

1.A. Find a F-augmenting path for the flow F on the transport network in figure 1.

B. On the transport network, show how to increase the flow using your path in part A.

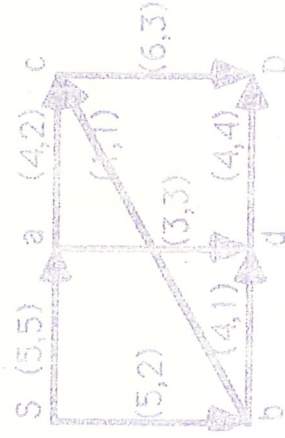


fig. 1

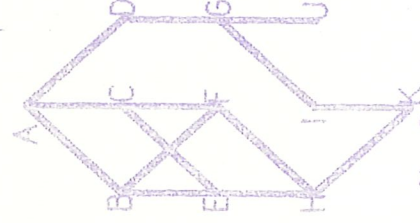


fig. 2

2. For the partial order "Klasse" graph in figure 2 answer:

- A. The maximal elements are: _____
- B. The minimal elements are: _____
- C. Find glb (F, I): _____
- D. Give a topological enumeration: _____
- E. Find lub (E, F): _____

3. The matrix (k=0) is an adjacency matrix of a digraph G. Use Warshall's algorithm to fill in the other matrices.

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$	$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$	$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$
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k = 0

k = 1

k = 2

k = 3

k = 4

4. Define the relation R on the set of non-zero rational numbers by $xRy \Leftrightarrow x/y$ is an integer. For each of the property below say "yes" if R has that property else say "no" and give a counterexample.

<u>reflexive</u>		transitive		symmetric		anti-symmetric

5. A. Company XYZ has 200 ditch diggers, 110 have round bottom shovels, 37 have both round and square bottom shovels and 22 don't have shovels. How many have square bottom shovels?

B. How many ways are there to put 202 people into 4 rooms with at least one person in each room?

14. Prove by induction:

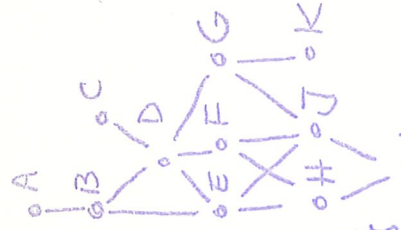
Given $a_n = \sqrt{6 + a_{n-1}}$ for $n \geq 1$, and $a_0 = \sqrt{3}$. Prove $a_n < 3$, for $n \geq 0$.

15. Prove a partial order is a total order if and only if its digraph is unilaterally connected.

16. Prove: If $G = (V, E)$ is a digraph and $n > 0$, then (x, y) is in E^n if and only if there is a directed path of length n from x to y in G .

1. For the pos. "graph" to the right

- A. List all maximal elements. _____
- B. List all minimal elements. _____
- C. List all upper bounds to the set {H, J, K} _____
- D. Find lub {H, G} _____
- E. Find glb {A, D} _____



2. The relation R is defined on the set $\{1, 2, 3, \dots\}$ by $xRy \iff x+z \leq y$. For each property below say "yes" if R has that property else say "no" and give a counter example

- A. Reflexive _____
- B. Transitive _____
- C. Symmetric _____
- D. Anti-symmetric _____

3. Draw Digraphs showing relations with the required properties

A. A relation which isn't transitive

B. A relation which is neither symmetric nor anti-symmetric

C. An equivalence relation on 3 or more vertices which isn't the relation " $=$ " and has more than one equivalence class.

4. A recurrence relation has characteristic polynomial $(x^2+1)(x-1)(x+3)^2$. Write the general solution to the homo problem

BCDE For the given forcing fn $f(n)$, write the correct guess for the particular solution

B, $8 \cdot 2^n$ _____ C, $100n$ _____ D, $6i^n$ _____ E, $(n+7)3^n$ _____

5. Solve $a_n + 2a_{n-1} = 10 \cdot 3^n$ $a_0 = 20$.

6. A. Find a pair of elements in the ^{Pos.} DAG in Prob 1 which do not have a lub!

B. Find a counterexample to "a poset with a least element is well-ordered"

C. Find a counterexample "if $R \notin S$ are irreflexive, then RS is irreflexive"

D. Find two positive integers x, y so that $\text{lem}(x, y)$ isn't x, y or xy .

E. Subsets $A \neq E$ so that $(A - E) \cup E \neq E$

7. Given $a_1 = 5$ and $a_n = 9 - 2(1 - \frac{1}{n})a_{n-1}$ for $n \geq 2$

Prove by induction $a_n = 3 + \frac{2}{n}$ for $n \geq 1$.

