independent spanning for $\ker A^*$.

<u>Lemma</u>. $\exists\, x_0\in\mathfrak{X}$ & $x_0^*\in\mathfrak{X}^*\ni j<n\Rightarrow$

$x_0^*(x_j)=0,\ x_0^*(x_n)\neq 0,$

$j<m\Rightarrow x_j^*(x_0)=0,\ x_m^*(x_0)\neq 0.$

Assume the Lemma is true:

$K_0:\mathfrak{X}\to\mathfrak{X}$

$K_0(x)=x_0^*(x)\,x_0$ - compact

$K_0^*(x^*)=x^*(x_0)\,x_0^*$

Consider $A-K_0=I-(K+K_0)$.

Note $x\notin$ range A, $x_0^*\notin$ range A^*

for $x_0\in$ range $A\Rightarrow x_0\perp\ker A^*$ # contradiction since $x_m^*(x_n) \neq 0$.

$x_0^* \in$ Range $A^* \Rightarrow x_0^* \perp \ker A$ # contradiction since $x_0^*(x_n) \neq 0$.

Consider $\ker(A - K_0)$. Let $x \in \mathfrak{X}$,

$Ax - K_0 x = 0$

$Ax = K_0 x = x_0^*(x)\, x_0$

$x \in \ker(A - K_0)$ iff $x_0^*(x) = 0$ & $Ax = 0$.

$\ker(A - K_0) = \text{span}\{x_1 \cdots x_{n-1}\}$. Consider $\ker(A^* - K_0^*)$. $A^* x^* = x^*(x_0)\, x_0^*$,

Then $x^* \in \ker(A^* - K_0) = \text{span}\{x_1 \cdots x_{m-1}\}$ iff $x^*(x_0) = 0$ and $A^* x^* = 0$.

Thus $\dim \ker(A - K_0) = n - 1$ $\dim \ker(A - K_0)^* = m - 1$ the theorem now follows by induction.

1

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MAT 534
157 12-1-76
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Lemma Ⓐ:- IF $x_1, \ldots, x_n \in X$, are linearly independent, then, $\exists\ f \in X^*$ such that $f(x_j) = 0 \quad j = 1, 2, \ldots, n-1$.
AND $f(x_n) = 1 \neq 0$.

Ⓑ:- IF $f_1, f_2, \ldots f_n \in X^*$, are linearly independent, then, $\exists\ x_1, x_2, \ldots x_n \in X$ such that $f_i(x_j) = \delta_{ij}$

Proof

Ⓐ: Define $g(x_i) = \begin{cases} 0 & i < n \\ 1 & i = n \end{cases}$.

Extend g to span of $\{x_i\}_1^n$ linearly.
g is continuous on span $\{x_i\}_1^n$, then by using HB theorem it can be extended to $f \in X^*$.

Ⓑ proof by induction

First consider the case when,
$n = 1$, f_1 indep $\Rightarrow f_1 \neq 0$, then
$\exists\ y \in X$ with $f_1(y) \neq 0$,
take $x_1 = \dfrac{1}{f_1(y)} y$ (works).

Now

Assume Lemma Ⓑ true for n, and,
let $f_1, \ldots f_{n+1} \in X^*$ and linearly independent.

Let $y_1, \ldots, y_n \in X$ be such that
$f_i(y_j) = \delta_{ij}$ $\quad i,j \leq n$

Define: $\quad g = \sum_{i=1}^{n} f_{n+1}(y_i) f_i$
Now, $\quad f_{n+1} - g = 0 \quad$ on span $\{y_1, \ldots y_n\}$.

