Problem set 2 Due Oct 8, 1979

1. Let $B \subseteq$ in the vector space $\mathfrak{X}$ and suppose that either B is circled or B is convex and $0 \in B$, Show that for $x \in \mathfrak{X}$, the set $\Theta = \{\lambda \geq 0 : \lambda x \in B\}$ is one of the sets $\{0\}$, $[0,\infty)$, $[0,\alpha)$ or $[0,\alpha]$ for some $\alpha \in \mathbb{R}$, $\alpha > 0$.

2. If $B \subset \mathfrak{X}$ satisfies the conclusion of (1), then define $\rho_B : \mathfrak{X} \to [0,\infty]$ by $\rho_B(x) = \inf\{\lambda \geq 0 : x \in \lambda B\}$ (note $\inf \phi = \infty$)
Show: (a) $0 \leq \rho_B(x) \leq \infty$; (b) $\alpha \geq 0 \Rightarrow \rho_B(\alpha x) = \alpha \rho_B(x)$;
(c) for all $x \in \mathfrak{X}$, $\rho_B(x) < \infty \Longleftrightarrow B$ is absorbent.

3. If $A, B \subset \mathfrak{X}$ are as in (2) Show: (a) $A \subset B \Rightarrow \rho_B \leq \rho_A$;
(b) $\rho_{A \cap B}(x) = \max\{\rho_A(x), \rho_B(x)\}$.

4. $B \subset \mathfrak{X}$ as above, define $B_i = \{x : \rho_B(x) < 1\}$ and $B_c = \{x : \rho_B(x) \leq 1\}$
Show: (a) $B_i \subset B \subset B_c$; (b) $\rho_{B_i}(x) = \rho_B(x) = \rho_{B_c}(x)$ for each $x \in \mathfrak{X}$;
(c) If $\rho_A \equiv \rho_B$ then $B_i \subset A \subset B_c$.

5. If $B \subset \mathfrak{X}$ & B is circled, then $\alpha \in \mathbb{K} \Rightarrow \rho_B(\alpha x) = |\alpha| \rho_B(x)$ [using $0 \cdot \infty = 0$]
conversely if B is as in (2) and $\rho_B(\alpha x) = |\alpha| \rho_B(x)$, then both B_i and B_c are circled.

6. If $0 \in B \subset \mathfrak{X}$ & B is convex, then $\rho_B(x+y) \leq \rho_B(x) + \rho_B(y)$, conversely if B is as in (2) and $\rho_B(x+y) \leq \rho_B(x) + \rho_B(y)$, then both B_i and B_c are convex.

7. If B is convex circled & absorbent, then ρ_B is a semi-norm,
If ρ is a semi-norm and $B = \{x : \rho(x) \leq 1\}$ then $\rho = \rho_B$.
(state & prove ~~a~~ similar statements for B circled & absorbent)

8 A semi-norm ρ is continuous on the TVS $(\mathfrak{X}, \mathfrak{I})$ $\Longleftrightarrow$ it's continuous at zero $\Longleftrightarrow$ $\{x : \rho(x) < 1\}$ is a neighborhood of zero,

9. ~~If~~ Let $f, f_n, g \in C^1[0,1]$ - the set of continuous differentiable real-value fcns on $[0,1]$. Suppose $f_n \to f$ uniformly and $f_n' \to g$ uniformly Show that $f' = g$ [HINT: INTEGRATE] and that $C^1[0,1]$ is Complete with the norm $\|f\| = \max\left(\sup_{x \in [0,1]} |f(x)|, \sup_{x \in [0,1]} |f'(x)|\right)$

Problem Set #6 Due Wed Nov 14, 1979

1 If A, B are precompact in the TVS $(\mathfrak{X}, \mathcal{J})$ and $C \subset A$ then C, $\mathrm{cl}A$, αA, $A+B$, circled hull A, and (if $\mathfrak{X}$ is LCS) convex hull A are precompact. If $T: X \to Y$ is cont & linear $T(A)$ is precompact.

2. Find an $f \in l_2^* \setminus l_2$ and a non-closed hyperplane $\subset l_2$

3. Let $\varphi : [0,1] \to L_2$ be defined via $\varphi(t) = \chi_{[0,t]}$. Show φ is cont. If $t_1 < t_2 \leq t_3 < t_4$ then $\varphi(t_4) - \varphi(t_3) \perp \varphi(t_2) - \varphi(t_1)$. $\varphi((0,1])$ is linearly independent.