8. Prove: if the finite poset A has exactly one minimal element s , then s is the least element of A . (You may use the fact that each finite poset has a minimal element).

6. Solve $a_n = 2a_{n-1} + 6a_{n-2}$; $a_0 = 1$, $a_1 = 2$.

7. A recurrence relation has characteristic polynomial $(x^2+1)(x-1)(x-2)(x+3)$.
A. Write the general solution to the homogeneous problem.

BCDE. For the given forcing function, write the correct guess for the form of the particular solution:

- B. $8 \cdot 4^n$ C. $10n$ D. $6i^n$ E. $n(-3)^n$

8. Solve $a_n = 3a_{n-1} + 2n$; $a_1 = 5$;

9. A. Find the coefficient of x^{40} in:

$$(1 + x + x^2 + \dots)^5 (x^5 + x^6 + \dots + x^9)^3 (x^3 + x^4 + x^5 + \dots)^4$$

B. Write (but do NOT solve) a generating function and tell which coefficient we need to find in order to count the number of ways of putting 50 identical balls into 20 boxes so that odd numbered boxes have either 1, 3 or 7 balls, boxes 2, 4, 6, 8 and 10 each have at least 2 balls and the rest of the boxes have no more than 3 balls.

10. Solve by generating functions: $a_0 = 16$; and for $n > 0$, $a_n = 3a_{n-1} + 4 \cdot 5^n$.

11. Give examples:

A. A circuit which isn't a cycle.

B. A digraph of a relation which isn't reflexive or irreflexive.

C. A digraph of a symmetric relation which isn't transitive.

D. Find all the unilaterally connected components of



E. Give the run time complexity of a binary search in an array of length n .

12 & 13. Give counterexamples:

A. A non-trivial closed path is a circuit.

B. A maximal matching which isn't a complete matching.

C. A maximal directed path is a directed path of maximal length.

D. A transport network with a unique minimal cut has a unique maximal flow.

E. A transport network with a unique maximal flow has a unique minimal cut.

F. A partial order on more than 2 elements is never an equivalence relation.

G. For each choice of X being one of the words "weakly", "unilaterally" or "strongly", each X connected digraph G , has a vertex v , so that $G-v$ is X connected.

H. If F is a maximal flow and $k(X, X^c) = F(X, X^c)$, then (X, X^c) is a minimal S-D cut.

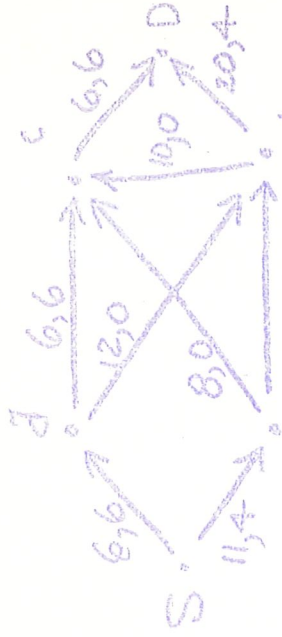
I. A partially ordered set with only one maximal element has a maximum element.

J. The lexicographic ordering on strings of the usual 26 letters is a well ordering.

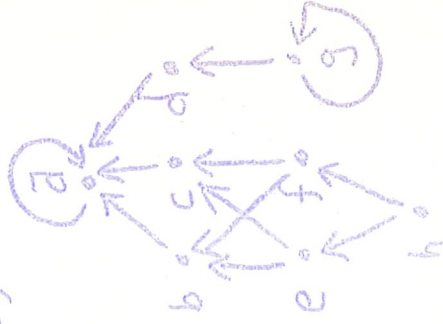
PM 2 - Test 2 Show ALL work for credit B)

