Chapter 13 §2 Permutations.

Suppose we have n-distinct objects named: $1, 2, \ldots, n$. Each permutation of these objects can be thought of as a function $\pi(i : 1..n) : 1..n$ where $\pi(i)$ is the position of the i^{th}-object in the permutation. For instance we list π for $n = 16$ as

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 16 & 3 & 4 & 2 & 10 & 11 & 8 & 9 & 13 & 6 & 5 & 14 & 7 & 12 & 15 \end{pmatrix}$$

where the top row is i and the bottom row is $\pi(i)$

This notation is rather long winded and is easy to cut in half (just delete the top row). Our first aim is to radically simplify it. We need to define orbits.

Let π be a permutation on $1..n$ and i w/ $1 \le i \le n$. The orbit of $i = \{ i, \pi(i), \pi(\pi(i)), \pi(\pi(\pi(i))), \ldots \}$. For example if $i = 1$, $\pi(i) = 16$, $\pi\pi(i) = \pi(16) = 1$, $\pi\pi\pi(1) = \pi(1) = 16, \ldots$ so the Orbit of 1 is $\{1, 16\}$. Similarly we obtain

Orbit of 5 $= \{5, 10, 6, 11\}$; Orbit of 15 $= \{15\}$;
Orbit of 6 $= \{6, 11, 5, 10\}$; Orbit of 2 $= \{2, 3, 4\}$

<u>LEMMA 1</u>: $\pi(i) = \pi(j) \iff i = j$

pf. $\Leftarrow$: if $i = j$ then $\pi(i) = \pi(j)$; (the i^{th} object & j^{th} object are the same, so they go to the same place)
$\Rightarrow$: if $\pi(i) = \pi(j)$ then $i = j$: (If $i \ne j$ then the i^{th}-object & j^{th} object go to different places. Thus $\pi(i) \ne \pi$

<u>LEMMA 2</u>: The list

$i, \pi(i), \pi(\pi(i)), \ldots$

is a collection of distinct numbers until some $\pi(\pi(\ldots\pi(i)\ldots)) = i$, then it endlessly repeats itself.

pf: Let $x_1 = i$, $x_2 = \pi(i) = \pi(x_1)$ and in general $x_{k+1} = \pi(x_k)$
So we can write the list as

$x_1, x_2, x_3, \ldots\ldots, x_n, x_{n+1}, \ldots$

Now the pigeonhole principle says two of the numbers x_k, x_j must be the same. (there are only n possible values $1,..n$) Let k be the smallest number s.t. there is a $j < k$ with $x_j = x_k$. If $j \neq 1$, then $x_j = \pi(x_{j-1}) = x_k = \pi(x_{k-1})$. This would imply (by Lemma 1) $x_{j-1} = x_{k-1}$ contradicting our choice of k. $\therefore$ $j = 1$. Also by our choice of k: $x_1, x_2, .. x_{k-1}$ are distinct. Now it starts repeating $x_k = x_1$, $x_{k+1} = x_2$ ... $x_{2k-2} = x_{k-1}$, $x_{2k-1} = x_1$, ...

LEMMA 3: If i is in the orbit of j, then
Orbit of i = Orbit of j

pf: As in lemma 2 write $x_1 = j$ $x_{k+1} = \pi(x_k)$ there is some l with $x_l = i$. Let $y_1 = i$ and $y_{k+1} = \pi(y_k)$. Then $y_1 = x_l$, $y_2 = x_{l+1}$, ... $y_k = x_{l+k-1}$ and since x_k's endlessly repeat themselves the two orbits are the same.