Since,
$$f_{n+1}(y_j) - \sum_{i=1}^{n} f_{n+1}(y_i) f_i(y_j)$$
$$= f_{n+1}(y_j) - f_{n+1}(y_j) \, .$$
since $f_i(y_j) = 0 \quad i \neq j, \; i = 1,2,\ldots,n$
$\quad = 1 \quad i = j$

Claim :- $\exists Z$ s.t
$f_{n+1}(Z) \neq 0$ but $g(Z) = 0$. :
i.e $Z \in \ker g \setminus \ker f_{n+1}$

Sublemma:
If $\ker f = \ker g$
Then $\exists \lambda \neq 0$ s.t $f = \lambda g$

Proof :-
Suppose $\ker f = X$, then $f = g = 0$.
Otherwise, let $x_0 \in X \setminus \ker f$.
Let $\lambda \neq 0$ s.t $f(x_0) = \lambda g(x_0)$.
Then for $x \in X$, $\exists \mu$ scalar, $m \in \ker f$
Such that $x = \mu x_0 + m$, i.e
$f(x) = f(\mu x_0 + m)$
$= \mu f(x_0) + f(m) = \mu f(x_0)$ $\quad (m \in \ker f)$

Also,

$$\lambda g(x) = \lambda g(\mu x_0 + m) = \mu \lambda g(x_0).$$

(done with sublemma)

Now

$\ker f_{n+1} \neq X$ if $\ker g = X$

$f_{n+1}(y_j) = 0$ if $j \leq n$,

then,

$z \in X \setminus \ker f_{n+1}$ (works)

Suppose now $\ker f \neq X \neq \ker g$,

then, $\ker g \setminus \ker f$ is non empty.

Otherwise $\ker g \subseteq \ker f. \Longrightarrow \ker g = \ker f$

$\Longrightarrow f = \lambda g$, i.e., $f = \lambda \sum_{i=1}^{n} f_{n+1}(y_i) f_i$

$\Longrightarrow f_{n+1} \in \operatorname{Span}\{f_i\}_{i=1}^{n}$,

Which contradict that, $f_1, f_2, \ldots f_n, f_{n+1}$, are linearly independent.

Now,

let $w = z - \sum_{i=1}^{n} f_i(z) y_i$ then

$$f_{n+1}(w) = f_{n+1}(z) - \sum_{i=1}^{n} f_i(z) f_{n+1}(y_i)$$

$$= f_{n+1}(z) - g(z) \neq 0.$$

On the other hand,

$$f_j(w) = f_j(z) - \sum_{i=1}^{n} f_i(z) f_j(y_j)$$

$$= f_j(z) - f_j(z) = 0.$$

Let $X_{n+1} = \dfrac{1}{f_{n+1}(w)} w$ and

$f_j(X_{n+1}) = \delta_{j(n+1)}$.

Let $x_i = y_i - f_{n+1}(y_i) X_{n+1}$

$f_{n+1}(x_i) = f_{n+1}(y_i) - f_{n+1}(y_i) = 0$,

For $j < n$ $\quad f_j(x_i) = f_j(y_i) - f_{n+1}(y_i) f_j(X_{n+1})$.
$= \delta_{ij} - 0$ done.

Now, we will use some simille argument to show,

$N_M \cap R_M = \{0\}$

$N_M + R_M = X$

where $N_1 \subseteq N_2 \subseteq \cdots N_M = N_{M+1} \cdots$,
$R_1 \supseteq R_2 \cdots R_M = R_{M+1} \cdots$.

Let
$x \in N_M \cap R_M$ then $x \in R_M = R_{2m}$, then
$\exists\, y \in R_M$ s.t $A^M y = x$.
We know
$x \in N_M$, $0 = A^M x = A^{2M} y \implies$

$\Longrightarrow y \in N_{2M} \Longrightarrow y \in N_M \Longrightarrow 0 = A^M y = x$

Then $N_M \cap R_M = \{0\}$.

Now
let $x \in X$, $A^M x \in R_M = R_{2M}$

$\Longrightarrow \exists y \in R_M$ with $A^M y = A^M x$

i.e. $A^M(x-y) = A^M y - A^M x = 0$,
i.e. $x - y \in N_M$ then,
$x \in N_M + R_M$.

Then $N_M + R_M = X$.

<u>Claim</u> :-
$A(R_M) = R_{M+1} = R_M$.
So, $A(R_M) \subset R_M$, and
$A|_{R_M}$ is onto.

let $x \in R_M$ $Kx = x - Ax \in R_M$
$K(R_M) \subseteq R_M$.

