Functional Analysis Problem Set 1 Due Oct. 1, 1979

1. Give examples in $\mathbb{R}^2$ to show that
   (a) the convex hull of a closed set need not be closed
   (b) the circle hull of a convex set need not be convex

2. In a TVS show that
   (a) the closure of a convex set is convex
   (b) the convex hull of a circled set is circled
   (c) the interior of a proper subspace is empty
   (d) the set A is absorbent iff A is radial at 0 [Dn is on page 14 of text]
   (e) If $\mathcal{U}$ is any basis of nghd of zero, then $\cap \mathcal{U} = cl(\{0\})$ and this set is a closed subspace

3. A TVS is said to be a LCS (locally convex space) if it has a basis of nghd of zero each of which is convex. Show that a LCS has a nghd basis of ~~zi~~ zero consisting of closed circled ~~and~~ convex sets.

4. Let X be a vector space and $\{\mathcal{J}_\alpha : \alpha \in I\}$ be a collection of TVS topologies on X. Define

$$\mathcal{S} = \bigcap_{\alpha \in I} \mathcal{J}_\alpha \qquad \mathcal{R} = \{ U_1 \cup U_2 \cup \dots U_n : \text{each } U_i \text{ is in some } \mathcal{J}_\alpha \}$$

Show that both $\mathcal{S}$ and $\mathcal{R}$ are TVS top on X; $\mathcal{S} \subset \mathcal{J}_\alpha \subset \mathcal{R}$ for each $\alpha \in I$ and if $\mathcal{P}, \mathcal{Q}$ are top on X with $\mathcal{P} \subset \mathcal{J}_\alpha \subset \mathcal{Q}$ for each $\alpha \in I$ then $\mathcal{P} \subset \mathcal{S}$, $\mathcal{R} \subset \mathcal{Q}$.

※1	※2	※3	※4	※5	※6	※7	※8
A/A⁻	A	A	B⁺/A⁻	A	A⁻	A⁻/B⁺	B⁺
A	A⁻	A	A⁻/B⁺	A⁻	A	A⁻	
A/A⁻	B⁺/A⁻	C					

1 Let X be a normed space

a) If X is separable so is its completion $\tilde{X}$

b) If Y is a subspace of separable X, then Y is separable

c) If Y is a closed subspace of separable X, then X/Y is separable

d) If Y is a closed subspace of X and both Y and X/Y are separable, then X is separable.

e) X is separable if and only if there is a sequence in X with dense linear span.

f) Show that L_∞ is not separable, but L_1 is separable

Hint use x_m^n with $1/m$ of norming f_n

g) Show that if X^* is separable, then X is separable

h) Show that if X is not separable then $\exists \delta > 0$ and an uncountable set $\{x_\alpha ; \alpha \in I\}$ s.t. $\|x_\alpha\| \le 1$ for each $\alpha \in I$ and $\|x_\alpha - x_\beta\| \ge \delta$ for $\alpha, \beta \in I$ with $\alpha \ne \beta$. [Hint. Show for each $\delta > 0$ there are maximal sets of this type using zorn's lemma, then prove it by contradiction]

i) If $\{x_n\} \subset X$ has the property: $f \in X^*$ $f(x_n) = 0$ for each n $\Rightarrow f = 0$ and $\|x_n\| = 1$, Then the map $T : X^* \to l_\infty$ which sends $f \in X^*$ to the sequence $\{f(x_n)\} \in l_\infty$ is continuous [HINT CLOSED GRAPH] suppose for each $f \in X^*$ $\{x_n\}$ has the additional property that for $f \in X^*$ $\|f\| = \sup |f(x_n)|$, Then T is an isometry into.

j) Show that each separable X has such a sequence $\{x_n\}$ required for both parts of (i)

Problem Set #5 Due Mon Nov. 5, 1979

1 Let $\dim X = n$ and let $\{e_i\}_{i=1}^n$ be a Hamel basis for the vector space X, then $\{e_j'\}_{j=1}^n$ is a Hamel basis for X^* where e_j' is the unique linear map so that $e_j'(e_i) = 0$ if $i \neq j$ and $e_j'(e_j) = 1$. If $\{e_i\}_{i=1}^n$ (resp. $\{f_j\}_{j=1}^m$) is a Hamel basis for X (resp Y) and $T: X \to Y$ is linear, then the matrix representation of $T': Y^* \to X^*$ with resp to basis $\{e_i'\}$ and $\{f_j'\}$ is the matrix transpose of the matrix representation of $T: X \to Y$ with resp to bases $\{e_i\}$ and $\{f_j\}$.

2. If X is normed space with cont dual X' and if $x_n \to y$ in the $\sigma(X, X')$-topology then there are integers $0 < N_1 < N_2 < \dots$, and $z_k \in$ convex hull $\{x_n : N_k \leq n < N_{k+1}\}$ so that $z_k \to y$ in norm.

3. Let $X = C_c^\infty(\mathbb{R})$ and let $T: X \to X$ be the map $Tf = f'$ (the derivative) Show T is 1-1 and cont., Show $T(X)$ is a (proper) closed hyperplane (hint consider the local-L_∞ fcn which is constantly 1). Use this to show that if the distribution $\varphi \in X'$ satisfies the D.E. $\frac{d}{dt}\varphi \equiv 0$ then φ is a constant function.

4. Let (E, F) & (H, G) be dual pairs and let $T: E \to H$ be linear so that $T'(G) \subset F$. Show T is 1-1 $\Leftrightarrow$ $T'(G)$ is $\sigma(F, E)$-dense in F and T' is 1-1 $\Leftrightarrow$ $T(E)$ is $\sigma(H, G)$ dense in H.

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Problem set 8 Due Wed 5 Dec 1979

1. X is Banach and $Y \subset X$ is dense and co-dimension 1 in X. Show Y is barrelled but not complete.

2. X is an infinite dimensional vector space and p, q are two norms on X. Let $S = \{q(x) : x \in X \;\; p(x) = 1\}$ find all possible S's, show that each of them can be realized.

3. Suppose $\langle\;\rangle$, $\langle\langle\;\rangle\rangle$ are two inner products on H. Set $|x| = \langle x, x\rangle^{1/2}$ and $\|x\| = \langle\langle x, x\rangle\rangle^{1/2}$. Suppose $(H\; |\cdot|)$ and $(H, \|\cdot\|)$ are Hilbert spaces and that $|\cdot|$ and $\|\cdot\|$ are equivalent. Show that $\forall_{\varepsilon > 0}$ there is an infinite dimensional subspace $X_\varepsilon \subset H$ s.t. $x, y \in X$ $|x| = 1 = |y|$ implies $|\,\|x\| - \|y\|\,| < \varepsilon$ [Hint find sequence x_n orthogonal in both $\langle\;\rangle$, $\langle\langle\;\rangle\rangle$]

4. If X is infinite dimensional p, q are norms on X and if $Y \subset X$ show that if p, q are equivalent on Y they are equivalent on $\mathrm{cl}\, Y$. Show that if p, q equivalent on Y and $\dim X/Y < \infty$ then p, q are equivalent on X.

5. If X is connected, $\{U_\alpha\}_{\alpha \in A}$ is open cover of X with the property that $\forall \alpha \in A$ there are only finitely many $\beta \in A$ so that $U_\alpha \cap U_\beta \neq \emptyset$, then A is countable