Define $i \sim j$ if i is in the orbit of j

Proposition: "$\sim$" is an equilvence relation.

pf Reflective: $i \sim i$ since i is the first element of the orbit of i

Symmetric: if $i \sim j$ then i is in orbit of j by Lemma 3 orbit of j = orbit of i and j is in the Orbit of j thus $j \sim i$

transitive: if $i \sim j$ and $j \sim k$, then by Lemma 3 Orbit of i = Orbit of j = Orbit of k. Thus $i \sim k$

Thm: For i, j Either (Orbit of i) $\cap$ (Orbit of j) $= \emptyset$ (empty set not zero) or Orbit of i = Orbit of j

pf: If the intersection is non-empty, let k belong to both orbits. By Lemma 3, Orbit i = Orbit k = Orbit j.

Remark: Say that a collection of subsets $A_1, A_2 \dots A_p$ of X is a PARTITION if $A_1 \cup A_2 \cup \dots \cup A_p = X$ and if $i \neq j \Rightarrow A_i \cap A_j = \emptyset$ or $A_i = A_j$. The theorem says Orbit 1, Orbit 2, ... Orbit n is a partition of $\{1 .. n\}$.

This is actually a special case of a general result: Equivalence relations yield partitions (see exercise)

Define the cycle $(x_1, x_2, \dots x_{k-1})$ means $\pi(x_1) = x_2$, $\pi(x_2) = x_3 \dots$ $\pi(x_{k-2}) = x_{k-1}$ and $\pi(x_{k-1}) = x_1$ and $x_1, x_2 \dots x_{k-1}$ are all distinct

We can rewrite π on page 1 in the product of cycles form

$$\pi = (1,16)(2,3,4)(5,10,6,11)(7,8,9,13)(12,14)$$

note if i is listed in a cycle we don't repeat the cycle also (15) is omitted since π doesn't move 15.

Suppose $\pi' = (1,16)(4,5,6)(13,11,14)$

What happens if I do π, then do π' to the rearrange list. The resulting list is also a permutation so we want its product of cycles form.

1 goes to 16 by π, 16 goes to 1 by π' which yields (1) which is omitted. $2 \xrightarrow{\pi} 3 \xrightarrow{\pi'} 3 \xrightarrow{\pi} 4 \xrightarrow{\pi'} 5 \xrightarrow{\pi} 10 \xrightarrow{\pi'} 10 \xrightarrow{\pi} 6 \xrightarrow{\pi'} 4 \xrightarrow{\pi} 2 \xrightarrow{\pi'} 2$. Yields $(2,3,5,10,4)$. $6 \xrightarrow{\pi} 11 \xrightarrow{\pi'} 14 \xrightarrow{\pi} 12 \xrightarrow{\pi'} 12 \xrightarrow{\pi} 14 \xrightarrow{\pi'} 13 \xrightarrow{\pi} 7 \xrightarrow{\pi'} 7 \xrightarrow{\pi} 8 \xrightarrow{\pi'} 8 \xrightarrow{\pi} 9 \xrightarrow{\pi'} 9 \xrightarrow{\pi} 13 \xrightarrow{\pi'} 11 \xrightarrow{\pi} 5 \xrightarrow{\pi'} 6$ Yields $(6,14,12,13,7,8,9,11)$ $15 \xrightarrow{\pi} 15 \xrightarrow{\pi'} 15$; $16 \xrightarrow{\pi} 1 \xrightarrow{\pi'} 16$ Yields omitted $(15)(16)$. Thus π followed by π' is $(2,3,5,10,4)(6,14,12,13,7,8,9,11)$

Remark some people write this as $\pi\pi'$ others as $\pi'\pi$ (Confusing, yes?)