1-4 one worth 10pts each, 5-8 15pts each, No

1. A. Find a F-augmenting path for the flow F on the transport network to right S.



B. On the transport network show how ~~to~~ increase the flow using your path in A.
 2. The matrix ($k=0$) is an adjacency matrix of a digraph G. Use Warshall's algorithm to fill in the other matrices



3. A. Construct the adjacency matrix for the digraph to the right
 B. Give any topological enumeration of this digraph

C. Give another topological enumeration different from the one given in B

4. Solve $a_n - 5a_{n-1} + 6a_{n-2} = 0$ $a_0 = 1$ $a_1 = 0$.

5. A. give an example of a maximal directed path which isn't a directed path of maximal length.

B. draw a digraph of a non-reflexive relation whose transitive closure is reflexive

C. give a permutation of $\langle 4, 2, 3, 4, 5 \rangle$ which has a disorder of

D. what is runtime complexity of interchange sort?

E. List the unilaterally connected components of $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$

DMZ FR 1 1-4 10pts 5-8 15pts * problems are from old tests ** problems are "close" to old tests

1. *A. In Z_{13} find i with $0 \leq i < 13$ so that $[i] = \frac{[2]}{[9]}$
**B. In the ordering $N, 1$ ($1 = \text{divides}$) find $\text{lub}(1462, 132)$

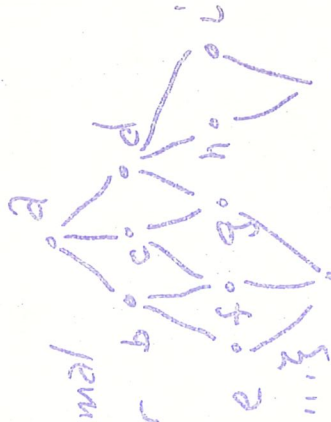
*2. The roots of ~~the~~ characteristic poly are $2, 2, 1, 5$ write A. write the general solution to the homo Prob.

B-E for the given forcing function fns give the correct guess for the particular solution

B. $f(n) = 3 \cdot 4^n$ C. $f(n) = n^2 + 7$ D. $f(n) = \frac{1}{2} \cdot 5^n$ E. $f(n) = 7 \cdot 2^n$

*3. For the p.o. 'graph' to right

A list all maximal elements B list all minimal elements. C list all elements "less than or equal to d" D $\text{glb}(b, g)$ E $\text{lub}(j, h)$



*4 The relation R is defined on $\{1, 2, 3, \dots\}$ by $xRy \iff x | (y+1)$. For each property below either say yes or say no and give a counterexample

A. ref B. irref. C. trans. D anti-sym

*5. Solve $a_n + a_{n+1} = 6a_n - 2$ $a_0 = 5$ $a_1 = 25$

6. Give counterexamples

*A an equivalence relation on a set with 2 or more elements is never a p.o. (draw digraph)

**B a relation which is not reflexive is irreflexive

C. if R & S are symmetric relations then $S \cdot R$ is symmetric

D. two strings have the same relationship in both the lexicographic and the enumeration orderings.

E. If the relation $R \subseteq R^2$ then R is transitive.
*7. K_n (complete graph on n vertices) [not digraph] has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. Prove this by induction. P.O.

B. We have proved that each finite poset has a maximal element. Prove that a finite poset with only one maximal element has a greatest element!

MORE PROBLEMS FROM OLD TESTS

10 pt problems

1. $a_0 = 0$, $a_n = 2a_{n-1} + 2^{n-2}$ for $n \geq 1$. Prove by induction $a_n = 4n2^n$ $n \geq 0$
2. $a_1 = 2$, $a_n = \frac{1}{a_{n-1}} + \frac{2}{n}$ for $n \geq 2$. Prove by induction $a_n = \frac{n+1}{n}$ $n \geq 1$
3. $(x+1)(x-3)^2 = x^3 - 5x^2 + 3x + 9$. For the recurrence relation $a_n = 5a_{n-1} + 3a_{n-2} + 9a_{n-3} = f(n)$

AB: write the gen sol when $f(n) = 0$. CDE write the correct guess for the particular sol when $f(n) = 4n^2 + 1$, $D = 6$, $E = 2n(-1)^n$ (do not solve for the part. sol.)

