Problem Set #1 535 Due 21 JAN 1977

Pts

5 1 The interior of a balanced set is balanced.

5 2. The interior of a convex set is convex

5 3. Use continuity of vector addition at $(0,0)$ to show: For each open neighborhood of 0, U there is an open neighborhood of 0 V s.t. $V+V \subseteq U$

5 4. If a TVS is T_1 then it is T_2

5 5. If U, V are absolutely convex & absorbing then $\| \cdot \|_{U \cap V} \equiv \max(\| \cdot \|_U, \| \cdot \|_V)$

5 6. If U is absolutely convex & absorbing and $\bar{U}$ is the closure of U then $\| \cdot \|_U = \| \cdot \|_{\bar{U}}$

7. Let ρ be a semi-norm on E

#2 a) Show $\ker \rho$ is a subspace of E and $\ker \rho$ is closed if ρ is continuous on E.

#1 b.) Show that if $x \in y + \ker \rho$ ~~in E~~ then $\rho(x) = \rho(y)$

#3 c). Show $\dot{\rho}(x + \ker \rho) = \rho(x)$ is a norm on $E/\ker \rho$

#4 d). Let E_ρ be the normed space $E/\ker\rho$ with norm $\dot{\rho}$. let $\varphi_\rho : E \to E/\ker\rho = E_\rho$ be the (algebraic) quotient map show φ_ρ is continuous if and only if ρ is continuous on E.

10 8. If A is a set of continuous semi-norms which generate the T_2 topology on E then

$$T : E \longrightarrow \prod_{\rho \in A} E_\rho \text{ is a homeomorphism (into)}$$

where $T(e)$ is the element of the product whose ρ^{th} co-ordinate is $\varphi_\rho(e)$ where

$\varphi_\rho : E \to E_\rho$ is given in 7d (For each $\rho \in A$)

PLACE IN BELLENOT'S MAILBOX IN 206 BY 5:00pm

MIDTERM MATH 535 DO EACH PROBLEM

ALL PROBLEMS ARE WORTH 10 POINTS

1. If $\{e_n\}$ is the usual basis for ℓ_1, then $\{e_n\} \longrightarrow 0$ in the $\sigma(\ell_1, c_0)$ topology but $\{e_n\} \not\rightarrow 0$ in the $\sigma(\ell_1, \ell_\infty)$ topology.

2. If the norm space $\mathfrak{X}$, $\|\cdot\|$ is strictly convex and $f \in \mathfrak{X}^*$ w/ $\|f\| = 1$, then there is __at most__ one $x \in \mathfrak{X}$ w/ $\|x\| = 1$ & $f(x) = 1$.

3. If $\mathfrak{X}$ & $\mathcal{Y}$ are normed spaces and $T: \mathfrak{X} \to \mathcal{Y}$ is a bounded linear map, then $T: \mathfrak{X} \to \mathcal{Y}$ is continuous if $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ is the topology on $\mathfrak{X}$ & $\sigma(\mathcal{Y}, \mathcal{Y}^*)$ is the topology on $\mathcal{Y}$ ($\equiv$ Definition of T is weakly continuous)

4. If the topologies $\sigma(\mathfrak{X}^*, \mathfrak{X})$ & $\sigma(\mathfrak{X}^*, \mathfrak{X}^{**})$ agree on $U^0 = \{x^* \in \mathfrak{X}^* \mid \|x^*\| \le 1\}$, then $\mathfrak{X}$ is reflexive.

5. If $\mathfrak{X}$ is a B-space and $\mathcal{Y}$ is a normed space and $T: \mathfrak{X} \to \mathcal{Y}$ is a weakly continuous (see Problem 3) linear map, Then T is norm cont (i.e. If $\mathfrak{X}$ & $\mathcal{Y}$ has the norm topology.

[HINT: Let $U_\mathcal{Y} = \{y \in \mathcal{Y} \mid \|y\| \le 1\}$ and show $T^{-1}(U_\mathcal{Y})$ is absorbent absolutely convex and normed closed (Then you can quote the result that such sets are neighborhoods (proved in class from Baire Category)).]

Problem set # 3 Due 18 Feb 1977.

points

10 1. Show ℓ_∞^* (real) is the set of real valued functions on $\beta(\mathbb{N})$ that are finitely additive & of bounded variation (i.e. $\|\mu\| = \sup\{\sum_{i=1}^{n} |\mu(A_i)| : A_i \text{ disjoint}\} < +\infty$)

5 2. If X is separable & reflexive then every bounded sequence has a weakly convergent subsequence.

5 3. If X is separable & reflexive and $T: X \to \ell_1$ is continuous then T is compact.

10 4. If $\varphi : \ell_1 \to \ell_2$ is a quotient map & $X = \ker \varphi$ show that there is no continuous projection $P: \ell_1$ onto $X \subseteq \ell_1$. [HINT: Show $\ker(I-P)$ isomorphic to ℓ_2]

5. 5. Show that in $C[0,1]$ (real valued) the unit ball has exactly two extreme points.

5. 6. Show that If $\{e_n\}$ is a basis for X and $Y \subseteq X$ has $\dim Y = +\infty$ then $\forall N \; \exists y \in Y$ s.t. $y = \sum_{i=1}^{\infty} \alpha_i e_i$ and $\alpha_i = 0$ $i = 1, \dots, N$.

10. 7. Show that every infinite dim subspace of ℓ_1 has a subspace isomorphic to ℓ_1.

LAST PROBLEM SET (#4) DUE 4 MARCH 1977

5 1. IF X_0 is a finite dimensional subspace of X and $\varepsilon > 0$ then $\exists$ finite set $x_1, \dots x_n$ of norm one elements of X_0 s.t. $\forall x \in X_0$ $\exists i$; $1 \le i \le n$ with $\|x - x_i\| < \varepsilon$

5 2. Fix $X_0, X, \varepsilon, x_1 \dots x_n$ as in 1, let $f_1, \dots f_n \in X^*$ s.t. $\|f_i\| = f_i(x_i) = 1$ $i = 1, 2, \dots, n$, let $Y = \bigcap_{i=1}^{n} \ker f_i$ Show that for $x \in X_0$, $y \in Y$, $\|x\| < \frac{1}{1-\varepsilon}\|x+y\|$.

5 3. A block basic sequence of a shrinking basis is a shrinking basic sequence

5. 4. A block basic sequence of a boundedly complete basis is a boundedly complete basic seq.

10 5. We know $x_n = (\overbrace{1, 1, \dots 1}^{n \text{ times}}, 0, 0, 0 \text{—})$ is a basis for c_0 show that $\{x_n\}$ is not shrinking and not boundedly complete

10 6. Let $\{x_n\}$ be a basis for the B-space X. Then X is reflexive $\Longleftrightarrow$ $\{x_n\}$ is shrinking and boundedly complete.

10 7. Let $\{x_n\}$ be a boundedly complete monotone basis for the B-space X Let $\{f_n\}$ be the coefficient functiondls, let Y = closed linear span $\{f_n\}$ in X^*, Show $\{f_n\}$ is a shrinking basis for Y and if $\{\varphi_n\}$ are the coefficent functionals to $\{f_n\}$ in Y^* then $T: Y^* \to X$ given by $T(\sum_1^\infty \alpha_n \varphi_n) = \sum_1^\infty \alpha_n x_n$ is a well-defined isomorphim onto X.