<u>Definition</u> :-

If $T : \mathbb{X} \to \mathbb{X}$ is linear op.
ψ is said to be an invariant subspace for T if $T(\psi) \subseteq \psi$.

Now
$A(N_M) = A(N_{M+1})$ since $N_M = N_{M+1}$
but
$A(N_{M+1}) \subseteq N_M$,
since
if $x \in N_{M+1}$ $A^M(Ax) = 0$ so
$Ax \in N_M$, then
$K(N_M) \subseteq N_M$

$A|_{R_M}$ is 1-1. suppose $x \in R_M$
$Ax = 0$ $x \in R_M = R_{2M} \Rightarrow \exists\, y \in R_M$ with
$A^M y = x$,
this till us that

$A^{M+1} y = AA^M y = Ax = 0$, $y \in N_{M+1} = N_M$
$\Longrightarrow y = 0$ since $y \in N_M, R_M$
hence $x = A^M y = 0$.

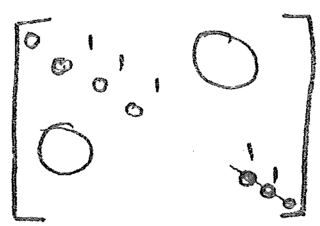
Now we have
R_M closed N_M f.dim closed, $N_M \cap R_M = \{0\}$, $N_M + R_M = \mathbb{X}$

The we can have,

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, equal] & X \arrow[d, equal] \\ N_M \times R_M \arrow[r, "{A|_{N_M},\ A|_{R_M}}"] & N_M \times R_M \end{tikzcd}$$

And, $A|_{R_M}$ is an isomorphism by open map Theorem

Also, $A|_{N_M}$ is Nilpotent i.e $\left(A|_{N_M}\right)^n = 0$ for some n



Also,

$$\begin{tikzcd} X \arrow[r, "K"] \arrow[d, leftrightarrow] & X \arrow[d, leftrightarrow] \\ N_M \times R_M \arrow[r, "{K|_{N_M},\ K|_{R_M}}"] & N_M \times R_M \end{tikzcd}$$

N_M includes all eigen vectors $= (M)$ for eigen value one. since $Kx = x$, $Ax = x - Kx = 0$.

Some times N_M is called the generalized eigen space.

Let us suppose:-

If $\lambda \neq 1$ and x is eigen vector for K w/eigen value λ then $x \in R_M$.

To see that

$$A^M x = (I-K)^M x = (1-\lambda)^M x \neq 0$$

$$x \in \text{span } A^M x \subseteq R_M \text{ .}$$

<u>Corollary</u>: If K is compact, $\dim X = +\infty$, then the spectrum $\sigma(K) = \{\lambda_1, \ldots \lambda_n, 0\}$ = Pt.sp. or $\underbrace{\{\lambda_n\}}_{\text{Pt.sp}} \cup \underbrace{\{0\}}_{\text{may or may not be in Pt sp}}$ where $\lambda_n \to 0$.

Suppose K is compact & $\sigma(K)$ contains a sequence λ_n with limit λ, $\lambda \neq 0$

WLOG we may assume $\lambda = 1$

via $K_1 = \lambda^{-1} K$.

Now assume $\lambda_n \neq 0$ $\exists x_n \ni$ s.t $\|x_n\| = 1$ & $K_{\lambda_n} = \lambda_n x_n$.

Since K is compact $\exists$ subseq. $\{x_n\}$ s.t K_{x_n} conv.

$$\lim \lambda_n x_n = \lim K_{x_{k_n}} = w$$
$$\|$$
$$\lim \lambda_n \lim x_n = \lim x_n$$

$$K\omega = \lim K x_n = \lim \lambda_n x_n = \omega .$$

$$\left\{ \begin{aligned} \omega &= \lim \lambda_n x_n \\ &= \frac{\lim \lambda_n x_n}{\lim \lambda_n} \\ &= \lim x_n \end{aligned} \right.$$

ω is eigen vector for $\lambda = 1$

$\|\omega\| = \lim \|x_n\| = 1$ $\qquad \omega \in N_M$, $(x_n) \in R_M$

$\Longrightarrow \omega \in R_M \Longrightarrow \omega = 0$. #