4. If $(\mathfrak{X}_n, \|\cdot\|_n)$ is a sequence of Banach spaces, define $l_2\text{-sum}(\mathfrak{X}_n) = \{(x_n) : x_n \in \mathfrak{X}_n \text{ \& } \|(x_n)\| = (\sum \|x_n\|_n^2)^{1/2} < \infty\}$. Show that the l_2-sum$(\mathfrak{X}_n)$ is complete and $(l_2\text{-sum}(\mathfrak{X}_n))' = l_2\text{-sum}(\mathfrak{X}_n')$

5. If $\{x_\alpha\}_{\alpha \in A} \subset \mathbb{R}$ is a net, define $\liminf\limits_\alpha x_\alpha = \lim\limits_{\alpha \in A} (\inf\limits_{\substack{\beta \geq \alpha \\ \beta \in A}} x_\beta)$. Show that this always exists.

If $\mathfrak{X}$ is normed and the net $\{x_\alpha'\} \subset \mathfrak{X}'$ so that $x_\alpha' \to x' \in \mathfrak{X}$ in $\sigma(\mathfrak{X}', \mathfrak{X})$ then $\|x'\| \leq \liminf\limits_\alpha \|x_\alpha'\|$

Problem set #7 Due Wed 21 Nov 1979.

1. Let E, F be complete metric TVS's $T: E \to \omega$, $S: F \to \omega$ continuous 1-1 linear maps ($\omega = \prod_{n=1}^{\infty} \mathbb{K}$). Show $T(E) \subset S(F)$ implies that $S^{-1} \circ T: E \to F$ is continuous and $T(E) = S(F)$ implies that E and F are isomorphic

2. Let $\mathfrak{X}$ be a barrelled TVS, $\mathcal{Y}$ LCS and $T_n: X \to \mathcal{Y}$ be cont & linear and suppose for each $x \in \mathfrak{X}$, $\lim_{n \to \infty} T_n x$ exists (call it Tx). Show that T is continuous and linear

3. If $\mathfrak{X}$ is locally convex, $\mathcal{Y}$ a closed subspace of $\mathfrak{X}$, then $j: \mathcal{Y} \to \mathfrak{X}$ induces an isomorphism $j': (\mathfrak{X}', \sigma(\mathfrak{X}', \mathfrak{X}))/\mathcal{Y}^\circ$ onto $\mathcal{Y}'$ and the quotient $\varphi: \mathfrak{X} \to \mathfrak{X}/\mathcal{Y}$ induces an isomorphism $\varphi'\ ((\mathfrak{X}/\mathcal{Y})', \sigma((\mathfrak{X}/\mathcal{Y})', (\mathfrak{X}/\mathcal{Y})^*))$ onto $\mathcal{Y}^\circ$ with topology $\sigma(\mathfrak{X}', \mathfrak{X})/\mathcal{Y}^\circ$ [note $\mathcal{Y}^\circ = \{x' \in \mathfrak{X}' : |\langle y, x' \rangle| \leq 1\ y \in \mathcal{Y}\} = \{x' \in \mathfrak{X}' : x'(y) = 0, y \in \mathcal{Y}\}$

4. Let E, F be complete metric LCS with cont duals E', F' and let $T: E \to F$ be continuous and linear. Show

(A) If T is 1-1 & onto, Then T' is 1-1 and onto

(B) If T is 1-1, $T(E)$ is dense in F but not onto, then there is an unbounded sequence $\{e_n\} \subset E$ so that $\{Te_n\}$ is bounded in F.

(C) If (B) is true, $\exists\, e' \in E'$ so that $\{e'(e_n)\}$ is unbounded and no $f' \in F'$ satisfies $T'f' = e'$.

(D) $T(E)$ is closed in F if and only if $T'(F')$ is $\sigma(E', E)$ closed in E'

Hint write T ~~and~~ as $E \to E/\ker T \xrightarrow[\hat{T}]{} F_0 \to F$ where F_0 is the closed subspace of F given by $cl(T(E))$ and use (3).

5. Show that the closed graph theorem implies the open mapping theorem.