Exercises

1. Let $\phi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\psi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$
   - A. Write φ & ψ as product of cycles
   - B. Write $\varphi\psi$ & $\psi\varphi$ as product of cycles ($\varphi\psi$ = do φ then do ψ)
   - C. Write φ^2 & ψ^2 as product of cycles ($\varphi^2 = \varphi\varphi$)

2. Let $\varphi_i = (1, i)$ in product of cycles form
   - A. Find $\varphi_2\varphi_{1,\lambda}$ ($\varphi_1\varphi_2$), $\varphi_3\varphi_2\varphi_{1,\lambda}$ ($\varphi_1\varphi_2\varphi_3$), $\varphi_4\varphi_3\varphi_2\varphi_1$ & $\varphi_1\varphi_2\varphi_3\varphi_4$
   - B. A cycle of the form (i, j) $(i \neq j)$ is called a <u>transposition</u> what does A. say about every cycle?
   - C. Prove each permutation can be written as $\pi_n \cdots \pi_2\pi_1$ where the π_i's are transpositions

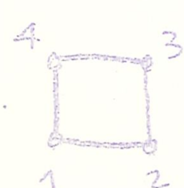
3. A permutation is even if n is even in 2c else it is odd.
   - A. In problem one which of $\varphi, \psi, \psi^2, \varphi\psi, \psi\varphi$ are even and which are odd.
   - B. If $(x_1, \ldots x_k)$ is a cycle when is it odd and when is it even?

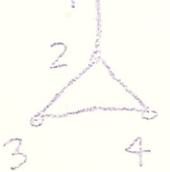
4. Let $\varphi = (1, 2, 3, 4)$ $\psi = (1, 4, 3, 2)$ compute $\varphi\psi$ and $\psi\varphi$

5. Let X be a set and let $\approx$ be an equivalence relation on X. For $x \in X$ let $E_x = \{y : y \approx x\}$. Show $\{E_x : x \in X\}$ is a partition of X

6. Let $A_1, \ldots A_k$ be a partition of X define a relation $\sim$ on X by $x \sim y$ if $x \in A_i$ implies $y \in A_i$ (for each i). Show $\sim$ is an equivalence relation.

7. Let X = integers $x \approx y \iff x - y$ is divisible by 5 Show $\approx$ is Equivalence relation, what are $E_0, E_1, \ldots E_4$ (in notation of 5)

8.
   - A. 

   list the permutations ~~tot~~ on $\{1..4\}$ which are isomorphisms of this graph
   - B. 

   same for this graph

TEST 4 MAD 3105 By ____________

SHOW ALL WORK & BE NEAT for CREDIT. 1-4: 10pts

1 A. Define a connected graph:

B. Define a spanning tree for a graph G:

2. Draw a Venn diagram for each statement to right: 1. All trees have paths
1. 2. 3. 2. All connected graphs have
3 ∴ All trees are connected

Is the logic valid or not? Draw a picture which supports your conclusion

3. Draw all trees with 5 edges
(Do not include duplicates (i.e isomorphic graphs)) ~~more than once~~)
Exactly one of each type.

4. $(x+1)(x-3)^2 = x^3 - 5x^2 + 3x + 9$. For the recurrence relation
$a_n - 5a_{n-1} + 3a_{n-2} + 9a_{n-3} = f(n)$ do the following:
AB: Write the general solution to homogeneous equation:

CDE: Write the correct guess for the form of a particular solution when $f(n)$ is the given function
C. $f(n) = 4n^2 - 1$ D. $f(n) = 6 \cdot 3^n$ E. $f(n) = 2n(-1)^n$

5. Solve (find the solution) $a_n + a_{n-1} = 6a_{n-2}$; $a_0 = 5$; $a_1 = 25$.

6. Draw the picture (i.e. graph) required in each part

A. A path with minimal number of edges B. A circuit with minim

C.D In C&D Find the rooted tree with 5 edges which has

C. The fewest leaves D. With the smallest height

E. Using the fewest edges possible, find a connected graph which isn't a tree but has a cut edge

7. Prove: An edge in a tree is a cut-edge.

8. The complete graph on n vertices (called K_n) has an edge between any pair of vertices x & y with $x \neq y$. Clearly, K_n has exactly $\binom{n}{2} = \frac{n(n-1)}{2}$ edges by the counting techniques of finite 1. YOUR JOB is to prove K_n has $n(n-1)/2$ edges BY INDUCTION on n (the number of vertices).