15 pt problems

4. $a_n = 7a_{n-1} + 12a_{n-2} = 12n - 4$ $a_0 = 5$ $a_1 = 8$
5. $a_0 = 3$ $a_1 = 4$ $a_n = 4a_{n-1} - 4a_{n-2}$ ($n \geq 2$)
Prove by induction $a_n = 3 \cdot 2^n - n2^n$ $n \geq 0$

Other Problems

A. Define lattice, total order, well order, equiv rel...

B. Give examples: 1. total order which isn't a well order
2. a lattice which isn't totally ordered. 3. A partially ordered set which isn't a lattice.

C. Give counterexamples: 1. lexicographic order is a well ordering. 2. \mathbb{H} subset B of a p.o. set A has an upper bound then B has a greatest element 3. $B \subset A \subseteq$ has a upper bound then it has a lub.
D. Prove:

1. If $R^2 \subseteq R$ then R is trans (R is a relation)
2. If R is symmetric, transitive and has the property that $\forall x, y$ then xRy or yRx , then R is a total order.

3. If R is symmetric, prove by induction that R^n is symmetric for $n \geq 1$

DMZ PT2 Some Old Problems on old tests:

1. (15pts) A $4 \log_2 n = n^k$ so $k = ?$

BCDE Solve $a_n = 2a_{n/3} + 4$, $a_1 = 5$

2. (10pts) Define $A, R(\mathbb{R}, \mathbb{R})$ (on a network) B, A circuit-

3. (4pts) Give counter examples: A. If E is an edge $x \rightarrow y$ in the connected graph G so that each vertex of G is a cut-edge of G then the only path $x \rightarrow y$ in G is E ~~is E~~ If v, w

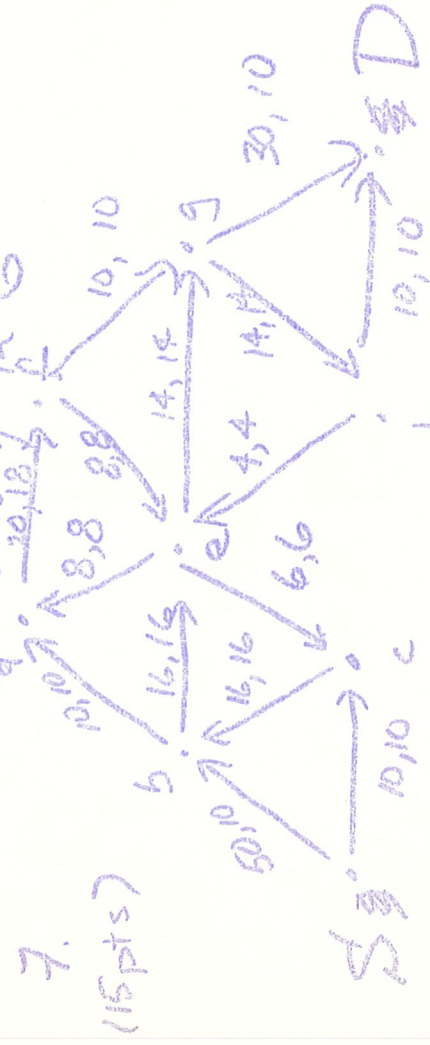
B. (4pts) If P, Q $x \rightarrow y$ are different ^{simple} paths then P followed by Q backwards is a circuit.

C. (4pts) A connected graph with G vertices and F edges has exactly two circuits (Indicate the Arcs)

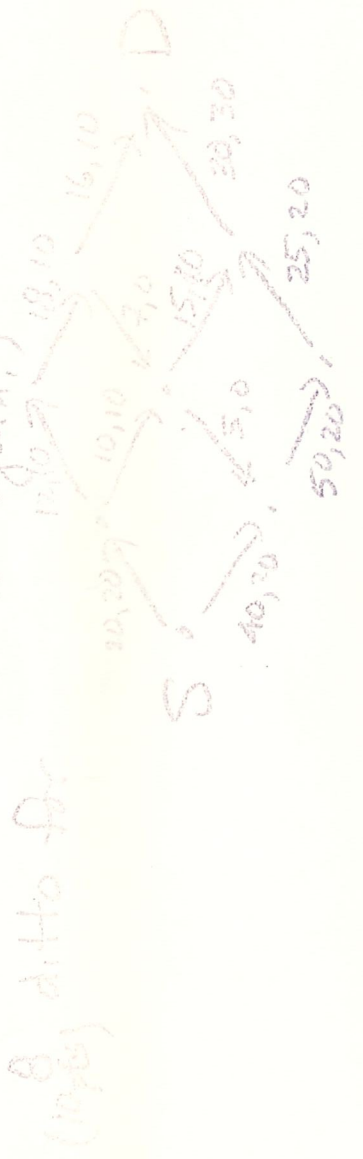
4. (15pts) Solve $a_n = 3a_{n/3} + 2n$, $a_1 = 1$

5. (15pts) G is a conn graph and E is an edge in G . Prove E is a cut-edge iff E is no circuits of G

6. (15pts) Prove by induction (on the crossing no.) If there is a path from x to y in G , then there is a simple path from x to y in G



A. Find a F- augmenting path B. Show how to increase the flow using your path in A. C. Suddenly there is a break in the edge (ed), without changing the value of the flow and changing as few edges as possible, correct the (i.e. make it a flow again.)



1. Give

Counterexamples: Each circuit is a cycle. Eachsimple closed path is a cycle. Each closed path with at least one edge is a circuit. An F-augmenting path can have no saturated edges. If $k(X, \bar{X})$ isa min cut and F is a flow so that each edge from a vertex in X to \bar{x} vertex in \bar{X} is saturated, thenF is a max flow. Interchanging a_i and a_j decreasesthe disorder of the array a_1, a_2, \dots, a_5 when $1 \leq i < j \leq 5$.merging two list with 7 elements each takes at most 8 comparisons, at least ~~8~~ comparisons2. Solve $a_n - 3a_{n-1} + 6a_{n-2} = 0$ $a_1 = a_2 = 10$ 3. Illustrate: Merge sort on the list $\langle 8, 1, 4, 3, 5, 6, 2 \rangle$ Binsearch (25) on $\langle 5, 10, 15, 20, 25, 30, 35 \rangle$ Top sort on G belowWarshall's Algorithm on G below
(Choose the vertex with smaller lexicographic order when paths pass.)

4. What is the run time complexity of Quicksort, Top sort, merge sort, binsearch?

5. Give examples: A unilaterally connected graph which has 3 strongly connected components. A weakly connected graph which isn't unilaterally connected. Two strongly unilaterally connected components which have a non-void intersection. An edge which is in no strongly conn. comp.

6. Prove by induction $a_n = 2^{3^n + (-1)^n}$ is the solution to $a_n = (a_{n-1})^2 \cdot (a_{n-2})^3$ $a_0 = a_1 = 4$ [hint take logs]

7. Prove the digraph G is strongly connected if and only if G contains a closed path with all the vertices of G

Part I: From old tests 1 - 7 were 10 points, 8 - 9 were 15 points.

1. Write (but do not solve) a generating function and tell which coefficient we need to find the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 27$$

with $0 \leq x_1, x_2, x_3; 3 \leq x_4 \leq 7$ and x_5 is an even integer between 1 and 7.

2. Find the coefficient of x^{24} in $(x^3 + x^4 + x^5 + x^6 + x^7)^4 (1 + x + \dots)^{12}$.

3. Find the generating function with $a_r = 4r^2$.

4. Solve $a_n = 4a_{n-1} + 6^n$ (for $n \geq 1$), $a_0 = 2$ by generating functions.

5. Use generating functions to find how many ways there are to put 30

identical balls into 9 distinct boxes so that the first three boxes have at least two balls each and the last three boxes have no more than 10 balls each.

6. Use Inclusion Exclusion to find the number of ways in which you can pick 20 electronic games from Santa's 50 kinds if you can pick up to 5 games of the same kind.

7. How many ways are there to roll 100 distinct dice with at least one of each kind of face (i.e. each of the numbers 1 through 6) showing?

8. Solve by GENERATING FUNCTIONS $a_n = 5a_{n-1} + 2^n$ (for $n \geq 1$); $a_0 = 2$.

9. A. Find the number of arrangements of ALGORITHM with the A before the L or the L before the G or the G before the A.

B. Find the number of 7 card poker hands with at least one card in each suit.

Part II: More questions

10. Solve for a_n given

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{30 - 2x^2}{(1-2x)(1+3x)(1-10x)}$$

11. Given $|A| = 100, |B| = 112, |C| = 95, |A \cap B| = 50, |A \cap C| = 47, |B \cap C| = 35$, and

$|A \cap B \cap C| = 13$; find $|A \cup B \cup C|$.

12. Give Counterexamples:

A. If a flow F and a S-D cut (X, X^C) on a transport network satisfy $F(X, X^C) = k(X, X^C)$ then F is a maximal flow.

B. A maximal matching is a complete matching.

C. If G is a directed bipartite graph with $|A| \leq |B|$, then a maximal matching is a complete matching.

D. A transport network with only one minimal cut has a unique maximal flow.

E. A transport network with a unique maximal flow has a unique minimal cut.

13. Suppose (G, k) is a transport network with integer edge capacities.

Suppose further that $e_1 = (v_0, v_1), e_2 = (v_1, v_2), \dots, e_k = (v_{k-1}, v_0)$ is a

directed k -cycle where $k(e_i) > 0$ for $1 \leq i \leq k$. Prove or disprove: there is a

maximal flow F on G with the property that $F(e_i) > 0$ for all $1 \leq i \leq k$.

14. Prove:

$$\binom{2n+k}{n+k} = \sum_{i=0}^{2n} \binom{1+k+i}{i}$$

Hint: Count the non-negative integer solutions to $x_1 + \dots + x_k < n$ in two ways.

Solve $a_n = 2a_{n-1} + 3^n$ ($n \geq 1$)^{*} $a_0 = 5$ by generating functions [we know answer is $a_n = 2 \cdot 2^n + 3^n$]

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ (the generating fun!)
 since $(a_0 + a_1 x + a_2 x^2 + \dots) = a_0 + (a_1 + a_2 x + \dots)$ we need to do this because of *

$$= a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n$$

since $a_0 = 5$ is given & $a_n = 2a_{n-1} + 3^n$ is given for $n \geq 1$

$$= 5 + \sum_{n=1}^{\infty} (2a_{n-1} + 3^n) x^n$$

distributive & commutative law

$$= 5 + \sum_{n=1}^{\infty} 2a_{n-1} x^n + \sum_{n=1}^{\infty} 3^n x^n$$

factor $2x$ out of 1st sum. Use $(b_1 + b_2 + \dots) = (b_0 + b_1 + \dots) - b_0$ on 2nd sum

$$= 5 + 2x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=0}^{\infty} 3^n x^n - 3^0 x^0$$

$$5 - 1 = 4; \quad (a_{-1} x^{-1} + a_{-2} x^{-2} + \dots) = (a_0 x^0 + a_1 x^1 + \dots); \quad \frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$$

$$= 4 + 2x \sum_{n=0}^{\infty} a_n x^n + \frac{1}{1-3x}$$

$$A = \frac{4-12x}{1-3x}; \quad A(x) = \sum_{n=0}^{\infty} a_n x^n \text{ Subst.}$$

$$= 2x A(x) + \frac{4-12x}{1-3x}$$

or we can re write this as

$$A(x)(1-2x) = \frac{4-12x}{1-3x}$$

or via partial fractions

$$A(x) = \frac{5-12x}{(1-2x)(1-3x)} = \frac{c}{1-2x} + \frac{d}{1-3x}$$

we obtain [multiply by $(1-2x)$ & subst $\frac{1}{2}$] $c = 2$ [in $(1-3x)$ & $x = \frac{1}{3}$] $d = 3$

$$\sum_{n=0}^{\infty} a_n x^n = A(x) = \frac{2}{1-2x} + \frac{3}{1-3x} = 2 \sum_{n=0}^{\infty} 2^n x^n + 3 \sum_{n=0}^{\infty} 3^n x^n$$

$$\text{or} \quad \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (2 \cdot 2^n + 3 \cdot 3^n) x^n$$

$$\therefore a_n = 2 \cdot 2^n + 3 \cdot 3^n$$

$$\frac{1}{1-bx} = \sum_{n=0}^{\infty} b^n x^n \quad \frac{1}{1+bx} = \sum_{n=0}^{\infty} (-1)^n b^n x^n$$

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